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ASYMPTOTIC BEHAVIOR OF THE LAPLACE TRANSFORM NEAR THE ORIGIN

DENNIS NEMZER

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ABSTRACT. The unilateral Laplace transform is extended to a space of generalized functions $\mathcal{B}_L$ which contains the space of transformable distributions supported on the interval $[0, \infty)$. The Mittag-Leffler functions are found to be useful in comparing the asymptotic behavior of an element of $\mathcal{B}_L$ at infinity to the asymptotic behavior of its transform at a singularity.

1. INTRODUCTION

A class of generalized functions $\mathcal{B}$ known as Boehmians is constructed algebraically. In [3], a subspace $\mathcal{B}_L$ of $\mathcal{B}$ is used to extend the classical Laplace transform. The object of this note is to present a final value theorem for the unilateral Laplace transform. The final value theorem relates the asymptotic behavior of a transformable Boehmian at infinity to the asymptotic behavior of its transform at a singularity.

The Mittag-Leffler functions are used to generate differential operators of infinite order on $\mathcal{B}$ which will be used in the final value theorem.

The construction of the space $\mathcal{B}$ utilizes convolution and delta sequences or approximate identities. The space is quite general. Indeed, the space of Schwartz distributions supported on the interval $[0, \infty)$ can be identified with a proper subspace of $\mathcal{B}$.

2. PRELIMINARIES

The space of all $f \in C(\mathbb{R})$ such that $f(t) = 0$ for $t < 0$ will be denoted by $C_+(\mathbb{R})$. The convolution product of two functions $f, g \in C_+(\mathbb{R})$ is given by

$$(f * g)(t) = \int_0^t f(t-u)g(u)du. \tag{2.1}$$

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A sequence of continuous nonnegative functions $\{\varphi_n\}$ will be called a *delta sequence* provided:

(i) $\int_{-\infty}^{\infty} \varphi_n(t)dt = 1$ for $n = 1, 2, \ldots$; and
(ii) supp $\varphi_n \subseteq [0, \varepsilon_n], \varepsilon_n \to 0$ as $n \to \infty \quad (\varepsilon_n > 0)$.

A pair of sequences (f_n, φ_n) is called a *quotient of sequences*, if $f_n \in C_+(\mathbb{R})$ for $n = 1, 2, \ldots, \{\varphi_n\}$ is a delta sequence, and $f_k * \varphi_m = f_m * \varphi_k$ for all k and m. Two quotients of sequences (f_n, φ_n) and (g_n, σ_n) are said to be equivalent if $f_k * \sigma_m = g_m * \varphi_k$ for all k and m. A straightforward calculation shows that this is an equivalence relation. The equivalence classes are called *Boehmians*. The space of all Boehmians will be denoted by $\mathcal{B}$ and a typical element of $\mathcal{B}$ will be written as $W = \left[\frac{f_n}{\varphi_n}\right]$.

Consider the map $i : C_+(\mathbb{R}) \to \mathcal{B}$ given by

$$i(f) = \left[\frac{f * \varphi_n}{\varphi_n}\right], \text{ where } \{\varphi_n\} \text{ is any fixed delta sequence.}$$

We can identify $C_+(\mathbb{R})$ with a proper subspace of $\mathcal{B}$. Likewise, $D'_+(\mathbb{R})$, the space of Schwartz distributions [7] supported on $[0, \infty)$, can be identified with a proper subspace of $\mathcal{B}$. For example, $i(\delta) = \left[\frac{\delta * \varphi_n}{\varphi_n}\right] = \left[\frac{\varphi_n}{\varphi_n}\right]$.

The *differentiation operator* D is defined on $\mathcal{B}$ as follows. $D\left[\frac{f_n}{\varphi_n}\right] = \left[\frac{f_n * \psi'_n}{\varphi_n * \psi_n}\right]$, where $\{\psi_n\}$ is any infinitely differentiable delta sequence.

