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UNIFIED APPROACH TO UNIVALENCY OF THE DZIOK-SRIVASTAVA AND THE FRACTIONAL CALCULUS OPERATORS

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ABSTRACT. In the Geometric Function Theory (GFT) much attention is paid to various linear integral operators mapping the class S of the univalent functions and its subclasses into themselves. In [10],[11] Hohlov obtained sufficient conditions that guarantee such mappings for the operator defined by means of Hadamard product with the Gauss hypergeometric function. In our earlier papers as [18], [17], [14], [15], etc., we extended his method to the operators of the Generalized Fractional Calculus (GFC, [13]). These operators have product functions of the forms ${}_{m+1}F_m$ and ${}_{m+1}\Psi_m$ and integral representations by means of the Meijer G- and Fox H-functions.

It happens that the used techniques can be extended to propose sufficient conditions that guarantee mapping of the univalent, respectively of the convex functions, into univalent functions in the case of the more general Dziok-Srivastava operator from [8], defined as a Hadamard product with an arbitrary generalized hypergeometric function ${}_pF_q$. We suggest similar conditions also for its extension involving the Wright ${}_p\Psi_q$-function and called the Srivastava-Wright operator, [29].

Since the Dziok-Srivastava operaror includes the above-mentioned GFC operators and many their special cases (operators of the classical FC), from the results proposed here one can derive univalence criteria for many named operators in the GFT, as the operators of Hohlov, Carlson and Shaffer, Saigo, Libera, Bernardi, Erdélyi-Kober, etc., by giving particular values to the orders $p \leq q+1$ of the generalized hypergeometric functions and to their parameters.

1. PRELIMINARIES

As usually in GFT, A denotes the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

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which are *analytic* in the unit disk $U = \{z : |z| < 1\}$, and let S be the subclass of A of the *univalent functions* in U. Further, a function $f(z)$ belonging to S is said to be *convex*, if it satisfies the inequality

$$\Re\Big\{1 + \frac{zf''(z)}{f'(z)}\Big\} > 0 \qquad (z \in U)$$

and this subclass of S is denoted by K.

The papers studying univalent functions and the subclasses of the starlike, convex etc. functions involve various linear integral or integro-differential operators, among them the well-known operators of Biernacki, Libera, Bernardi, Komatu, Rusheweyh, Saigo, Hohlov, Srivastava and Owa, etc. Among the important problems treated there is the construction of linear integral operators preserving the class S and some of its subclasses, that is, finding conditions under which the above mentioned operators do this. Examples of such results can be found in [1], [5], [19], [10], [11], [23], etc. In [13], [18], [17], [14], [15], etc. we have shown that all these named operators are special cases of the operators of the generalized fractional calculus [13] and found univalence criteria for them in terms of the values at $z = 1$ of the generalized hypergeometric functions ${}_{m+1}F_m$ and ${}_{m+1}\Psi_m$.

Recently, we have extended our study to the more complicated and general cases of the Dziok-Srivastava and Srivastava-Wright operators, defined by means of arbitrary ${}_pF_q$-function, $p \le q + 1$, or by the Wright function ${}_p\Psi_q$, studied in many recent papers as [8], [9], [20], [29], [28], [2], etc. The sufficient conditions we obtain for mapping of the univalent, respectively of the convex functions, into univalent functions supply corollaries for all the mentioned particular operators used in GFT.

The *Hadamard product (convolution)* of two analytic functions f, g in U is defined by

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,\ g(z) = \sum_{k=0}^{\infty} b_k z^k \ \mapsto f * g(z) := \sum_{k=0}^{\infty} a_k b_k z^k. \tag{1.2}$$

We briefly remind also the definitions of some special functions used in this paper.

Definition 1.1. *The Wright generalized hypergeometric functions ${}_p\Psi_q(z)$, called also Fox-Wright functions are defined as:*

$$ {}_p\Psi_q\left[\begin{array}{c} (\alpha_1, A_1), \dots, (\alpha_p, A_p) \\ (\beta_1, B_1), \dots, (\beta_q, B_q) \end{array}\middle| z\right] = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha_1 + kA_1)\dots\Gamma(\alpha_p + kA_p)}{\Gamma(\beta_1 + kB_1)\dots\Gamma(\beta_q + kB_q)} \frac{z^k}{k!}. \tag{1.3}$$

