

# AN EXTREME FUNCTION FOR A CERTAIN CLASS OF ANALYTIC FUNCTIONS

JUNICHI NISHIWAKI AND SHIGEYOSHI OWA

*Presented at the 8<sup>th</sup> International Symposium GEOMETRIC FUNCTION THEORY AND APPLICATIONS, 27-31 August 2012, Ohrid, Republic of Macedonia.*

**ABSTRACT.** Let  $\mathcal{A}$  be the class of analytic functions  $f(z)$  in the open unit disk  $\mathbb{U}$ . Furthermore, the subclass  $\mathcal{B}$  of  $\mathcal{A}$  concerned with the class of uniformly convex functions or the class  $S_p$  is defined. By virtue of some properties of uniformly convex functions and the class  $S_p$ , an extreme function of the class  $\mathcal{B}$  and its power series are considered.

## 1. INTRODUCTION

Let  $\mathcal{A}$  be the class of functions  $f(z)$  of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk  $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ . A function  $f(z) \in \mathcal{A}$  is said to be in the class of uniformly convex (or starlike) functions denoted by  $\mathcal{UCV}$  (or  $\mathcal{UST}$ ) if  $f(z)$  is convex (or starlike) in  $\mathbb{U}$  and maps every circle or circular arc in  $\mathbb{U}$  with center at  $\zeta$  in  $\mathbb{U}$  onto the convex arc (or the starlike arc) with respect to  $f(\zeta)$ . These classes are introduced by Goodman [1] (see also [2]). For the class  $\mathcal{UCV}$ , it is defined as the one variable characterization by Rønning [4] and [5], that is, a function  $f(z) \in \mathcal{A}$  is said to be in the class  $\mathcal{UCV}$  if it satisfies

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in \mathbb{U}).$$

It is independently studied by Ma and Minda [3]. Further, a function  $f(z) \in \mathcal{A}$  is said to be the corresponding class denoted by  $S_p$  if it satisfies

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \left| \frac{zf'(z)}{f(z)} - 1 \right| \quad (z \in \mathbb{U}).$$

This class  $\mathcal{S}_p$  was introduced by Rønning [4]. We easily know that the relation  $f(z) \in \mathcal{UCV}$  if and only if  $zf'(z) \in \mathcal{S}_p$ . In view of these classes, we introduce the subclass  $\mathcal{B}$  of  $\mathcal{A}$  consisting of all functions  $f(z)$  which satisfy

$$\operatorname{Re} \left( \frac{z}{f(z)} \right) > \left| \frac{z}{f(z)} - 1 \right| \quad (z \in \mathbb{U}).$$

We try to derive some properties of functions  $f(z)$  belonging to the class  $\mathcal{B}$ .

**Remark 1.1.** For  $f(z) \in \mathcal{B}$ , we write  $w(z) = \frac{f(z)}{z} = u + iv$ , then  $w$  lies in the domain which is the part of the complex plane which contains  $w = 1$  and is bounded by a kind of teardrop-shape domain such that

$$u^4 - 2u^3 + 2u^2v^2 - 2uv^2 + v^4 + v^2 < 0.$$

## 2. AN EXTREME FUNCTION FOR THE CLASS $\mathcal{B}$

In this section, we would like to exhibit an extreme function of the class  $\mathcal{B}$  and its power series. For our results, we need to recall here some properties of the class  $\mathcal{S}_p$ .

**Lemma 2.1. (Rønning [4]).** *The extremal function  $f(z)$  for the class  $\mathcal{S}_p$  is given by*

$$\frac{zf'(z)}{f(z)} = 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2.$$

By using the expansion of logarithmic part of  $\frac{zf'(z)}{f(z)}$  in Lemma 2.1, we get

**Lemma 2.2. (Rønning [4]).** *The power series of  $\frac{zf'(z)}{f(z)}$  is following*

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 \\ &= 1 + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \left( \sum_{k=1}^n \frac{1}{2k-1} \frac{1}{2n+1-2k} \right) z^n. \end{aligned}$$

From Remark 1.1 and Lemma 2.1, we have the first result for the class  $\mathcal{B}$ .