Definition 2.1. *Let $W = \left[\frac{f_n}{\varphi_n}\right] \in \mathcal{B}$. Then W vanishes in a neighborhood of infinity, denoted $W(t) \overset{e}{\sim} 0$ as $t \to \infty$, provided there exists $b > 0$ such that $f_n \to 0$ uniformly on compact subsets of (b, ∞).*

Remark 2.1. *Let $W, V \in \mathcal{B}$, $W(t) \overset{e}{\sim} V(t)$ as $t \to \infty$, provided $(W - V)(t) \overset{e}{\sim} 0$ as $t \to \infty$.*

Example 2.1. *Let $\delta = \left[\frac{\varphi_n}{\varphi_n}\right]$. Then, $\delta \overset{e}{\sim} 0$ as $t \to \infty$. This follows by observing that supp $\varphi_n \subseteq [0, \varepsilon_n]$, where $\varepsilon_n \to 0$. Thus, $\varphi_n \to 0$ uniformly on compact subsets of $(0, \infty)$.*

Definition 2.2. *A sequence $\{W_n\}$ of Boehmians is said to converge to the Boehmian W, denoted by δ -$\lim_{n\to\infty} W_n = W$, if there exists a delta sequence $\{\varphi_n\}$ such that for each n and k, $W_n * \varphi_k$, $W * \varphi_k \in C_+(\mathbb{R})$, and for all k, $(W_n - W) * \varphi_k \to 0$ uniformly on compact sets as $n \to \infty$.*

For more on Boehmians and convergence, see [2].

The *Mittag-Leffler functions* are defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad z \in \mathbb{C}, \tag{2.2}$$

where $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\, dt$.

For each $\alpha > 1$, the Mittag-Leffler function generates a differential operator of infinite order.

$$E_\alpha(D) = \sum_{n=0}^{\infty} \frac{D^n}{\Gamma(\alpha n + 1)}. \tag{2.3}$$

$E_\alpha(D) : \mathcal{B} \to \mathcal{B}$, where $E_\alpha(D)W = \delta\text{-}\lim_{n\to\infty} \sum_{k=0}^{n} \frac{D^k W}{\Gamma(\alpha k+1)} = \sum_{n=0}^{\infty} \frac{D^n W}{\Gamma(\alpha n+1)}$.

By using Theorem 4 in [5], it follows that the above series converges in $\mathcal{B}$.

3. Transformable Boehmians

$W \in \mathcal{B}$ is called *transformable* (see [3]) provided there exist a delta sequence $\{\varphi_n\}$ and $\sigma > 0$ such that $W * \varphi_n \in C_+(\mathbb{R})$ and $(W * \varphi_n)(t) = O(e^{\sigma t})$ as $t \to \infty$, for all $n \in \mathbb{N}$. The space of transformable Boehmians will be denoted $\mathcal{B}_L$.

The *Laplace transform* of $W = \left[\frac{f_n}{\varphi_n}\right] \in \mathcal{B}_L$ is given by

$$\mathcal{L}(W)(z) = \mathcal{W}(z) = \lim_{n\to\infty} \mathcal{L}(f_n)(z), \tag{3.1}$$

where $f_n(t) = O(e^{\sigma t})$ as $t \to \infty$, for all $n \in \mathbb{N}$, and $\mathcal{L}(f_n)(z) = F_n(z) = \int_0^\infty e^{-zt} f_n(t)dt$.

Remark 3.1 *1) $\mathcal{W}(z)$ is independent of the representation.*
2) The convergence is uniform on compact subsets in the half-plane $Re\, z > \sigma$.
3) $\mathcal{W}(z)$ is an analytic function in the half-plane $Re\, z > \sigma$.
4) $\mathcal{L}(\varphi_n) \to 1$ uniformly on compact sets as $n \to \infty$.

