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MAPPING PROPERTIES OF SOME CLASSES OF ANALYTIC FUNCTIONS UNDER NEW GENERALIZED INTEGRAL OPERATORS

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Abstract. In this paper we study the mapping properties with respect to some generalized integral operators which was studied recently.

1. Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U,

$$\mathcal{A} = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$$

and $S = \{f \in \mathcal{A} : f \text{ is univalent in } U\}$.

In [12] the subfamily T of S consisting of functions f of the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j,\ a_j \geq 0, j = 2, 3, ...,\ z \in U \tag{1.1}$$

was introduced.

Thus we have the subfamily $S - T$ consisting of functions f of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j,\ a_j \geq 0, j = 2, 3, ...,\ z \in U \tag{1.2}$$

A function $f(z) \in \mathcal{A}$ is said to be spiral-like if there exists a real number λ, $|\lambda| < \pi/2$, such that

$$Re\, e^{i\lambda} \frac{z f'(x)}{f(X)}\ (z \in U).$$

The class of all spiral-like functions was introduced by L. Spacek ([10]) and we denote it by $S^{\star}_{\lambda}$. Later, Robertson ([9]) considered the class C_{λ} of analytic functions in U for which $z f'(z) \in S^{\star}_{\lambda}$.

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Let $P_k^\lambda(\rho)$ be the class of functions $p(z)$ analytic in U with $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{Re\, e^{i\lambda} p(z) - \rho \cos\lambda}{1-\rho} \right| d\theta \leq k\pi \cos\lambda\,,\ z = re^{i\theta}\,, \tag{1.3}$$

where $k \geq 2$, $0 \leq \rho < 1$, λ is real with $|\lambda| < \frac{\pi}{2}$. In case that $k = 2$, $\lambda = 0$, $\rho = 0$, the class $P_k^\lambda(\rho)$ reduces to the class P of functions $p(z)$ analytic in U with $p(0) = 1$ and whose real part is positive.

we recall the well-known classes

$$R_k^\lambda(\rho) = \left\{ f(z) :\ f(z) \in \mathcal{A} \ and\ \frac{zf'(z)}{f(z)} \in P_k^\lambda(\rho)\,,\ 0 \leq \rho < 1 \right\}\,,$$

$$V_k^\lambda(\rho) = \left\{ f(z) :\ f(z) \in \mathcal{A} \ and\ \frac{(zf'(z))'}{f'(z)} \in P_k^\lambda(\rho)\,,\ 0 \leq \rho < 1 \right\}\,.$$

These classes are introduced and studied in [8].

The propose of this paper is to develop the mapping properties with respect to a new generalized integral operator.

2. Preliminary results

Prof. Breaz ([3]) has introduced the following integral operators on univalent function spaces:

$$J(z) = \left\{ \beta \int_0^z [f_1'(t^n)]^{\gamma_1} \cdot \ldots \cdot [f_p'(t^n)]^{\gamma_p}\, dt \right\}^{\frac{1}{\beta}}\,, \tag{2.1}$$

$$H(z) = \left\{ \beta \int_0^z t^{\beta-1} [f_1'(t)]^{\gamma_1} \cdot \ldots \cdot [f_p'(t)]^{\gamma_p}\, dt \right\}^{\frac{1}{\beta}}\,, \tag{2.2}$$

$$F(z) = \int_0^z \left(\frac{f_1(t)}{t} \right)^{\gamma_1} \cdot \ldots \cdot \left(\frac{f_p(t)}{t} \right)^{\gamma_p} dt, \tag{2.3}$$

$$G(z) = \left[\beta \int_0^z \left(\frac{f_1(t)}{t} \right)^{\gamma_1} \cdot \ldots \cdot \left(\frac{f_p(t)}{t} \right)^{\gamma_p} dt \right]^{\frac{1}{\beta}}\,, \tag{2.4}$$

$$F_{\gamma,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \left(\frac{f_1(t)}{t} \right)^{\frac{1}{\gamma_1}} \cdot \ldots \cdot \left(\frac{f_p(t)}{t} \right)^{\frac{1}{\gamma_p}} dt \right\}^{\frac{1}{\beta}} \tag{2.5}$$

and

$$G_{\gamma,p}(z)=\left\{[p(\gamma-1)+1]\int_0^z g_1^{\gamma-1}(t)\cdot\ldots\cdot g_p^{\gamma-1}(t)dt\right\}^{\frac{1}{p(\gamma-1)+1}}, \tag{2.6}$$

where $\gamma_i\,,\ \gamma\,,\ \beta\in\mathbb{C}\,,\ \forall i=\overline{1,p}\,,\ p\in\mathbb{N}-\{0\}\,,\ n\in\mathbb{N}-\{0,1\}$.