When all $A_1 = \dots = A_p = 1, B_1 = \dots = B_q = 1$, these are reduced to the more popular generalized hypergeometric ${}_pF_q$-function, namely:

$$ {}_p\Psi_q\left[\begin{array}{c} (\alpha_1, 1), \dots, (\alpha_p, 1) \\ (\beta_1, 1), \dots, (\beta_q, 1) \end{array}\middle| z\right] = \omega^{-1}\ {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q;\ z)\,,$$

$$ {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q;\ z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_p)_k}{(\beta_1)_k \dots (\beta_q)_k} \frac{z^k}{k!}\,, \tag{1.4}$$

where

$$\omega := \left[\prod_{j=1}^{q} \Gamma(\beta_j) / \prod_{i=1}^{p} \Gamma(\alpha_i)\right], \quad (\alpha)_k := \Gamma(\alpha + k)/\Gamma(\alpha).$$

The series ${}_pF_q$ is generally considered for parameters $\alpha_i, \beta_j \in \mathbb{C}$, $\beta_j \neq 0, -1, -2, \dots$ $(i = 1, \dots, p; j = 1, \dots, q)$ and it is absolutely convergent for all $|z| < \infty$ if $p \leq q$. If $p = q + 1$, it is absolutely convergent in the unit disk $U = \{z : |z| < 1\}$, and diverges for all $z \neq 0$ if $p > q + 1$. If $|z| = 1$ in ${}_{q+1}F_q$, we require the condition (see [7], §4.1)

$$\Re\{\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{q+1} \alpha_i\} > 0. \tag{1.5}$$

As we suppose the parameters α_i, β_j are real positive, the sign $\Re$ is further omitted.

The ${}_p\Psi_q$-functions are special cases of the *Fox H-functions* $H_{p,q}^{m,n}\left[z \left| \begin{array}{c} (\alpha_j, A_j)_1^p \\ (\beta_k, B_k)_1^q \end{array} \right. \right]$ (see details in [30], [25], [13]), which for all $A_1 = \dots = A_p = 1, B_1 = \dots = B_q = 1$ reduce to the *Meijer G-functions* defined by means of the Mellin-Barnes type contour integral (see definition and details in [7], [25], [13])

$$G_{p,q}^{m,n}\left[z \left| \begin{array}{c} (\alpha_j, A_j)_1^p \\ (\beta_k, B_k)_1^q \end{array} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\prod_{k=1}^{m} \Gamma(\beta_k + s) \prod_{j=1}^{n} \Gamma(1 - \alpha_j - s)}{\prod_{k=m+1}^{q} \Gamma(1 - \beta_k - s) \prod_{j=n+1}^{p} \Gamma(\alpha_j + s)} z^{-s} ds, \quad z \neq 0. \tag{1.6}$$

It is analytic function of z with a branch point at the origin. Specially, in the case of the kernel-function of the operators of the generalized fractional calculus [13], discussed in Section 3, $G_{m,m}^{m,0}$ is analytic in the unit disc $|z| < 1$ and vanishes identically in $|z| > 1$.

2. The Dziok-Srivastava and Srivastava-Wright operators and univalence conditions

By means of the generalized hypergeometric function ${}_pF_q(z)$ in its general form (1.4), Dziok and Srivastava [8] introduced a linear operator known today in GFT as the Dziok-Srivastava operator. In that paper and in next ones as [20], [29], [28], [2], many authors studied this operator and its further extension by means of the Wright g.h.f. (1.3), in view of the mapping properties, coefficient estimates, distortion theorems, extreme points etc.

Definition 2.1. *The Dziok-Srivastava operator (*[8]*) is defined on the class A by means of the Hadamard convolution* (1.2) *involving an arbitrary ${}_pF_q$-function:*

$$\mathcal{F}_{p,q}f(z) = h(z) * f(z) \ , \qquad h(z) := h_{p,q}(z) = z\,{}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z). \tag{2.1}$$

Thus, the D-S operator (2.1) *that can be denoted also by $\mathcal{F}_{p,q}(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q)$, maps a function of the form (1.1) into a function of the same form:*

$$\mathcal{F}_{p,q}f(z) = z + \sum_{k=2}^{\infty} \Psi(k) \ a_k \, z^k \tag{2.2}$$

with a multiplies' sequence

$$\Psi(k) = \frac{(\alpha_1)_{k-1} \dots (\alpha_p)_{k-1}}{(\beta_1)_{k-1} \dots (\beta_q)_{k-1} \, (k-1)!} \, . \tag{2.3}$$

