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SERIES IN MITTAG-LEFFLER FUNCTIONS: GEOMETRY OF CONVERGENCE

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ABSTRACT. We consider series, defined by means of the Mittag-Leffler functions, find the domains of convergence and study the behaviour on the boundaries of these domains. We give analogues of the classical theorems for the power series like Cauchy-Hadamard, Abel as well as Fatou type theorems. The asymptotic formulae for the Mittag-Leffler functions in the cases of "large" values of indices that are used in the proofs of the convergence theorems for the considered series are also provided.

1. INTRODUCTION

We consider series in functions related to the Mittag-Leffler functions E_α and $E_{\alpha,\beta}$:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0,\ \beta > 0. \tag{1.1}$$

We study their geometry of convergence, more precisely, we determine where these series converge and where do not, and moreover, where the convergence is uniform and where is not. Their disks of convergence have been found and studied the behaviour on the boundaries of these domains, proving theorems of Cauchy-Hadamard, Abel and Fatou type. Such kind of results are provoked by the fact that the solutions of some fractional order differential and integral equations can be written in terms of series (or series of integrals) of Mittag-Leffler functions (as for example in Kiryakova [2] and Sandev, Tomovski and Dubbeldam [8]).

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2. Inequalities and asymptotic formulae

First of all, denoting

$$\theta_n(z)=\sum_{k=1}^{\infty}\frac{z^k}{\Gamma(kn+1)},\quad \theta_{n,\beta}(z)=\sum_{k=1}^{\infty}\frac{\Gamma(\beta)\,z^k}{\Gamma(kn+\beta)},\quad \theta_{\alpha,n}(z)=\sum_{k=1}^{\infty}\frac{\Gamma(n)\,z^k}{\Gamma(\alpha k+n)},$$

we give some inequalities and asymptotic formulae for "large" values of indices as follows (see for details [6]).

Lemma 2.1. *Let $n\in\mathbb{N}$, $z\in\mathbb{C}$ and $K\subset\mathbb{C}$ be a nonempty compact set. Then there exists a constant $\widetilde{C}$, $0<\widetilde{C}<\infty$, such that*

$$|\theta_n(z)|\le \widetilde{C}/n!,\quad |\theta_{n,\beta}(z)|\le \widetilde{C}/(n-1)!,\quad |\theta_{\alpha,n}(z)|\le \widetilde{C}\frac{\Gamma(n)}{\Gamma(\alpha+n)},\tag{2.1}$$

for all the natural numbers n and each $z\in K$.

Theorem 2.1. *For the Mittag-Leffler functions E_n, $E_{n,\beta}$, $E_{\alpha,n}$ $(n\in\mathbb{N})$, the following asymptotic formulae*

$$E_n(z)=1+\theta_n(z),\ \ z\in\mathbb{C},\quad \theta_n(z)\to 0\quad as\ n\to\infty\tag{2.2}$$

$$E_{n,\beta}(z)=\frac{1}{\Gamma(\beta)}\,(1+\theta_{n,\beta}(z)),\ \ z\in\mathbb{C},\quad \theta_{n,\beta}(z)\to 0\quad as\ n\to\infty\tag{2.3}$$

$$E_{\alpha,n}(z)=\frac{1}{\Gamma(n)}\,(1+\theta_{\alpha,n}(z)),\ \ z\in\mathbb{C},\quad \theta_{\alpha,n}(z)\to 0\quad as\ n\to\infty\tag{2.4}$$

are valid. The functions $\theta_n(z)$, $\theta_{n,\beta}(z)$, $\theta_{\alpha,n}(z)$ are holomorphic for $z\in\mathbb{C}$. The convergence is uniform on the compact subsets of the complex plane $\mathbb{C}$.

Note 2.1. According to the asymptotic formulae (2.2) - (2.4), it follows there exists a natural number N_0 such that the functions E_n, $\Gamma(n)E_{\alpha,n}$, $\Gamma(\beta)E_{n,\beta}$ have not any zeros at all for $n>N_0$.