Let D^n be the Sălăgean differential operator (see [11]) $D^n:\mathcal{A}\to\mathcal{A}$, $n\in\mathbb{N}$, defined as:

$$D^0f(z)=f(z)\,,\ D^1f(z)=Df(z)=zf'(z)\,,\ D^nf(z)=D(D^{n-1}f(z)) \tag{2.7}$$

and $D^k\,,\ D^k:\mathcal{A}\to\mathcal{A}\,,\ k\in\mathbb{N}\cup\{0\}$, of form:

$$D^0f(z)=f(z)\,,\ \ldots\,,\ D^kf(z)=D(D^{k-1}f(z))=z+\sum_{n=2}^{\infty}n^ka_nz^n\,. \tag{2.8}$$

Definition 2.1. [2] *Let $\beta,\lambda\in\mathbb{R}$, $\beta\ge0$, $\lambda\ge0$ and $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$. We denote by D_λ^β the linear operator defined by*

$$D_\lambda^\beta:A\to A\,,\quad D_\lambda^\beta f(z)=z+\sum_{j=n+1}^{\infty}[1+(j-1)\lambda]^\beta a_jz^j\,. \tag{2.9}$$

Remark 2.1. *In ([1]) we have introduced the following operator concerning the functions of form (1.1):*

$$D_\lambda^\beta:A\to A\,,\quad D_\lambda^\beta f(z)=z-\sum_{j=n+1}^{\infty}[1+(j-1)\lambda]^\beta a_jz^j\,. \tag{2.10}$$

The neighborhoods concerning the class of functions defined using the operator (2.10) is studied in [5].

Remark 2.2. *Let consider the following operator concerning the functions $f\in S\,,\ S=\{f\in\mathcal{A}: f$ is univalent in $U\}$:*

$$D_{\lambda_1,\lambda_2}^{n,\beta}f(z)=(h*\psi_1*f)(z)=z\pm\sum_{k\ge2}\frac{[1-\lambda_1(k-1))]^{\beta-1}}{[1-\lambda_2(k-1))]^{\beta}}\cdot\frac{1+c}{k+c}\cdot C(n,k)\cdot a_k\cdot z^k\,, \tag{2.11}$$

where $C(n,k)=\frac{(n+1)_{k-1}}{(1)_{k-1}}$; $(\cdot)$. is the Pochammer symbol; $k\ge2\,,\ c\ge0$.

The following integral operator is studied in [4], where $f_i\,,\ i=1\ldots n\,,\ n\in\mathbb{N}$, is considered to be of form (1.2):

Definition 2.2. *We define the general integral operator $I_{k,n,\lambda,\mu}:\mathcal{A}_n\to\mathcal{A}$ by*

$$I_{k,n,\lambda,\mu}(f_1,\ldots,f_n)=F\,, \tag{2.12}$$

$$D^kF(z)=\int_0^z\left(\frac{D_1^\lambda f_1(t)}{t}\right)^{\mu_1}\cdot\ldots\cdot\left(\frac{D_n^\lambda f_n(t)}{t}\right)^{\mu_n}dt\,,$$

where $f_i\in\mathcal{A}\,,\ i\in\mathbb{N}-\{0\}\,,\ \lambda=(\lambda_1,\ldots,\lambda_n)\in\mathbb{N}_0^n\,,\ \mu=(\mu_1,\ldots,\mu_n)\in\mathbb{N}^n\,,\ n\in\mathbb{N}$ and $k\in\mathbb{N}_0$.