Example 3.1. *Let $f \in C_+(\mathbb{R})$ having at most exponential growth σ as $t \to \infty$. Then, for $W_f = \left[\frac{f*\varphi_n}{\varphi_n}\right]$, $\mathcal{W}_f(z) = \lim_{n\to\infty} \mathcal{L}(f * \varphi_n)(z) = \lim_{n\to\infty} \mathcal{L}(f)(z)\,\mathcal{L}(\varphi_n)(z) = \mathcal{L}(f)(z)$, $Re\, z > \sigma$.*

Example 3.2. *Consider $W = \delta' = \left[\frac{\varphi_n'}{\varphi_n}\right]$.*
$\mathcal{W}(z) = \lim_{n\to\infty} \mathcal{L}(\varphi_n')(z) = \lim_{n\to\infty}[z\mathcal{L}(\varphi_n)(z) - (\varphi_n)(0)] = z$, $z \in \mathbb{C}$.

Not every function that is analytic in some half-plane is the Laplace transform of a Boehmian. One example is $g(z) = e^{z}$. However, we have the following theorem.

Theorem 3.1. *(see [4]) Suppose that $g(z)$ is an analytic function in some half-plane $Re\, z > \sigma$, and for some integer k and all $\varepsilon > 0$, $z^k g(z) = O(e^{\varepsilon z})$ as $z \to \infty$, $Re\, z > \sigma$. Then, there exists a $W \in \mathcal{B}_L$ such that $\mathcal{W}(z) = g(z)$, $Re\, z > \sigma$.*

Remark 3.2. *The Laplace transform of every transformable distribution supported on $[0, \infty)$ is bounded in some half-plane by a polynomial. Thus, the space of transformable distributions supported on $[0, \infty)$ can be identified with a subspace of $\mathcal{B}_L$.*

For each $\alpha > 1$, $E_\alpha(D) : \mathcal{B}_L \to \mathcal{B}_L$.

Theorem 3.2. *Let $W \in \mathcal{B}_L$ where $\mathcal{W}(z)$ exists for $Re\, z > \sigma$. Then,*

$$\mathcal{L}(E_\alpha(D)W)(z) = E_\alpha(z)\mathcal{W}(z), Re\, z > \sigma. \tag{3.2}$$

Theorem 3.3. Final Value Theorem. *Let $W \in \mathcal{B}_L$, $\alpha > 1$, and $\xi, z_0 \in \mathbb{C}$. If $W(t) \overset{e}{\sim} E_\alpha(D)f(t)$ as $t \to \infty$, where f is a locally integrable function and $\lim_{t\to\infty} \frac{f(t)}{t^\lambda e^{z_0 t}} = \xi$ $(Re\,\lambda > -1)$, then $\mathcal{W}(z)$ exists for $Re\, z > Re\, z_0$. Moreover, $\mathcal{W}(z)$ has the following asymptotic behavior.*

(I) *When $E_\alpha(z_0) \neq 0$,*
$\frac{(z-z_0)^{\lambda+1}\mathcal{W}(z)}{\Gamma(\lambda+1)} \sim \xi E_\alpha(z)$ *as $z \to z_0$ in $|arg(z-z_0)| \leq \psi < \frac{\pi}{2}$.*
(That is, $$\lim_{\substack{z\to z_0\\ |arg(z-z_0)|\leq\psi<\frac{\pi}{2}}} \frac{(z-z_0)^{\lambda+1}\mathcal{W}(z)}{\Gamma(\lambda+1)E_\alpha(z)} = \xi.)$$

(II) *If $E_\alpha^{(k)}(z_0) = 0$, for $0 \leq k \leq n-1$ and $E_\alpha^{(n)}(z_0) \neq 0$, for some $n \in \mathbb{N}$, then:*
 (i) *For $n < Re\,\lambda + 1$,*
 $\frac{(z-z_0)^{\lambda+1}\mathcal{W}(z)}{\Gamma(\lambda+1)}) \sim \xi E_\alpha(z)$ *as $z \to z_0$ in $|arg(z-z_0)| \leq \psi < \frac{\pi}{2}$.*
 (ii) *For $n \geq Re\,\lambda + 1$,*
 $\frac{(z-z_0)^{\lambda+1}(\mathcal{W}(z) - \sum_{k=0}^{m} \frac{\mathcal{U}^{(k)}(z_0)}{k!}(z-z_0)^k)}{\Gamma(\lambda+1)} \sim \xi E_\alpha(z)$ *as $z \to z_0$ in $|arg(z-z_0)| \leq \psi < \frac{\pi}{2}$ (where $U = W - E_\alpha(D)f$, $m = n - [Re\,\lambda] - 1$, and $[\cdot]$ is the greatest integer function).*