Note 2.2. Note that each of the functions $E_n(z), E_{n,\beta}(z), E_{\alpha,n}(z)$, $(n\in\mathbb{N})$, being an entire function, no identically zero, has no more than finite number of zeros in the closed and bounded set $|z|\le R$ ([3], vol.1, ch. 3, §6, 6.1, p.305). Moreover, because of Note 2.1., no more than finite number of these functions have some zeros.

3. Series in Mittag-Leffler functions. Theorems of Cauchy-Hadamard and Abel type

We introduce auxiliary functions, related to Mittag-Leffler's functions, adding $\widetilde{E}_0(z)$, $\widetilde{E}_{0,\beta}(z))$ and $\widetilde{E}_{\alpha,0}(z)$ just for completeness, namely:

$$\begin{aligned}\widetilde{E}_0(z)=1;\quad &\widetilde{E}_n(z)=z^nE_n(z),\ n\in\mathbb{N},\\ \widetilde{E}_{0,\beta}(z)=1;\quad &\widetilde{E}_{n,\beta}(z)=\Gamma(\beta)z^nE_{n,\beta}(z),\ n\in\mathbb{N};\quad \beta>0,\\ \widetilde{E}_{\alpha,0}(z)=1;\quad &\widetilde{E}_{\alpha,n}(z)=\Gamma(n)z^nE_{\alpha,n}(z),\ n\in\mathbb{N};\quad \alpha>0,\end{aligned}\tag{3.1}$$

and consider the series in these functions, respectively:

$$\sum_{n=0}^{\infty} a_n \widetilde{E}_n(z), \qquad \sum_{n=0}^{\infty} a_n \widetilde{E}_{n,\beta}(z), \qquad \sum_{n=0}^{\infty} a_n \widetilde{E}_{\alpha,n}(z), \tag{3.2}$$

with complex coefficients a_n $(n = 0, 1, 2, ...)$.

We give some previous results for series of the kind (3.2) and consider their behaviour on an arc of the unit circle $|z| = 1$, all the points of which (including the ends) are regular to the sum of the series. Such type of results are also obtained for series in other special functions, for example, for series in Laguerre and Hermite polynomials [7], and resp. by the author for systems of some other representatives of the SF of FC, which are fractional indices analogues of the Bessel functions and also multi-index Mittag-Leffler functions (in the sense of [1]) in the previous papers [4] - [5].

In the beginning we give a theorem of Cauchy-Hadamard type and a corollary for every one of the above series.

Theorem 3.1. *(of Cauchy-Hadamard type)The domain of convergence of each of the series (3.2) with complex coefficients a_n is the disk $|z| < R$ with a radius of convergence $R = 1/\Lambda$, where*

$$\Lambda = \limsup_{n\to\infty} (\ |a_n|\ )^{1/n}. \tag{3.3}$$

More precisely, the series (3.2) are absolutely convergent on the disk $|z| < R$ and divergent on the domain $|z| > R$. The cases $\Lambda = 0$ and $\Lambda = \infty$ can be included in the general case, provided $1/\Lambda$ means ∞, respectively 0.

Corollary 3.1. *Let anyone of the series (3.2) converges at the point $z_0 \neq 0$. Then it is absolutely convergent on the disk $D = \{z : |z| < |z_0|,\ z \in \mathbb{C}\}$. Inside of the disk $|z| < 1/\Lambda = R$, i.e. on each closed disk $|z| \leq r < R$ (Λ defined by (3.3)), the convergence is uniform.*