Definition 2.2. *The Srivastava's extension of (2.1) from* [29]*, called in* [16] *as Srivastava-Wright operator, involves the more general Wright function (1.3):*

(2.4) $\mathcal{W}_{p,q}f(z) = h(z) * f(z)$, *with* $h(z) := w_{p,q}(z) = \omega\ z\ {}_p\Psi_q \left[\begin{array}{c} (\alpha_i, A_i)_1^p \\ (\beta_j, B_j)_1^q \end{array} \middle| z \right]$,

with

$$\omega := \Big[\prod_{j=1}^{q} \Gamma(\beta_j) / \prod_{i=1}^{p} \Gamma(\alpha_i) \Big],$$

that is, for $f \in A$ *(see e.g. (3)-(4) in* [2]*),*

$$\mathcal{W}_{p,q}f(z) = z + \sum_{k=2}^{\infty} \Psi(k)\ a_k\, z^k, \tag{2.5}$$

$$\textit{with} \quad \Psi(k) = \omega\, \frac{\Gamma(\alpha_1 + A_1(k-1)) \dots \Gamma(\alpha_p + A_p(k-1))}{\Gamma(\beta_1 + B_1(k-1)) \dots \Gamma(\beta_q + B_q(k-1))\,(k-1)!}\,.$$

Many authors proposed recently various results for the Dziok-Srivastava and Srivastava-Wright operators (2.1), (2.4), as convolutional and distortion theorems, extreme points, coefficient estimates, etc. Here we state some univalency results for the Dziok-Srivastava (D-S) operator in its simpler form (2.1), but they go in similar way with heavier denotations for the Srivastava-Wright (S-W) operator (2.4). From the above definitions, it is easy to derive the following result mentioned in most of the papers.

Lemma 2.1. *The Dziok-Srivastava operator* (2.1) *and the Srivastava-Wright operator* (2.4) *map the function class* A *into itself.*

Idea of Proof. As defined by the Hadamard products in (2.1) and (2.4), both operators transform a function of the form (1.1) into an image-function having series representation of the same form. It remains to prove that this series has the same radius of convergence ($R' = R = 1$), using the Cauchy-Hadamard formula and the Stirling asymptotic formula for the involved Γ-functions and Pochhammer symbols, in a way similar to [12], [18]. □

The aim of the next theorems is to propose some sufficient conditions on the parameters of the Dziok-Srivastava ensuring that a univalent function, respectively a convex function, will be transformed into a univalent function.

Theorem 2.1. *Suppose that for the parameters of the generalized hypergeometric function* $h_{p,q}(z) = z_pF_q(z)$ *in the product* (2.1) *the following conditions are satisfied:*

$$\alpha_i > 0, \quad \beta_j > 0, \quad i = 1, \dots, p;\ j = 1, \dots, q; \quad p \le q+1, \tag{2.6}$$

and if $p = q+1$, *let additionally*

$$\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > 0\,. \tag{2.7}$$

If the inequality

(2.8)

$$\left[\frac{\prod_{i=1}^{p} \alpha_i(\alpha_i+1)}{\prod_{j=1}^{q} \beta_j(\beta_j+1)} \right] {}_pF_q \left[\begin{array}{c} (\alpha_i+2)_1^p \\ (\beta_j+2)_1^q \end{array} \middle| 1 \right] + 3 \left[\frac{\prod_{i=1}^{p} \alpha_i}{\prod_{j=1}^{q} \beta_j} \right] {}_pF_q \left[\begin{array}{c} (\alpha_i+1)_1^p \\ (\beta_j+1)_1^q \end{array} \middle| 1 \right] + {}_pF_q \left[\begin{array}{c} (\alpha_i)_1^p \\ (\beta_j)_1^q \end{array} \middle| 1 \right] < 2$$

holds true, then for each univalent function f *in the class* A, *the Dziok-Srivastava image* $\mathcal{F}_{p,q}f$ *is also univalent, that is,* $\mathcal{F}_{p,q} : S \mapsto S$.