Proof. Since $W = U + E_\alpha(D)f$ (where U has compact support), $\mathcal{W}(z)$ exists for $\mathrm{Re}\, z > \mathrm{Re}\, z_0$. Now,

$$\frac{(z-z_0)^{\lambda+1}\mathcal{W}(z)}{\Gamma(\lambda+1)E_\alpha(z)} = \frac{(z-z_0)^{\lambda+1}(\mathcal{U}(z) + E_\alpha(z)F(z))}{\Gamma(\lambda+1)E_\alpha(z)}$$

$$= \frac{(z-z_0)^{\lambda+1}\mathcal{U}(z)}{\Gamma(\lambda+1)E_\alpha(z)} + \frac{(z-z_0)^{\lambda+1}F(z)}{\Gamma(\lambda+1)} \tag{3.3}$$

By a classical final value theorem [1], the second term converges to ξ as $z \to z_0$ in $|\arg(z-z_0)| \leq \psi < \frac{\pi}{2}$.

(I) Assume $E_\alpha(z_0) \neq 0$.
Then, $\frac{(z-z_0)^{\lambda+1}\mathcal{U}(z)}{\Gamma(\lambda+1)E_\alpha(z)} \to 0$ as $z \to z_0$. This proves (I).

(II) Assume $E_\alpha^{(k)}(z_0) = 0, 0 \leq k \leq n-1$ and $E_\alpha^{(n)}(z_0) \neq 0$.
 (i) Suppose $n < \mathrm{Re}\,\lambda + 1$. Then,

$$\frac{(z-z_0)^{\lambda+1}\mathcal{U}(z)}{\Gamma(\lambda+1)E_\alpha(z)} = \frac{(z-z_0)^{\lambda+1}\mathcal{U}(z)}{\Gamma(\lambda+1)(z-z_0)^n Q(z)}$$

$$= \frac{(z-z_0)^{\lambda+1-n}\mathcal{U}(z)}{\Gamma(\lambda+1)Q(z)}, \tag{3.4}$$

where $Q(z_0) \neq 0$. This term converges to zero as $z \to z_0$. Thus, the proof of part (i) is complete.

(ii) Suppose $n \geq \operatorname{Re}\lambda + 1$. Then,

$$\frac{(z-z_0)^{\lambda+1}(\mathcal{W}(z)-\sum_{k=0}^{m}\frac{\mathcal{U}^{(k)}(z_0)}{k!}(z-z_0)^k)}{\Gamma(\lambda+1)E_\alpha(z)}$$

$$= \frac{(z-z_0)^{\lambda+1}(\mathcal{U}(z)-\sum_{k=0}^{m}\frac{\mathcal{U}^{(k)}(z_0)}{k!}(z-z_0)^k)}{\Gamma(\lambda+1)E_\alpha(z)} + \frac{(z-z_0)^{\lambda+1}F(z)}{\Gamma(\lambda+1)}. \tag{3.5}$$

As before, the second term converges to ξ as $z \to z_0$ in $|\arg(z-z_0)| \leq \psi < \frac{\pi}{2}$. And,

$$\frac{(z-z_0)^{\lambda+1}(\mathcal{U}(z)-\sum_{k=0}^{m}\frac{\mathcal{U}^{(k)}(z_0)}{k!}(z-z_0)^k)}{\Gamma(\lambda+1)E_\alpha(z)} = \frac{(z-z_0)^{\lambda+1}(\mathcal{U}(z)-\sum_{k=0}^{m}\frac{\mathcal{U}^{(k)}(z_0)}{k!}(z-z_0)^k)}{\Gamma(\lambda+1)(z-z_0)^n Q(z)}$$

(Q is entire and $Q(z_0) \neq 0$)

$$= \left(\frac{(z-z_0)^{\lambda-[\operatorname{Re}\lambda]}}{\Gamma(\lambda+1)Q(z)}\right)\frac{(\mathcal{U}(z)-\sum_{k=0}^{m}\frac{\mathcal{U}^{(k)}(z_0)}{k!}(z-z_0)^k)}{(z-z_0)^m} \tag{3.6}$$

$\to 0$ as $z \to z_0$ in $|\arg(z-z_0)| \leq \psi < \frac{\pi}{2}$. This completes the proof of part (ii) and the theorem.