Theorem 2.1. *Let* α, γ_1, γ_2, $\beta \in \mathbb{C}$, $Re\alpha = a > 0$ *and* $D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z) \in \mathcal{A}$, λ_1, λ_2, $\kappa \geq 0$, $\sigma \in \mathbb{R}$, $j = \overline{1,p}$, $p \in \mathbb{N}$, $D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n)$ *of form (2.11). If*

$$\left|\frac{(D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n))''}{(D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n))'}\right| \leq \frac{1}{n} \text{ and } \left|\frac{(D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n))'}{(D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n))}\right| \leq \frac{1}{n},\ \forall z \in U,\ j = \overline{1,p},$$

$$\frac{\sum\limits_{j=1}^{p} [|\delta_j^1| \cdot (|2\gamma_1 - 1| - |\sigma|) + |\delta_j^2| \cdot (|2\gamma_2 - 1| - |\sigma|)]}{|\sigma \cdot (2\gamma_1 - 1) \cdot (2\gamma_2 - 1) \cdot (\prod\limits_{j=1}^{p} \delta_j^1 \cdot \delta_j^2)|} \leq 1$$

and

$$|\sigma \cdot (2\gamma_1 - 1) \cdot (2\gamma_2 - 1) \cdot (\prod_{j=1}^{p} \delta_j^1 \cdot \delta_j^2)| \leq \frac{n+2a}{2} \cdot \left(\frac{n+2a}{n}\right)^{\frac{1}{n+2a}},$$

then for $\forall \delta$, δ_j^1, $\delta_j^2 \in \mathbb{C}$, $j = 1 \ldots p$, $Re(\beta) \geq a$, $Re(\beta\delta) \geq a$, *the function*

(2.13)

$$I^1(z) = \left\{\beta \int_0^z t^{\beta\delta-1} \cdot \prod_{j=1}^{p} \left[\frac{((D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(t^n)')^{2\gamma_1-1}}{t^\sigma}\right]^{\delta_j^1} \cdot \left[\frac{(D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(t^n))^{2\gamma_2-1}}{t^\sigma}\right]^{\delta_j^2} dt\right\}^{\frac{1}{\beta}}$$

is univalent for all $n \in \mathbb{N} - \{0\}$.

If we consider the operator $D^\beta_\lambda f(z)$ of form (2.10) we obtain the following Corolary, whose proof is similar with the prove of Theorem 2.1.

Corollary 2.1. *Let* α, γ_1, γ_2, $\chi \in \mathbb{C}$, $Re\alpha = a > 0$ *and* $D^\beta_\lambda f_j(z) \in \mathcal{A}$, $\beta \geq 0$, $\lambda \geq 0$, $\sigma \in \mathbb{R}$, $D^\beta_\lambda f(z^n)$ *of form (2.10). If*

$$\left|\frac{(D^\beta_\lambda f_j(z^n))''}{(D^\beta_\lambda f_j(z^n))'}\right| \leq \frac{1}{n} \text{ and } \left|\frac{(D^\beta_\lambda f_j(z^n))'}{(D^\beta_\lambda f_j(z^n))}\right| \leq \frac{1}{n},\ \forall\ z \in U,\ j = \overline{1,p},$$

$$\frac{\sum\limits_{j=1}^{p} [|\delta_j^1| \cdot (|2\gamma_1 - 1| - |\sigma|) + |\delta_j^2| \cdot (|2\gamma_2 - 1| - |\sigma|)]}{|\sigma \cdot (2\gamma_1 - 1) \cdot (2\gamma_2 - 1) \cdot (\prod\limits_{j=1}^{p} \delta_j^1 \cdot \delta_j^2)|} \leq 1$$

and

$$|\sigma \cdot (2\gamma_1 - 1) \cdot (2\gamma_2 - 1) \cdot (\prod_{j=1}^{p} \delta_j^1 \cdot \delta_j^2)| \leq \frac{n+2a}{2} \cdot \left(\frac{n+2a}{n}\right)^{\frac{1}{n+2a}},$$

then for all δ, δ_j^1, $\delta_j^2 \in \mathbb{C}$, $j = 1 \ldots p$, $Re(\chi) \geq a$, $Re(\chi\delta) \geq a$, *the function*

$$I^2(z) = \left\{\chi \int_0^z t^{\chi\delta-1} \prod_{j=1}^{p} \left[\frac{((D^\beta_\lambda f_j(t^n)')^{2\gamma_1-1}}{t^\sigma}\right]^{\delta_j^1} \left[\frac{(D^\beta_\lambda f_j(t^n))^{2\gamma_2-1}}{t^\sigma}\right]^{\delta_j^2} dt\right\}^{\frac{1}{\chi}} \quad (2.14)$$

is univalent for $\forall n \in \mathbb{N} - \{0\}$.