Idea of Proof. For the operator $\mathcal{F}_{p,q}$ to preserve the class S of univalent functions, we require for the image-function

$$\mathcal{F}_{p,q}f(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

where $b_k = \Psi(k)a_k$ and $\Psi(k)$ is as in (2.3), that the following condition (see Avhadiev and Alksent'ev [3]) is satisfied:

$$\sigma_1 := \sum_{k=2}^{\infty} k|b_k| = \sum_{k=2}^{\infty} k\Psi(k)|a_k| < 1\,. \tag{2.9}$$

Using the known estimate for the coefficients of an univalent function $f(z) = z + \sum\limits_{k=1}^{\infty} a_k z^k \in A$, given by de Branges' theorem [6]: $|a_k| < k$, we can estimate the parameter σ_1 as:

$$\sigma_1 = \sum_{k=2}^{\infty} k\Psi(k)|a_k| \leq \sum_{k=2}^{\infty} k^2\Psi(k) = \sum_{k=2}^{\infty} \frac{k^2}{(1)_{k-1}}[\Psi(k)(1)_{k-1}] = \sum_{k=2}^{\infty} \frac{k^2}{(1)_{k-1}}\,\Theta(k) < 1,$$

with the denotation (compare with (2.3))

$$\Theta(k) := \Psi(k)(1)_{k-1} = [\prod_{i=1}^{p}(\alpha_i)_{k-1}]/[\prod_{j=1}^{q}(\beta_j)_{k-1}]. \tag{2.10}$$

Then, the estimate (2.9), giving a sufficient condition for the univalency of the Dziok-Srivastava image $\mathcal{F}_{p,q}f$, takes the form:

$$\sigma_1 \leq \sum_{k=2}^{\infty}\left[\frac{k-1}{(1)_{k-2}} + \frac{2}{(1)_{k-2}} + \frac{1}{(1)_{k-1}}\right]\Theta(k)$$

$$= \sum_{k=2}^{\infty} \frac{k-1}{(1)_{k-2}}\Theta(k) + 2\sum_{k=2}^{\infty} \frac{1}{(1)_{k-2}}\Theta(k) + \sum_{k=2}^{\infty} \frac{1}{(1)_{k-1}}\Theta(k) := A + 2B + C < 1\,,$$

where the constants A, B and C are represented by the values at $z = 1$ of the ${}_pF_q$-functions in (2.8). The proof uses essentially the techniques of the generalized hypergeometric functions and the details can be seen in Kiryakova [16]. $\square$

Similarly, we obtain the following

Theorem 2.2. *Assume the same conditions* (2.6) (*and* (2.7) *if* $p = q + 1$) *for the parameters of the generalized hypergeometric function* ${}_pF_q$ *in the Hadamard product* (2.1). *If the inequality*:

$$\left[\prod_{i=1}^{p}\alpha_i \,/ \prod_{j=1}^{q}\beta_j\right] {}_pF_q\left[\begin{array}{c}(\alpha_i+1)_1^p\\ (\beta_j+1)_1^q\end{array}\middle|\,1\right] + {}_pF_q\left[\begin{array}{c}(\alpha_i)_1^p\\ (\beta_j)_1^q\end{array}\middle|\,1\right] < 2 \tag{2.11}$$

is satisfied, then the Dziok-Srivastava operator $\mathcal{F}_{p,q}$ *maps a convex function* $f(z)$ *into a univalent function, that is,* $\mathcal{F}_{p,q} : K \mapsto S$.

Idea of Proof. The proof is much akin to that of Theorem 2, again requiring (according to a criterium from [3]) the condition (2.9). But in this case, instead of the estimate $|a_k| \leq k$, we use the estimate $|a_k| \leq 1$ for the coefficients of convex functions $f(z)$ defined by (1.1) (see, for example [24]). Thus, we require that $\sigma_1 = \sum\limits_{k=2}^{\infty} k\Psi(k)|a_k| \leq \sum\limits_{k=2}^{\infty} k\Psi(k) := D < 1$, and by similar manipulations and arguments as in the earlier proof, we obtain for $D+1$ the expression from (2.11). $\square$

Similarly to Theorems 2.1 and 2.2, we can derive *criteria for the Srivastava-Wright operator* (2.4) so to have $\mathcal{W}_{p,q} : S \mapsto S$ and $\mathcal{W}_{p,q} : K \mapsto S$. We use the same techniques, including the values of the Wright generalized hypergeometric functions ${}_p\Psi_q(1)$, and to ensure their existence we require the additional conditions on the parameters, as: $\sum\limits_{i=1}^{p} A_i \le \sum\limits_{j=1}^{q} B_j + 1$, and also $\sum\limits_{j=1}^{q} b_j - \sum\limits_{i=1}^{p} a_i + \frac{p-q}{2} > \frac{1}{2}$, if $\sum\limits_{i=1}^{p} A_i = \sum\limits_{j=1}^{q} B_j + 1$.

