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HARMONIC FUNCTIONS FOR WHICH SECOND DILATATION HAS POSITIVE REAL PART

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Abstract. In this paper we will extend a fundamental property, first defined by M. S. Robinson [4] and then applied by R. J. Libera [3] to functions with positive real part, to harmonic functions and study the class of such functions.

1. Introduction

Let Ω be the class of functions $\phi(z)$ which are regular in the open unit disc $D = \{z| \quad |z| < 1\}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Denote by $\mathcal{P}$ the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ which are regular in $\mathbb{D}$ such that $p(z)$ is in $\mathcal{P}$ if and only if

$$p(z) = \frac{1 + \phi(z)}{1 - \phi(z)}$$

for some function $\phi(z) \in \Omega$ for all $z \in \mathbb{D}$. Next let S^* denote the family of functions $s(z) = z + \alpha_2 z^2 + \alpha_3 z^3 + \ldots$ which are regular in $\mathbb{D}$ such that $s(z)$ is in S^* if and only if

$$z\frac{s'(z)}{s(z)} = p(z)$$

for some $p(z) \in \mathcal{P}$ for all $z \in \mathbb{D}$, and let $s_1(z) = z + \beta_2 z^2 + \beta_3 z^3 + \ldots$ and $s_2(z) = z + \gamma_2 z^2 + \gamma_3 z^3 + \ldots$ be analytic functions in $\mathbb{D}$. If there exists $\phi(z) \in \Omega$ such that $s_1(z) = s_2(\phi(z))$, then we say that $s_1(z)$ is subordinate to $s_2(z)$ and we write $s_1(z) \prec s_2(z)$, then $s_1(\mathbb{D}) \subset s_2(\mathbb{D})$. Moreover, univalent harmonic functions are generalization of univalent analytic functions. The point of the departure is the canonical representation

$$f = h(z) + \overline{g(z)}, g(0) = 0$$

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of a harmonic function f in the open unit disc $\mathbb{D}$ as the sum of an analytic function $h(z)$ and conjugate of an analytic function $g(z)$. With the convention that $g(0) = 0$ the representation is unique. The power series expansions of $h(z)$ and $g(z)$ are denoted by

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, g(z) = \sum_{n=1}^{\infty} b_n z^n .$$

If f is sense-preserving harmonic mapping of $\mathbb{D}$ onto some other region , then by Lewy's Theorem its Jacobian is strictly positive, i. e.

$$J_{f(z)} = |h'(z)|^2 - |g'(z)|^2 > 0 .$$

Equivalently, the inequality $|g'(z)| < |h'(z)|$ hold for all $z \in \mathbb{D}$. This shows in particular that $h'(0) \neq 0$ and $h(0) = 1$. The class of all sense-preserving harmonic mapping of the disc with $a_0 = b_0 = 0$, $a_1 = 1$ will be denoted by S_H. Thus S_H contains the standard class S of analytic univalent functions. Although the analytic part $h(z)$ of a function $f \in S_H$ is locally univalent, it will become apparent that it need not be univalent. The class of functions $f \in S_H$ with $g'(0) = 0$ will be denoted by S_H^0. At the same time we note that S_H is a normal family and S_H^0 is a compact normal family (for details see [1]).

Finally, we consider the following class of harmonic mappings

$$S^*_{HPST} = \left\{ f = h(z) + \overline{g(z)} \mid f \in S_H, h(z) \in S^*, w(z) = \frac{g'(z)}{h'(z)} \in \mathcal{P} \right\}$$

In the present paper we will investigate the subclass S^*_{HPST}. We will need the following lemma and theorem in the sequel.

Lemma 1.1. *([3]) N and $\mathbb{D}$ are regular in $D = \{z| \quad |z| < 1\}$; $N(0) = \mathbb{D}(0) = 0$, $\mathbb{D}$ maps $\mathbb{D}$ onto a many-sheeted region which is starlike with respect to the origin and $(\frac{N'}{\mathbb{D}'}) \in \mathcal{P}$, then $(\frac{N}{\mathbb{D}}) \in \mathcal{P}$.*

Theorem 1.1. *([2]) Let $h(z)$ be an element of S^*, then*

$$\frac{r}{(1+r)^2} \leq |h(z)| \leq \frac{r}{(1-r)^2},$$
$$\frac{1-r}{(1+r)^3} \leq |h'(z)| \leq \frac{1+r}{(1-r)^3}.$$