Lemma 2.1. [7] *Let* $u = u_1 + iu_2$, $v = v_1 + iv_2$ *and* $\Psi(u,v)$ *be a complex valued function satisfying the conditions:*

(i) $\Psi(u,v)$ *is continuous in a domain* $D \in \mathbb{C}^2$,

(ii) $(1,0) \in D$ *and* $Re\,\Psi(1,0) > 0$,

(iii) $Re\,\Psi(iu_2, v_1) \leq 0$, *whenever* $(iu_2, v_1) \in D$ *and* $v_1 \leq -\frac{1}{2}(1+u_2^2)$.

If $h(z) = 1 + \sum\limits_{i\geq 1} c_i z^i$ is an analytic function in U such that $(h(z), zh'(z)) \in D$ and $Re\,\Psi(h(z), zh'(z)) > 0$ for $z \in U$, then $Re\,h(z) > 0$ in U.

Lemma 2.2. [6] *Let* $f(z) \in V_k^\lambda(\rho)$, $0 \leq \rho < 1$ *and* λ *is real with* $|\lambda| < \frac{\pi}{2}$. *Then* $f(z) \in R_k^\lambda(\beta)$, *where* β *is one of the root of*

$$2\beta^3 + (1-2\rho)\beta^2 + (3\sec^2\lambda - 4)\beta - (1+2\rho)\tan^2\lambda = 0\,. \tag{2.15}$$

Following we present the mapping properties of the general integral operator of form (2.13), giving also several examples which prove its relevance.

3. Main Results

Using Theorem 2.1 and making additional calculus, we obtain:

Theorem 3.1. *Let* $D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n) \in R_k^\lambda$, $D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n)$ *of form (2.11),* $n \in \mathbb{N}$, λ_1, λ_2, $\kappa \geq 0$, $\sigma \in \mathbb{R}$, $j = \overline{1,p}$, $p \in \mathbb{N}$, *for* $0 \leq \rho < 1$. *Also let* λ *be real,* $|\lambda| < \frac{\phi}{2}$. *If*

$$0 \leq [\rho - 1]\sum_{j=1}^{p} \delta_j^a + \beta\delta < 1\,,$$

then $I^1(z) \in V_k^\lambda(\eta)$, $I^1(z)$ *of form (2.13), with*

$$\eta = [\rho - 1]\sum_{j=1}^{p} \delta_j^a + \beta\delta\,, \tag{3.1}$$

β, δ, $\delta_j^a \in \mathbb{C}$, $a \in \{1,2\}$, $j = \overline{1,p}$, $Re(\beta\delta) > 0$.

Remark 3.1. *If we consider the operator* $D_\lambda^\beta f(z) \in R_k^\lambda(\rho)$ *of form (2.10) we obtain similar result as in Theorem 3.1.*

Remark 3.2. *If we apply the operator (2.7) to the integral operator* $F(z)$ *of form (2.3), we obtain the result from* [6].

Next we give few examples of particular cases which can be found in literature.

Let $\beta = 0$ in $D_\lambda^\beta f(z)$ of form (2.9) or (2.10). So we have that $D_\lambda^0 f(z) = f(z)$, $\forall \lambda \geq 0$. We will use this form of the integral operator, where the function f is of form (1.2) with respect to the operator (2.14). For further simplification, we consider that $\gamma_1 = \gamma_2 = 1$, and $\delta = 1$ (except of Example 3.4).

For the first four examples we consider $\delta_j^1 = 0$, $j = \overline{1,p}$, $p \in \mathbb{N} - \{0\}$, $n = 1$.