3. The generalized fractional calculus operators as special cases

The following operators of the *Generalized Fractional Calculus* (GFC) have been introduced in Kiryakova [13], where a full their theory is proposed, together with various applications to the special functions and integral transforms, hyper-Bessel operators, ODEs and Volterra integral equations, geometric function theory, etc. All the *classical FC operators* ([27]), and most of their generalizations by other authors fall in the GFC as very special cases, by taking "multiplicities" $m = 1, 2, \dots$ and some specific parameters. Specially, for $m = 1$ these are the *Erdélyi-Kober fractional integration operators* and the *Riemann-Liouville integrals*, and the respective fractional derivatives, see [27].

Definition 3.1. *(see in Kiryakova* [13], [12], [17], [15], *etc.) Let $m \ge 1$ be an integer; $\delta_i \ge 0, \gamma_i \in \mathbb{R}, i = 1, \dots, m$; $\beta > 0$. We consider $\delta = (\delta_1, \dots, \delta_m)$ as a multi-order of fractional integration; $\gamma = (\gamma_1, \dots, \gamma_m)$ as multi-weight, and β as additional parameter. The integral operators defined by means of the kernel $G^{m,0}_{m,m}$-function* (1.6) *as follows:*

$$(3.1)\quad If(z) = I^{(\gamma_i),(\delta_i)}_{\beta,m} f(z) = \int\limits_0^1 G^{m,0}_{m,m}\left[\sigma \left| \begin{matrix} (\gamma_i + \delta_i)_1^m \\ (\gamma_i)_1^m \end{matrix} \right.\right] f(z\sigma^{1/\beta}) d\sigma \quad \text{if} \quad \sum_{i=1}^m \delta_i > 0,$$

and as

$$If(z) = f(z) \quad \text{if} \quad \delta_1 = \delta_2 = \dots = \delta_m = 0,$$

are said to be multiple (m-tuple) Erdélyi-Kober fractional integration operators.

More generally, all the operators of the form

$$(3.2)\qquad \widetilde{I} f(z) = z^{\delta_0}\ I^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) \quad \text{with} \quad \delta_0 \ge 0,$$

are briefly called as generalized (m-tuple) fractional integrals. The corresponding generalized fractional derivatives are denoted by $D^{(\gamma_i),(\delta_i)}_{\beta,m}$ and defined by means of explicit differintegral expressions (Kiryakova [13, Ch.1]*), similarly to the idea for the classical Riemann-Liouville derivative (see e.g. in* [27]*).*

The operators of GFC have also more general variants when in the integral operator (3.1) *the kernel G-function is replaced by a Fox's $H^{m,0}_{m,m}$-function*, and the parameter $\beta > 0$ is generalized to a multi-index. See results in GFT for these H-function operators of GFC for example in Kiryakova [12], [17], etc.

However less havier for denotations, yet enough general and *most interesting for the purposes of GFT*, is the simpler case of the generalized fractional integral (3.1) when $\beta = 1$:

$$(3.3)\qquad I^{(\gamma_j),(\delta_j)}_{1,m} f(z) = \int\limits_0^1 G^{m,0}_{m,m}\left[\sigma \left| \begin{matrix} (\gamma_j + \delta_j)_1^m \\ (\gamma_j)_1^m \end{matrix} \right.\right] f(z\sigma)\, d\sigma, \quad \text{if} \quad \sum_{j=1}^m \delta_j > 0.$$

Note, that the conditions

$$(3.4)\qquad \delta_i \ge 0,\ \gamma_i \ge -1, \quad i = 1, \dots, m,$$

ensure that the "*normalized*" operator

$$(3.5)\quad \widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)} f(z) := \left[\prod_{j=1}^{m} \frac{\Gamma(\gamma_j+\delta_j+2)}{\Gamma(\gamma_j+2)}\right] I_{1,m}^{(\gamma_i),(\delta_i)} f(z), \quad \text{such that} \quad \widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)}\{z\} = z,$$

does preserve the function class A. The details can be seen in [12], [17], [15]. In terms of the Hadamard product (1.2), we have in A the following representation

$$(3.6)\qquad \widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)} f(z) = h(z) * f(z), \quad \text{with} \quad h(z) = z\ {}_{m+1}F_m\left[\begin{array}{c} 1, (\gamma_j+2)_1^m \\ (\gamma_j+\delta_j+2)_1^m \end{array}\middle| z\right]$$

$$\text{and the multipliers squence is:} \quad \Psi(k) = \prod_{j=1}^{m} \frac{(\gamma_j+2)_{k-1}}{(\gamma_j+\delta_j+2)_{k-1}} > 0.$$