□

Corollary 3.1. *Let $W \in \mathcal{B}_L$, $\alpha > 1$, and $\xi \in \mathbb{C}$. If $W(t) \overset{e}{\sim} E_\alpha(D)f(t)$ as $t \to \infty$, where f is a locally integrable function and $\lim_{t\to\infty}\frac{f(t)}{t^\lambda} = \xi \quad (Re\,\lambda > -1)$, then $\mathcal{W}(z)$ exists for $Re\,z > 0$ and $\frac{z^{\lambda+1}\mathcal{W}(z)}{\Gamma(\lambda+1)} \sim \xi E_\alpha(z)$ as $z \to 0$ in $|arg\,z| \leq \psi < \frac{\pi}{2}$.*

Since for $\alpha \geq 2$, the zeros of $E_\alpha(z)$ are on the negative real axis [6], the following corollary is immediate from the Final Value Theorem.

Corollary 3.2. *Let $W \in \mathcal{B}_L$, $\alpha \geq 2$, and $\xi, z_0 \in \mathbb{C}$ such that z_0 does not lie on the negative real axis. If $W(t) \overset{e}{\sim} E_\alpha(D)f(t)$ as $t \to \infty$, where f is a locally integrable function and $\lim_{t\to\infty}\frac{f(t)}{t^\lambda e^{z_0 t}} = \xi \quad (Re\,\lambda > -1)$, then $\mathcal{W}(z)$ exists for $Re\,z > Re\,z_0$ and $\frac{(z-z_0)^{\lambda+1}\mathcal{W}(z)}{\Gamma(\lambda+1)} \sim \xi E_\alpha(z)$ as $z \to z_0$ in $|arg\,(z-z_0)| \leq \psi < \frac{\pi}{2}$.*

Example 3.3. *Let $f(t) = \sqrt{t}$, and let φ be an infinitely differentiable function with compact support. Notice that $\frac{f(t)}{\sqrt{t}} \to 1$ as $t \to \infty$.*

Let

$$W = \varphi\left(pf\frac{1_+(t)}{t}\right) + E_2(D)f, \tag{3.7}$$

where $pf\frac{1_+(t)}{t}$ denotes the distributional derivative of $1_+(t)\log t$ (see [7]).

Since the support of $\varphi\left(pf\frac{1_+(t)}{t}\right)$ is bounded,

$$W(t) \overset{e}{\sim} E_2(D)f(t) \text{ as } t \to \infty. \tag{3.8}$$

By Corollary 1,

$$z^{3/2}\mathcal{W}(z) \sim \frac{\sqrt{\pi}}{2}\cosh\sqrt{z} \text{ as } z \to 0 \text{ in } |\arg z| \leq \psi < \frac{\pi}{2}. \tag{3.9}$$

That is, $$\lim_{\substack{z\to 0\\ |\arg z|\le\psi<\frac{\pi}{2}}} \frac{z^{3/2}\mathcal{W}(z)}{\cosh\sqrt{z}} = \frac{\sqrt{\pi}}{2}.$$

Remark 3.3. *It is sometimes possible to use a special function for comparison in the Final Value Theorem other than the Mittag-Leffler functions. One such example is* $\frac{J_\alpha(\sqrt{z})}{z^{\alpha/2}}$*, where* J_α *is the Bessel function of the first kind of order* α*.*

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Department of Mathematics
California State University
Stanislaus Turlock, CA 95382, USA
E-mail address: jclarke@csustan.edu