**Theorem 2.1.** *The extreme function  $f(z)$  for the class  $\mathcal{B}$  is given by*

$$f(z) = \frac{z}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2}.$$

*Proof.* Let us consider the function  $\frac{f(z)}{z}$  as given by

$$\frac{f(z)}{z} = \frac{1}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2}.$$

It suffices to show that  $\frac{f(z)}{z}$  maps  $\mathbb{U}$  onto the interior of the domain such that

$$u^4 - 2u^3 + 2u^2v^2 - 2uv^2 + v^4 + v^2 < 0,$$

implying that  $\frac{f(z)}{z}$  maps the unit circle onto the boundary of the domain. Taking  $z = e^{i\theta}$ , we obtain that

$$\begin{aligned} \frac{1}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2} &= \frac{1}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + e^{i\frac{\theta}{2}}}{1 - e^{i\frac{\theta}{2}}} \right) \right)^2} \\ &= \frac{1}{1 + \frac{2}{\pi^2} \left( \log i - \log \left( \tan \frac{\theta}{4} \right) \right)^2} \\ &= \frac{1}{\frac{1}{2} + \frac{2}{\pi^2} \left( \log \left( \tan \frac{\theta}{4} \right) \right)^2 - i \frac{2}{\pi} \log \left( \tan \frac{\theta}{4} \right)} \\ &= \frac{\frac{1}{2} + \frac{2}{\pi^2} \left( \log \left( \tan \frac{\theta}{4} \right) \right)^2}{\frac{1}{4} + \frac{6}{\pi^2} \left( \log \left( \tan \frac{\theta}{4} \right) \right)^2 + \frac{4}{\pi^4} \left( \log \left( \tan \frac{\theta}{4} \right) \right)^4} \\ &\quad + i \frac{\frac{2}{\pi} \log \left( \tan \frac{\theta}{4} \right)}{\frac{1}{4} + \frac{6}{\pi^2} \left( \log \left( \tan \frac{\theta}{4} \right) \right)^2 + \frac{4}{\pi^4} \left( \log \left( \tan \frac{\theta}{4} \right) \right)^4}. \end{aligned}$$

Writing  $\frac{f(z)}{z} = u + iv$ , we see that

$$\log \left( \tan \frac{\theta}{4} \right) = \frac{\pi(u \pm \sqrt{u^2 - v^2})}{2v}.$$

Thus we have

$$\begin{aligned}
 v &= \frac{\frac{2}{\pi} \log \left( \tan \frac{\theta}{4} \right)}{\frac{1}{4} + \frac{6}{\pi^2} \left( \log \left( \tan \frac{\theta}{4} \right) \right)^2 + \frac{4}{\pi^4} \left( \log \left( \tan \frac{\theta}{4} \right) \right)^4} \\
 &= \frac{\frac{2}{\pi} \frac{\pi(u \pm \sqrt{u^2 - v^2})}{2v}}{\frac{1}{4} + \frac{6}{\pi^2} \left( \frac{\pi(u \pm \sqrt{u^2 - v^2})}{2v} \right)^2 + \frac{4}{\pi^4} \left( \frac{\pi(u \pm \sqrt{u^2 - v^2})}{2v} \right)^4}.
 \end{aligned}$$

Therefore, we arrive that

$$u^4 - 2u^3 + 2u^2v^2 - 2uv^2 + v^4 + v^2 = 0.$$

This completes the proof of the theorem.  $\square$

Considering the power series of the function  $f(z)$  in Theorem 2.1, we derive

**Theorem 2.2.** *The power series of the extreme function for the class  $\mathcal{B}$  is given by*

$$\begin{aligned}
 f(z) &= \frac{z}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2} \\
 &= z + \sum_{n=2}^{\infty} \sum_{p=1}^{n-1} (-1)^p \left( \frac{8}{\pi^2} \right)^p \sum_{m_1=1}^{n-p} \left( \sum_{k=1}^{m_1} \frac{1}{2k-1} \frac{1}{2m_1+1-2k} \right) \\
 &\quad \times \sum_{m_2=1}^{n+1-p-m_1} \left( \sum_{k=1}^{m_2} \frac{1}{2k-1} \frac{1}{2m_2+1-2k} \right) \times \cdots \\
 &\quad \times \sum_{m_{p-1}=1}^{n-2-A_{p-2}} \left( \sum_{k=1}^{m_{p-1}} \frac{1}{2k-1} \frac{1}{2m_{p-1}+1-2k} \right) \left( \sum_{k=1}^{n-1-A_{p-1}} \frac{1}{2k-1} \frac{1}{2(n-A_{p-1})-1-2k} \right) z^n,
 \end{aligned}$$