2. Main Results

Lemma 2.1. *Let α be a real number with $\alpha > 0$ and let $p(z) = b_1 + p_1 z + p_2 z^2 + ...$ be analytic in $\mathbb{D}$ and satisfies the condition $\mathrm{Re}\, p(z) > 0$, then*

$$\frac{F(\alpha, \beta, -r)}{1+r^2} \leq |p(z)| \frac{F(\alpha, \beta, r)}{1-r^2}, \tag{2.1}$$

where

$$F(\alpha, \beta, r) = 2\alpha r + \sqrt{\alpha^2(1+\alpha^2) + \beta^2(1-r^2)^2}.$$

Proof. Since $p(z) = (\alpha + i\beta) + p_1 z + p_2 z^2 + ...$ is analytic in $\mathbb{D}$ and satisfies the condition $\mathrm{Re}\, p(z) > 0$, then the function

$$p_1(z) = \frac{1}{(\alpha + i\beta)}[p(z) - i\beta] \tag{2.2}$$

is an element of $P([\alpha,\beta])$. On the other hand, since $p_1(z)\in\mathcal{P}$, then we have

$$\left|p_1(z)-\frac{1+r^2}{1-r^2}\right|\le\frac{2r}{1-r^2}\,. \tag{2.3}$$

Considering (2.2) and (2.3) together, we can write

$$\left|\frac{1}{\alpha}(p(z)-i\beta)-\frac{1+r^2}{1-r^2}\right|\le\frac{2r}{1-r^2}\,. \tag{2.4}$$

After simple calculations from (2.4) we get (2.1). □

Theorem 2.1. *Let $f=h(z)+\overline{g(z)}$ be an element of S^*_{HPST}, then*

$$\frac{rF(\alpha,\beta,-r)}{(1+r)^2(1+r^2)}\le|g(z)|\,\frac{rF(\alpha,\beta,r)}{(1-r)^3(1+r)}\,, \tag{2.5}$$

and

$$\frac{(1-r)F(\alpha,\beta,-r)}{(1+r^2)(1+r)^3}\le|g'(z)|\,\frac{F(\alpha,\beta,r)}{(1-r)^4}\,, \tag{2.6}$$

where $b_1=\alpha+i\beta,\ \alpha>0$.

Proof. Since

$$h(z)=z+a_2z^2+a_3z^3+\ldots\Rightarrow h'(z)=1+2a_2z+3a_3z^3+\ldots$$

$$g(z)=b_1z+b_2z^2+b_3z^3+\ldots\Rightarrow g'(z)=b_1+2b_2z+3b_3z^2+\ldots$$

$h(0)=g(0)=0$, and $h(z)\in S^*$, if

$$w(z)=\frac{g'(z)}{h'(z)}\in\mathcal{P}$$

then the conditions of Lemma 1.1 are satisfied, therefore we have $\dfrac{g(z)}{h(z)}\in\mathcal{P}$.

$$w(0)=\frac{g'(0)}{h'(0)}=\frac{b_1+2b_2.0+\ldots}{1+2a_2.0+\ldots}=b_1\,.$$

If $b_1=\alpha+i\beta$, $\alpha>0$, then we can apply Lemma 2.1 to obtain

$$\frac{F(\alpha,\beta,-r)}{1+r^2}\le\left|\frac{g'(z)}{h'(z)}\right|\le\frac{F(\alpha,\beta,r)}{1-r^2}\,,$$

$$\frac{F(\alpha,\beta,-r)}{1+r^2}\le\left|\frac{g(z)}{h(z)}\right|\le\frac{F(\alpha,\beta,r)}{1-r^2}\,,$$

or

$$|h'(z)|\,\frac{F(\alpha,\beta,-r)}{1+r^2}\le|g'(z)|\le|h'(z)|\,\frac{F(\alpha,\beta,-r)}{1-r^2}\,, \tag{2.7}$$

$$|h(z)|\,\frac{F(\alpha,\beta,-r)}{1+r^2}\le|g(z)|\le|h(z)|\,\frac{F(\alpha,\beta,-r)}{1-r^2}\,. \tag{2.8}$$

