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HAUSDORFF GEOMETRY OF POLYNOMIALS

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Abstract. Let $D(c;r)$ be the smallest disk, with center c and radius r, containing all zeros of the polynomial $p(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$. Half a century ago, we conjectured that for every zero z_k of $p(z)$, the disk $D(z_k;r)$ contains at least one zero of the derivative $p'(z)$. More than 100 papers are devoted to this conjecture, in which it is proved for different special cases. But in general, this conjecture is proved only for the polynomials of degree $n \leq 8$.

In this lecture a stronger conjecture is discussed and proved for polynomials of degree $n = 3$. A number of other conjectures are presented, including a variation of the Smale's mean value conjecture.

1. Introduction

The geometrical relations between the set $Z(p)$ of the zeros of a polynomial $p(z) = (z-z_1)(z-z_2)\cdots(z-z_n)$ and the set $Z(p')$ of the zeros (critical points) of its derivative $p'(z) = n(z-\zeta_1)(z-\zeta_2)\cdots(z-\zeta_{n-1})$ is the main subject of the Geometry of Polynomials. In the Hausdorff Geometry of Polynomials one considers the estimation of the deviations and the Hausdorff distance between these two sets, see [9]. A polynomial $p(z)$ is identified with the set $Z(p) = \{z_1, z_2, \ldots, z_n\}$ of its zeros, not necessarily distinct. Two polynomials $p(z)$ and $q(z)$ are not considered different, if $Z(p) = Z(q)$.

Denote by $D\Big(C(p); R(p)\Big)$ - the smallest disk containing all the zeros of the polynomial $p(z)$ on the complex plane $\mathcal{C}$. Call

$$c(p) = \frac{1}{n}\sum_{k=1}^{n} z_k \qquad \text{and} \qquad r(p) = \frac{1}{n}\sum_{k=1}^{n} |z_k - c(p)| \leq R(p), \tag{1.1}$$

respectively *center of gravity* (or *centroid*) and *radius of gravity* of $p(z)$.

The *deviation* of a set A from an other set B is defined by

$$d(A;B) = \max\Big\{\min\Big\{|z-\zeta| : \ \zeta \in B\Big\} : \ z \in A\Big\}$$

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and the *deviation* of set B from set A, by

$$d(B;A) = \max\Big\{\min\Big\{|\zeta - z| :\ z \in A\Big\} :\ \zeta \in B\Big\}.$$

In the general case $d(A;B)$ is different from $d(B;A)$. The *Hausdorff distance* between the two sets is

$$h(A,B) = \max\Big\{d(A;B), d(B;A)\Big\}.$$

Half a century ago, we stated:

Conjecture 1. *For every zero z_k of $p(z)$, the disk $D(z_k;R(p))$ contains at least one zero of the derivative $p'(z)$. In other words, $d(Z(p);h(Z(p')) \le R(p)$.*

This conjecture was published for the first time in [5], under the name of my professor, Lyubomir Iliev. See also [7, p. 224-240]. Conjecture 1 is proved in the general case only for polynomials of degree $n \le 8$, see [2].

The problem for estimation of $d(Z(p');h(Z(p))$ was formulated and solved by A. Aziz [1].

Theorem 1.1. *For every polynomial $p(z)$, the inequality $d(Z(p');h(Z(p)) \le R(p)$ holds.*

Hence, Conjecture 1 is equivalent to the inequality $h(Z(p),h(Z(p')) \le R(p)$.

New Cojecture 1. *For every polynomial $p(z)$ of degree $n \ge 2$, the inequality*

$$h(Z(p),h(Z(p')) \le r(p)$$

holds. If $p(z)$ is not a linear transformation of $z^n - 1$, then the inequality is strict.

As $r(p) \le R(p)$, the New Conjecture 1 is stronger than the old one.

2. Generalized deviations and Hausdorff distance

Let $F(z_1, z_2, \dots, z_l, \zeta_1, \zeta_2, \dots, \zeta_m)$ be a nonnegative real valued function of $l+m$ complex variables and A, B be two point sets on the complex plane C. We call

$$d(F:A;B) = \max\Big\{\min\Big\{F(z_1,\dots,z_l,\zeta_1,\dots,\zeta_m) :\ \zeta_1,\dots,\zeta_m \in B\Big\} :\ z_1,\dots,z_l \in A\Big\}$$

F-deviation of A from B,

$$d(F:B;A) = \max\Big\{\min\Big\{F(z_1,\dots,z_l,\zeta_1,\dots,\zeta_m) :\ z_1,\dots,z_l \in A\Big\} :\ \zeta_1,\dots,\zeta_m \in B\Big\}$$

F-deviation of B from A and

$$h(F:A,B) = \max\Big\{d(F:A;B), d(F:B;A)\Big\}$$

F-Hausdorff distance between A and B. For $F(z_1,\zeta_1) = |z_1 - \zeta_1|$, the F-Hausdorff distance is equal to the Hausdorff distance $h(A,B)$ between A and B.