Evidently, the GFC operator (3.5) is a *special case of the Dziok-Srivastava operator* (2.1), with its parameters taken as:

$$(3.7)\qquad \begin{array}{c} p = m+1,\ q = m \ \ \text{(that is, we are in the case } p = q+1), \\ \alpha_{m+1} = 1,\ \ \alpha_i = \gamma_i + 2,\ i = 1, ..., m; \quad \beta_j = \gamma_j + \delta_j + 2,\ j = 1, ..., m. \end{array}$$

Then, Theorems 2.1, 2.2 from Section 2 yield as corollaries criteria for the generalized operators of fractional integration (3.1), (3.4) so $\widetilde{I} : S \mapsto S$ and $\widetilde{I} : K \mapsto S$. These are already proved in Kiryakova, Saigo and Srivastava [18] as Theorems 3 and 4 there.

4. More special cases of the Dziok-Srivastava and of the GFC operators

The results for the operators of Sections 2 and 3 can be specialized for a great number of linear integral operators used in Geometric Function Theory, starting from *the classical operators of Biernacki, Libera, Bernardi, Komatu, Rusheweyh, Saigo, Hohlov, Srivastava and Owa*, and going to the more general operators studied recently by different authors. It is enough to choose suitable particular parameters $m, \gamma_k, \delta_k, \beta$ for the operators of GFC. We list below examples, and their presentation in the denotation (3.3).

For $\underline{\mathbf{m = 1}}$, we have the examples (see longer list and more details in [13], [18]):

Biernacki operator: ([5]) $Bf(z) = I_{1,1}^{-1,1} f(z) = \log(\frac{1}{1-z}) * f(z) = \int\limits_0^z \frac{f(\sigma)}{\sigma} d\sigma;$

Libera operator: $Lf(z) = 2I_{1,1}^{0,1} f(z) = z\,{}_2F_1(1,2;3;z) * f(z) = \frac{2}{z}\int\limits_0^z f(\sigma)d\sigma;$

Generalized Libera operator: ([22]) $B_c f(z) = (c+1) I_{1,1}^{c-1,1} f(z) = \frac{c+1}{z^c}\int\limits_0^z \sigma^{c-1} f(\sigma) d\sigma$

$= z^{c+1}{}_2F_1(1, c+1; c+2; z) * f(z);$ For integer $c \in \mathbb{N}$, it is called *Bernardi operator* ([4]);

Carlson-Shaffer operator: $L(a,c)f(z) = \frac{\Gamma(c)}{\Gamma(a)} I_{1,1}^{a-2,c-a} f(z) = z\,{}_2F_1(1,a;c;z) * f(z)$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int\limits_0^1 (1-\sigma)^{c-a-1} \sigma^{a-2} f(z\sigma) d\sigma.$$

Examples of GFC operators for $\underline{\mathbf{m = 2}}$ are the so-called *hypergeometric integral operators of Hohlov and Saigo.* We pay here some more attention on them.

In [10],[11] Hohlov introduced the hypergeometric operator $\mathbf{F}$(a,b,c) (we call it as *Hohlov operator*) defined in the class A by means of the Hadamard product with the *Gauss hypergeometric function* ${}_2F_1(a,b;c;z)$:

$$\mathbf{F}(a,b,c)f(z)=\{z\ {}_2F_1(a,b;c;z)\}*f(z). \tag{4.1}$$

We can write (4.1) in terms of the GFC operators (3.3)-(3.5) with $m=2$, as:

$$\mathbf{F}(a,b,c)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\,I_{1,2}^{(a-2,b-2),(1-a,c-b)}=\widetilde{I}_{1,2}^{(a-2,b-2),(1-a,c-b)}. \tag{4.2}$$

Another class of hypergeometric fractional integration operators has been introduced by Saigo [26](see [14]) for solving the Euler-Darboux equation, and studied from view of univalent functions' theory in series of papers by Srivastava, Saigo and Owa, as for example [31]. This linear integral operator named as *Saigo operator* can be represented also as a generalized fractional integral in the sense of (3.3) with $m=2$ (details in [14]):