Proof. Indeed, since the considered series converges at the point $z_0 \neq 0$, its radius of convergence is the positive number R, and moreover the point z_0 lies either in the disk $|z| < R$ or on its boundary - the circle $|z| = R$. That is why, the disk D is either a part of the domain of convergence or coincide with it, whence the absolute convergence follows. To prove uniformity of the convergence inside of the disk $|z| < R$, it is sufficiently to show that the series is uniformly convergent on each closed disk $|z| \leq r < R$. To this purpose, choosing a point ζ, $|\zeta| = \rho$, $r < \rho < R$ and considering e.g. the first of the series (3.2), we estimate $|a_n \widetilde{E}_n(z)|$. First, mention that some of the values of $\widetilde{E}_n(\zeta)$, but only finite numbers of them, can be zero. Then there exists a number p such that

$$|a_n \widetilde{E}_n(z)| = |a_n \widetilde{E}_n(\zeta)| \frac{|\widetilde{E}_n(z)|}{|\widetilde{E}_n(\zeta)|} = |a_n \widetilde{E}_n(\zeta)|\ \frac{|z^n||1+\theta_n(z)|}{|\zeta^n||1+\theta_n(\zeta)|} \leq |a_n \widetilde{E}_n(\zeta)|\ \frac{|1+\theta_n(z)|}{|1+\theta_n(\zeta)|}$$

for all $n > p$ and $|z| \leq r$.

Because of (2.1) and the relations $\lim\limits_{n\to\infty} \frac{1}{n!} = 0$, $\lim\limits_{n\to\infty} (1 + \theta_n(\zeta))^{-1} = 1$, there exist numbers A and B such that $|1 + \theta_n(z)||1 + \theta_n(\zeta)|^{-1} \leq AB$ and hence $|a_n \widetilde{E}_n(z)| \leq AB|a_n \widetilde{E}_n(\zeta)|$, for all the values of $n > p$ and $|z| \leq r$. Since the series $\sum\limits_{n=0}^{\infty} a_n \widetilde{E}_n(\zeta)$ is absolutely convergent and by the criterium of comparing, the uniform convergence is proved. The proofs for the other two series go in the similar way. $\square$

The very disk of convergence is not obligatory a domain of uniform convergence and on its boundary the series may even be divergent.

Let $z_0 \in \mathbb{C}$, $0 < R < \infty$, $|z_0| = R$, g_φ be an arbitrary angular domain with size $2\varphi < \pi$ and with vertex at the point $z = z_0$, which is symmetric with respect to the straight line defined by the points 0 and z_0, and d_φ be the part of the angular domain g_φ, closed between the angle's arms and the arc of the circle with center at the point 0 and touching the arms of the angle. The following inequality can be verified inside d_φ

$$|z - z_0| \cos\varphi < 2(|z_0| - |z|). \tag{3.4}$$

Next theorem refers to the uniform convergence on the set d_φ and the convergence at the point z_0, provided $|z| < R$ and $z \in g_\varphi$.

Theorem 3.2. ***(of Abel type)*** *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers, Λ be the real number defined by (3.3), $0 < \Lambda < \infty$. Let $K = \{z : z \in \mathbb{C},\ |z| < R, R = 1/\Lambda\}$. If $f(z)$, $g(z;\beta)$, $h(z;\alpha)$ are the sums respectively of the first, second and third of the series (3.2) on the domain K, and these series converge at the point z_0 of the boundary of K, then the series (3.2) are uniformly convergent on the domain d_φ. and*

$$\lim_{z\to z_0} f(z) = \sum_{n=0}^{\infty} a_n \widetilde{E}_n(z_0), \quad \lim_{z\to z_0} g(z;\ \beta) = \sum_{n=0}^{\infty} a_n \widetilde{E}_{n,\ \beta}(z_0),$$
$$\lim_{z\to z_0} h(z;\ \alpha) = \sum_{n=0}^{\infty} a_n \widetilde{E}_{\alpha,\ n}(z_0).$$

provided $|z| < R$ and $z \in g_\varphi$.

The proofs of Theorems 3.1 and 3.2, excepting the uniformity, are given in [6].

Note 3.1. If some of the series (3.2) has a finite and non-zero radius of convergence R, it converges at the point $z_0 \in C(0,R)$ and F is the holomorphic function defined by this series in its domain of convergence, then by the Theorem 3.2. it follows that

$$\lim_{z\to z_0,\ z\in d_\varphi} F(z) = F(z_0),$$

i.e. the restriction of the function F to each set of the kind d_φ is continuous at the point z_0.