Example 3.1. *If* $\sigma = 1$, $\chi = 1$ *and we use the notation* $\delta_j^2 = \gamma_j$, $j = \overline{1,p}$, $p \in \mathbb{N} - \{0\}$, *we obtain the operator* $F(z)$ *of form (2.3).* $F(z) \in V_k^\lambda(\eta)$ *if* $0 \leq (\rho - 1)\sum\limits_{j=1}^{p} \gamma_j + 1 < 1$ *with* $\eta = (\rho - 1)\sum\limits_{j=1}^{p} \gamma_j + 1$.

Example 3.2. *If* $\sigma = 1$ *we obtain the operator* $G(z)$ *of form (2.4) for* $\delta_j^2 = \gamma_j$, $j = \overline{1,p}$, $p \in \mathbb{N} - \{0\}$. $G(z) \in V_k^\lambda(\eta)$ *if* $0 \leq (\rho - 1)\sum\limits_{j=1}^{p} \gamma_j + 1 < 1$ *with* $\eta = (\rho - 1)\sum\limits_{j=1}^{p} \gamma_j + 1$.

Example 3.3. *If* $\sigma = 1$ *and we use the notation* $\delta_j^2 = 1/\gamma_j$, $j = \overline{1,p}$, $p \in \mathbb{N} - \{0\}$, *we obtain the operator* $F_{\gamma,\beta}(z)$ *of form (2.15).* $F_{\gamma,\beta}(z) \in V_k^\lambda(\eta)$ *if* $0 \leq (\rho-1)\sum\limits_{j=1}^{p} \frac{1}{\gamma_j} + \beta < 1$ *with* $\eta = (\rho - 1)\sum\limits_{j=1}^{p} \gamma_j + \beta$.

Example 3.4. *If* $\sigma = 0$ *we obtain the operator* $G_{\gamma,p}(z)$ *of form (2.6) for* $\chi = [p(\gamma - 1) + 1]$, $\delta = \frac{1}{\chi}$ *and* $\delta_j^2 = \gamma - 1$, $G_{\gamma,p}(z) \in V_k^\lambda(\eta)$ *if*
$0 \leq (1 - \rho)\sum\limits_{j=1}^{p} \gamma_j + 1 < 1$ *with* $\eta = (\rho - 1)\sum\limits_{j=1}^{p} \gamma_j + 1$.

For the next two examples we consider $\delta_j^2 = 0$ $j = \overline{1,p}$, $p \in \mathbb{N} - \{0\}$, and $\sigma = 0$.

Example 3.5. *a) If* $\chi = 1, \delta = 1$, *we obtain a particular case of the function* $J(z)$ *of form (2.9), in which* $\beta = 1$, $\forall n \in \mathbb{N} - \{0\}$. $J(z) \in V_k^\lambda(\eta)$ *if* $0 \leq (1 - \rho)\sum\limits_{j=1}^{p} \gamma_j + 1 < 1$ *with* $\eta = (\rho - 1)\sum\limits_{j=1}^{p} \gamma_j + 1$.

b) If $\delta = \frac{1}{\chi}$, $\delta_j^1 = \gamma_j$, $j = \overline{1,p}$, $p \in \mathbb{N} - \{0\}$. *we obtain the operator* $J(z)$ *of form (2.9), in which* $\beta = 1$, $\forall n \in \mathbb{N} - \{0\}$. $J(z) \in V_k^\lambda(\eta)$ *if*
$0 \leq (1 - \rho)\sum\limits_{j=1}^{p} \gamma_j + 1 < 1$ *with* $\eta = (\rho - 1)\sum\limits_{j=1}^{p} \gamma_j + 1$.

Example 3.6. *If* $n = 1$, $\delta = \frac{1}{\chi}$, *we obtain the operator* $H(z)$ *of form (2.12) for* $\delta_j^1 = \gamma_j$, $j = \overline{1,p}$, $p \in \mathbb{N} - \{0\}$. $F(z) \in V_k^\lambda(\eta)$ *if* $0 \leq (1 - \rho)\sum\limits_{j=1}^{p} \gamma_j + \beta < 1$ *with* $\eta = (\rho - 1)\sum\limits_{j=1}^{p} \gamma_j + \beta$.

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