where  $A_p = \sum_{l=1}^p m_l$ .

*Proof.* Let us suppose that

$$\frac{f(z)}{z} = \frac{1}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2}$$

as the proof of Theorem 2.1. Then from Lemma 2.2, we have

$$\frac{f(z)}{z} = \frac{1}{1 + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \left( \sum_{k=1}^n \frac{1}{2k-1} \frac{1}{2n+1-2k} \right) z^n}$$

$$\begin{aligned}
&= 1 - \frac{8}{\pi^2} \left( \sum_{n=1}^{\infty} \left( \sum_{k=1}^n \frac{1}{2k-1} \frac{1}{2n+1-2k} \right) z^n \right) \\
&\quad + \left( \frac{8}{\pi^2} \right)^2 \left( \sum_{n=1}^{\infty} \left( \sum_{k=1}^n \frac{1}{2k-1} \frac{1}{2n+1-2k} \right) z^n \right)^2 \\
&\quad - \left( \frac{8}{\pi^2} \right)^3 \left( \sum_{n=1}^{\infty} \left( \sum_{k=1}^n \frac{1}{2k-1} \frac{1}{2n+1-2k} \right) z^n \right)^3 + \dots \\
&\quad + (-1)^n \left( \frac{8}{\pi^2} \right)^n \left( \sum_{n=1}^{\infty} \left( \sum_{k=1}^n \frac{1}{2k-1} \frac{1}{2n+1-2k} \right) z^n \right)^n + \dots \\
&= 1 - \frac{8}{\pi^2} \sum_{k=1}^1 \frac{1}{2k-1} \frac{1}{3-2k} z \\
&\quad + \left\{ -\frac{8}{\pi^2} \sum_{k=1}^2 \frac{1}{2k-1} \frac{1}{5-2k} + \left( \frac{8}{\pi^2} \right)^2 \left( \sum_{k=1}^1 \frac{1}{2k-1} \frac{1}{3-2k} \right) \left( \sum_{k=1}^1 \frac{1}{2k-1} \frac{1}{3-2k} \right) \right\} z^2 \\
&\quad + \left[ -\frac{8}{\pi^2} \sum_{k=1}^3 \frac{1}{2k-1} \frac{1}{7-2k} + \left\{ \left( \frac{8}{\pi^2} \right)^2 \left( \sum_{k=1}^1 \frac{1}{2k-1} \frac{1}{3-2k} \right) \left( \sum_{k=1}^2 \frac{1}{2k-1} \frac{1}{5-2k} \right) \right. \right. \\
&\quad \left. \left. + \left( \sum_{k=1}^2 \frac{1}{2k-1} \frac{1}{5-2k} \right) \left( \sum_{k=1}^1 \frac{1}{2k-1} \frac{1}{3-2k} \right) \right\} - \left( \frac{8}{\pi^2} \right)^3 \left( \sum_{k=1}^1 \frac{1}{2k-1} \frac{1}{3-2k} \right)^3 \right] z^3 \\
&\quad + \dots \\
&\quad + \left\{ -\frac{8}{\pi^2} \sum_{k=1}^n \frac{1}{2k-1} \frac{1}{2n+1-2k} \right. \\
&\quad \left. + \left( \frac{8}{\pi^2} \right)^2 \sum_{m_1=1}^{n-1} \left( \sum_{k=1}^{m_1} \frac{1}{2k-1} \frac{1}{2m_1+1-2k} \right) \left( \sum_{k=1}^{n-m_1} \frac{1}{2k-1} \frac{1}{2n-2m_1+1-2k} \right) \right. \\
&\quad \left. - \left( \frac{8}{\pi^2} \right)^3 \sum_{m_1=1}^{n-2} \left( \sum_{k=1}^{m_1} \frac{1}{2k-1} \frac{1}{2m_1+1-2k} \right) \sum_{m_2=1}^{n-1-m_1} \left( \sum_{k=1}^{m_2} \frac{1}{2k-1} \frac{1}{2m_2+1-2k} \right) \right. \\
&\quad \left. \times \left( \sum_{k=1}^{n-m_1-m_2} \frac{1}{2k-1} \frac{1}{2n-2m_1-2m_2+1-2k} \right) \right. \\
&\quad \left. + \dots \right. \\
&\quad + (-1)^p \left( \frac{8}{\pi^2} \right)^p \sum_{m_1=1}^{n+1-p} \left( \sum_{k=1}^{m_1} \frac{1}{2k-1} \frac{1}{2m_1+1-2k} \right) \sum_{m_2=1}^{n+2-p-m_1} \left( \sum_{k=1}^{m_2} \frac{1}{2k-1} \frac{1}{2m_2+1-2k} \right) \\
&\quad \times \dots \times \sum_{m_{p-1}=1}^{n-1-A_{p-2}} \left( \sum_{k=1}^{m_{p-1}} \frac{1}{2k-1} \frac{1}{2m_{p-1}+1-2k} \right) \left( \sum_{k=1}^{n-A_{p-1}} \frac{1}{2k-1} \frac{1}{2(n-A_{p-1})+1-2k} \right) \\
&\quad \quad \quad + \dots + \left( \sum_{k=1}^1 \frac{1}{2k-1} \frac{1}{3-2k} \right)^n \Big\} z^n \\
&\quad + \dots \quad \left( A_p = \sum_{l=1}^p m_l \right)
\end{aligned}$$