Using Theorem 1.1, (2.7) and (2.8) we get (2.5) and (2.6). □

Corollary 2.1. *Let $f=h(z)+\overline{g(z)}$ be an element of S^*_{HPST}, then*

$$\frac{(1-r^2)^2-[F(\alpha,\beta,r)]^2}{(1+r)^8}\le J_{f(z)}\le\frac{(1+r^2)^2-[(1+r^2)^2-F(\alpha,\beta,-r)]^2}{(1-r)^6(1+r^2)^2}\,. \tag{2.9}$$

Proof. This corollary is a simple consequence of Theorem 2.1 □

Corollary 2.2. *If $f = (h(z) + \overline{g(z)}) \in S^*_{HPST}$, then*

$$\int \frac{1-r^2}{(1+r)^4}dr - \int \frac{F(\alpha,\beta,r)}{(1+r)^4}dr \leq |f| \leq \int \frac{1-r^2}{(1-r)^4}dr + \int \frac{F(\alpha,\beta,r)}{(1-r)^4}dr \tag{2.10}$$

Proof. Using Theorem 1.1 and Theorem 2.1 and straightforward calculations we get,

$$\frac{(1-r^2)-F(\alpha,\beta,r)}{(1+r)^4} \leq |h'(z)|\,(1-|w(z)|) \leq \frac{(1+r)[(1+r^2)-F(\alpha,\beta,-r)]}{(1-r)^3(1+r^2)} \tag{2.11}$$

$$\frac{(1-r)[(1+r^2)-F(\alpha,\beta,-r)}{(1+r)^3(1+r^2)} \leq |h'(z)|\,(1+|w(z)|) \leq \frac{(1-r^2)+F(\alpha,\beta,r)}{(1-r)^4} \tag{2.12}$$

On the other hand we have

$$|h'(z)|\,(1-|w(z)|)\,|dz| \leq |df| \leq |h'(z)|\,(1+|w(z)|)\,|dz| \tag{2.13}$$

Using (2.11), (2.12), (2.13) and integrating from 0 to r we get (2.10). □

Theorem 2.2. *Let $f = h(z) + \overline{g(z)}$ be an element of S^*_{HPST}, then*

$$|b_n| \leq \frac{n(2n+1)}{3} + (n+1)\,.$$

Proof. Using the definition of S^*_{HPST}, we can write

$$p(z) = 1+p_1z+p_2z^2+...+p_nz^n+... = \frac{g'(z)}{h'(z)} = \frac{b_1+2b_2z+3b_3z^2+...+nb_nz^{n-1}+...}{1+2a_2z+3a_3z^2+...+na_nz^{n-1}+...} \Rightarrow$$

$$p_n + 2a_2p_{n-1} + 3a_3p_{n-2} + 4a_4p_{n-3} + ... + na_np_1 + (n+1)a_{n+1} = (n+1)b_n$$

using the coefficient inequalities for the class S^* and $\mathcal{P}$, we obtain

$$|(n+1)b_n| = |p_n + 2a_2p_{n-1} + 3a_3p_{n-2} + 4a_4p_{n-3} + ... + na_np_1 + (n+1)a_{n+1}|$$
$$\leq |p_n| + 2\,|a_2|\,|p_{n-1}| + 3\,|a_3|\,|p_{n-2}| + 4\,|a_4|\,|p_{n-3}| + ... + n\,|a_n|\,|p_1| + (n+1)\,|a_{n+1}|$$
$$2 + 2.2.2 + 3.3.2 + 4.4.2 + ... + n.n.2 + (n+1)(n+1)$$
$$= 2(1 + 2^2 + 3^2 + 4^2 + ... + n^2) + (n+1)^2$$
$$= 2\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{n(n+1)(2n+1)}{3} + (n+1)^2 \Rightarrow$$
$$|b_n| \leq \frac{n(2n+1)}{3} + (n+1)\,.$$

□

Remark 2.1. *We also note that the inequalities in this paper are sharp because for these inequalities the extremal function is obtained in the following manner, using Lemma 1.1 we have*

$$\frac{g(z)}{h(z)} = p(z) \Rightarrow g(z) = h(z).p(z) \Rightarrow g(z) = \frac{z}{(1-z)^2}\frac{1+z}{1-z} \Rightarrow$$

$$g(z) = \frac{z(1+z)}{(1-z)^3}\,.$$

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