2.1. Homogeneous analogue of Conjecture 1

We may consider a homogeneous analogue of Conjecture 1, using the function

$$G_1(z_1, z_2, \zeta_1) = |z_1 - \zeta_1|/|z_1 - z_2|.$$

In other words, we may ask "How big is the ratio between the distance from a zero to the closest critical point and the distance from this zero to the closest other zero?" The ratio $|z_1 - \zeta_1|/|z_1 - z_2|$ makes sense as a limit, when $z_2 = z_1$, as in this case $\zeta_1 = z_1$.

From the Grace-Heawood theorem, [4] and [6], see also [7, p. 126], it follows:

For every polynomial of degree $n \geq 2$, the inequality

$$d(G_1 : p) \leq \frac{1}{2}\left(1 + \cot\frac{\pi}{n}\right)$$

holds.

It is natural to expect that:

Conjecture 2. *For every polynomial of degree $n \geq 2$, the inequality*

$$d(G_1 : p) = \max\left\{\min\left\{\frac{|z_1 - \zeta_1|}{|z_1 - z_2|} \;:\; \zeta_1 \in Z(p')\right\} \;:\; z_1, z_2 \in Z(p)\right\} \leq \frac{1}{2\sin\frac{\pi}{n}}$$

holds.

We have a proof of Conjecture 2 only for $n = 2, 3$.

B. Ćurgus B. and V. Mascion [3] proved that

$$\min\left\{\min\left\{\frac{|z_1 - \zeta_1|}{|z_1 - z_2|} \;:\; \zeta_1 \in Z(p')\right\} \;:\; z_1, z_2 \in Z(p)\right\} \leq \frac{1}{2\sin\frac{\pi}{n}}, \tag{2.1}$$

and this equality is strict. Obviously, Conjecture 2 is stronger than (2.1).

2.2. Convex hull

G. Schmeisser [8] posed the following generalization of Conjecture 1:

Conjecture 3. *Let $K(p)$ be the convex hull of the zeros of the polynomial $p(z)$. Then for every $\zeta \in K(p)$, the disk $D(\zeta; R(p))$ contains at least one critical point of $p(z)$.*

In [8] is proved that Conjecture 3 is true if $K(p)$ is a triangular region. We may formulate Conjecture 3 in terms of the Hausdorff distance:

For every polynomial $p(z)$ of degree $n \geq 2$, the inequality $h(K(p); Z(p')) \leq R(p)$ holds.

It is natural to formulate

New Conjecture 2. *For every polynomial $p(z)$ of degree $n \geq 2$, the inequality*

$$h(K(p); Z(p')) \leq r(p)$$

holds.

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References

[1] A. Aziz: *On the zeros of a polynomial and its derivative*, Bull. Austr. Math. Soc., **31**(4) (1985), 245–255.
[2] J.E. Brown, G. Xiang: *Proof of the Sendov conjecture for polynomials of degree utmost eight*, J. Math. Anal. Appl., **232**(2) (1999), 272–292.
[3] B. Ćurgus, V. Mascioni: *On the location of critical points of polynomials*, Proc. of the Amer. Math. Soc., **131**(1) (2002), 253–264.
[4] J.H. Grace: *The zeros of a polynomial*, Proc. of the Cambridge Philos. Soc., **11** (1902), 352–357.
[5] W.K. Hayman: *Research Problems in Function Theory*, The Athlone Press of the University of London, 1967.
[6] P.J. Heawood: *Geometrical relations between the roots of* $f(x) = 0, f'(x) = 0$, Quarterly J. of Math., **38** (1907), 84–107.
[7] Q.I. Rahman, G. Schmeisser: *Analytic Theory of Polynomials*, Oxford Univ. Press Inc., New York, 2002.
[8] G. Schmeisser: *On Iliev's conjecture*, Math. Z., **156** (1977), 165–173.
[9] B. Sendov: *Hausdorff geometry of polynomials*, East J. on Appr., **7**(2), (2001), 11-56.

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