(4.3)

$$I_{0,z}^{\alpha,\beta,\eta}f(z)=\frac{z^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^z(z-\zeta)^{\alpha-1}{}_2F_1\left[\begin{array}{c}\alpha+\beta,-\eta\\ \alpha\end{array}\middle|1-\frac{\zeta}{z}\right]f(\zeta)d\zeta=z^{-\beta}I_{1,2}^{(\eta-\beta,0),(-\eta,\alpha+\eta)}f(z).$$

To preserve the class A, the Saigo operator is normalized as

$$\widetilde{I}_{0,z}^{\alpha,\beta,\eta}:=\frac{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)}{\Gamma(2-\beta+\eta)}\ z^{\beta}\ I_{0,z}^{\alpha,\beta,\eta}. \tag{4.4}$$

Operators (3.3) with "multiplicity" $\underline{\mathbf{m>2}}$ have been not so popular. Such one is the Saigo operator (see in [13], [14]) with the *Appel F_3-function* in the kernel, that is, an operator (3.3) with $\mathbf{m=3}$:

$$\begin{aligned}I(\alpha,\alpha',\beta,\beta';\gamma)f(z)&=z^{-\alpha}\int_0^z\frac{(z-\xi)^{\gamma-1}}{\Gamma(\gamma)}\xi^{-\alpha'}\ F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{\xi}{z},1-\frac{z}{\xi}\right)f(\xi)d\xi\\ &=z^{-\alpha-\alpha'+\gamma}\int_0^1 G_{3,3}^{3,0}\left[\sigma\,\middle|\,\begin{array}{c}\alpha-\alpha'+\beta,\gamma-2\alpha',\gamma-\alpha'-\beta'\\ \alpha-\alpha',\beta-\alpha',\gamma-2\alpha'-\beta'\end{array}\right]f(z\sigma)d\sigma\\ &=z^{-\alpha-\alpha'+\gamma}\ I_{1,3}^{(\alpha-\alpha',\beta-\alpha',\gamma-2\alpha'-\beta'),(\beta,\gamma-\alpha'-\beta,\alpha')}f(z).\end{aligned} \tag{4.5}$$

A typical example of (3.3) <u>with arbitrary $\mathbf{m>2}$</u> is given by the *integral operator* $L=z^{\beta}I_{\beta,m}^{(\gamma_1,\dots,\gamma_m),(1,1,\dots 1)}$ which is the linear right inverse to the so-called *hyper-Bessel differential operator, introduced by Dimovski* (see in [13, Ch.3]), of the form

$$B=z^{\alpha_0}\frac{d}{dz}z^{\alpha_1}\frac{d}{dz}\dots\frac{d}{dz}z^{\alpha_m}=z^{-\beta}\prod_{i=1}^{m}\left(z\frac{d}{dz}+\beta\gamma_k\right),\quad \beta>0,\ \alpha_i,\gamma_i\in\mathbb{R}. \tag{4.6}$$

In this relation, let us mention the *Salagean differential operator* (see e.g. [21]), defined in A for functions $f(z)$ of the form (1.1) and for $m=1,2,3,\dots$ by the recurrence relation

$$S_0f(z)=f(z);S_1f(z)=zf'(z),\dots,S_mf(z)=S_1\left(S_{m-1}f(z)\right)=z+\sum_{k=2}^{\infty}k^m a_kz^k. \tag{4.7}$$

This operator can be seen as an interesting case of hyper-Bessel differential operator with $\beta=1$ and all $\gamma_k=-1,\delta_k=1,\ k=1,\dots,m$. Its linear right inverse operator is the *integral operator of Alexander* ([1], see e.g. [21]): A_m, $m=1,2,3\dots$, (4.8) $$A_0 f(z)=f(z),\, A_1 f(z)=\int_0^1 \frac{f(\sigma)}{\sigma}d\sigma,\ \dots,\, A_m f(z)=A_1(A_{m-1}f(z))=z+\sum_{k=2}^{\infty}\frac{1}{k^m}a_k z^k,$$

which can be written in the form of *generalized fractional integral* (3.3), namely:

(4.9)
$$A_m f(z)=I_{1,m}^{(-1,-1,\dots,-1),(1,1,\dots,1)}f(z), \quad \text{put} \quad \Psi(k)=1/k^m=[\Gamma(k)/\Gamma(1+k)]^m \ \text{ in } \ (2.3).$$