$$\begin{aligned}
&= 1 + \sum_{n=2}^{\infty} \sum_{p=1}^n (-1)^p \left( \frac{8}{\pi^2} \right)^p \sum_{m_1=1}^{n+1-p} \left( \sum_{k=1}^{m_1} \frac{1}{2k-1} \frac{1}{2m_1+1-2k} \right) \\
&\quad \times \sum_{m_2=1}^{n+2-p-m_1} \left( \sum_{k=1}^{m_2} \frac{1}{2k-1} \frac{1}{2m_2+1-2k} \right) \times \cdots \\
&\quad \times \sum_{m_{p-1}=1}^{n-1-A_{p-2}} \left( \sum_{k=1}^{m_{p-1}} \frac{1}{2k-1} \frac{1}{2m_{p-1}+1-2k} \right) \left( \sum_{k=1}^{n-A_{p-1}} \frac{1}{2k-1} \frac{1}{2(n-A_{p-1})+1-2k} \right) z^n.
\end{aligned}$$

This completes the proof of the theorem.  $\square$

## REFERENCES

- [1] A. W. GOODMAN: *On uniformly convex functions*, Annal. Polon. Math. **56**(1991), 87 – 92.
- [2] A. W. GOODMAN: *On uniformly starlike functions*, J. Math. Anal. Appl. **155**(1991), 364 – 370.
- [3] W. MA, D. MINDA: *Uniformly convex functions*, Annal. Polon. Math. **57**(1992), 165 – 175.
- [4] F. RØNNING: *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc. **118**(1993), 189 – 196.
- [5] F. RØNNING: *On uniform starlikeness and related properties of univalent functions*, Complex Variables **24**(1994), 233 – 239.

DEPARTMENT OF MATHEMATICS AND PHYSICS  
 SETSUNAN UNIVERSITY  
 NEYAGAWA, OSAKA 572-8508 JAPAN  
*E-mail address*: jerjun2002@yahoo.co.jp

DEPARTMENT OF MATHEMATICS  
 KINKI UNIVERSITY  
 HIGASHI-OSAKA, OSAKA 577-8502 JAPAN  
*E-mail address*: shige21@ican.zaq.ne.jp