The above list of examples of operators of classical and generalized fractional calculus, that fall also as special cases of the Dziok-Srivastava operator, shows that the univalence criteria from Section 2 can be specified for each of them. We give examples only for the most interesting hypergeometric operators, when $p=2$, $q=1$. Namely, in [10], [11] Hohlov studied the problem for which values of the parameters a,b,c the operator (4.1) maps the class S of univalent functions into itself? His results, that can be obtained also, on the base of representation (4.2), from Theorems 2.1 and 2.2, state as follows:

Theorem 4.1. *([10],[11]) Let* $\mathbf{F}(a,b,c)$ *be the operator (4.1) in* A. *Suppose that the parameters* $a,b,c\in\mathbf{R}_+$ *satisfy the inequalities*

(4.10) $$a>0, \quad b>0, \quad c>a+b+2,$$

(4.11) $$\frac{\Gamma(c)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\ [(a)_2(b)_2+3ab(c-a-b-2)+(c-a-b-2)_2]<2.$$

Then for each univalent function f *in* A*, the image* $\mathbf{F}(a,b,c)f$ *is also univalent, i.e.* $\mathbf{F}(a,b,c):S\mapsto S$.

Theorem 4.2. *([10],[11]) Let* $\mathbf{F}(a,b,c)$ *be operator (4.1) in* A. *Suppose that the parameters* $a,b,c\in\mathbf{R}_+$ *satisfy*

(4.12) $$a>0, \quad b>0, \quad c>a+b+1,$$

(4.13) $$\frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\ [ab+c-a-b-1]<2.$$

Then, $\mathbf{F}(a,b,c)$ *maps any convex function into a univalent function,* $\mathbf{F}(a,b,c):K\mapsto S$.

From Theorems 2.1 and 2.2 we can derive also some univalence criteria for the normalized Saigo operator $\widetilde{I}_{0,z}^{\alpha,\beta,\eta}$, that involve values of the hypergeometric function ${}_3F_2$ at the point $z=1$. Similarly to the case of Hohlov, the conditions can be substantially simplified to involve the Gauss ${}_2F_1$-function at $z=1$, by taking in (4.3)-(4.4) some particular parameters α,β,η. For example, if $\beta=0$, our conditions for $\widetilde{I}^{\alpha,0,\eta}:S\mapsto S$ reduce to:

(4.14) $$-2<\eta\le 0, \quad \alpha>3, \quad \frac{(\alpha+\eta+1)(\alpha^2+3\alpha\eta+2\eta^2+\alpha+\eta)}{(\alpha-1)(\alpha-2)(\alpha-3)}<2\,;$$

and if $\beta=-\alpha$, the same conditions get the form

(4.15) $$\eta\le 0, \quad \alpha>3, \quad \alpha+\eta\ge 0, \quad \frac{\alpha(\alpha+1)^2}{(\alpha-1)(\alpha-2)(\alpha-3)}<2.$$

Although the conditions in the general cases of the Dziok-Srivastava operator (2.1) and generalized fractional integrals (3.3)-(3.5) may look somewhat abstract and too complicated, by the above special cases when taking specific values to the parameters for a particular operator, these criteria for univalence reduce to some easily checked numerical inequalities (as say, these in (4.14) and (4.15). One more example in this direction is to use the particular Hohlov's criteria (Theorems 4 and 5) for the operator of Biernacki and its generalization. Namely, the *operator of Biernacki*,

$$Bf(z) = I_{1,1}^{-1,1} f(z) = \log(\frac{1}{1-z}) * f(z) = \int_0^z \frac{f(\sigma)}{\sigma} d\sigma = \mathbf{F}(1,1,2)f(z)$$

does not map the class of univalent functions into itself, as observed by a counterexample given by Krzyz and Lewandowski [19]. The explanation is that in this case the set of parameters $(a,b,c) = (1,1,2)$ does not satisfy the Hohlov's inequalities in Theorem 4 (the series ${}_2F_1(1,1;2;1)$ is divergent). However, for the *generalized Biernacki operator*

$$\mathbf{F}(1,1,n+1)f(z) = n!\, z^{1-n} \int_0^z \int_0^{\sigma_n} \dots \int_0^{\sigma_2} \frac{f(\sigma_1)}{\sigma_1}\, d\sigma_1 \dots d\sigma_n \tag{4.16}$$

it follows that any univalent function is transferred into a univalent function for any $n > 8$. As a matter of fact, the operator $\mathbf{F}(1,1,c)$ maps the class S into S for any real $c > c_0 = (11+\sqrt{33})/2$, but the assumptions for a natural parameter n slightly increases the required value of the parameter.

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