Proof. Here we consider the first of the series (3.2) whose convergence have been proved in [6]. To prove its uniform convergence we use the inequality (3.4) that is the crucial point of the proof.

So, let $z \in d_\varphi$. Setting

$$S_k(z) = \sum_{n=0}^{k} a_n \widetilde{E}_n(z), \quad S_k(z_0) = \sum_{n=0}^{k} a_n \widetilde{E}_n(z_0), \quad \lim_{k\to\infty} S_k(z_0) = s, \tag{3.5}$$
$$\beta_n = S_n(z_0) - s, \quad \beta_n - \beta_{n-1} = a_n \widetilde{E}_n(z_0),$$

we obtain

$$S_{k+p}(z) - S_k(z) = \sum_{n=0}^{k+p} a_n \widetilde{E}_n(z) - \sum_{n=0}^{k} a_n \widetilde{E}_n(z) = \sum_{n=k+1}^{k+p} a_n \widetilde{E}_n(z).$$

According to Note 2.1, there exists a natural number N_0 such that $\widetilde{E}_n(z_0) \neq 0$ when $n > N_0$. Let $k > N_0$ and $p > 0$. Then, using the denotation $\gamma_n(z; z_0) = \widetilde{E}_n(z)/\widetilde{E}_n(z_0)$, we can write the difference $S_{k+p}(z) - S_k(z)$ as follows:

$$S_{k+p}(z) - S_k(z) = \sum_{n=k+1}^{k+p} a_n \widetilde{E}_n(z_0) \frac{\widetilde{E}_n(z)}{\widetilde{E}_n(z_0)} = \sum_{n=k+1}^{k+p} a_n \widetilde{E}_n(z_0) \gamma_n(z; z_0).$$

Now, by the Abel transformation (see in [3], vol.1, ch.1, p.32, 3.4:7), we obtain consecutively:

$$S_{k+p}(z) - S_k(z) = \sum_{n=k+1}^{k+p} (\beta_n - \beta_{n-1}) \gamma_n(z; z_0)$$

$$= \beta_{k+p}\gamma_{k+p}(z) - \beta_k \gamma_{k+1}(z) - \sum_{n=k+1}^{k+p-1} \beta_n(\gamma_{n+1}(z; z_0) - \gamma_n(z; z_0)),$$

$$S_{k+p}(z) - S_k(z) = (S_{k+p}(z_0) - s)\gamma_{k+p}(z) - (S_k(z_0) - s)\gamma_{k+1}(z)$$
$$+ \sum_{n=k+1}^{k+p-1} (S_n(z_0) - s) \times \left(\frac{\widetilde{E}_n(z)}{\widetilde{E}_n(z_0)} - \frac{\widetilde{E}_{n+1}(z)}{\widetilde{E}_{n+1}(z_0)} \right).$$

So, using last relation, we are going to estimate the module of the difference $S_{k+p}(z) - S_k(z)$ as follows:

$$|S_{k+p}(z) - S_k(z)| \leq |S_{k+p}(z_0) - s||\gamma_{k+p}(z)| + |S_k(z_0) - s||\gamma_{k+1}(z)|$$
$$+ \sum_{n=k+1}^{k+p-1} |S_n(z_0) - s| \times \left| \frac{\widetilde{E}_n(z)}{\widetilde{E}_n(z_0)} - \frac{\widetilde{E}_{n+1}(z)}{\widetilde{E}_{n+1}(z_0)} \right|. \tag{3.6}$$

Because of (2.1) and the relations $\lim\limits_{n\to\infty} \frac{1}{n!} = 0$, $\lim\limits_{n\to\infty} (1 + \theta_n(z_0))^{-1} = 1$, there exist numbers A and $N_1 > N_0$ such that $|1 + \theta_n(z)| \leq A/2$ for all the natural values of n and $|1 + \theta_n(\zeta)|^{-1} < 2$ for $n > N_1$, whence

$$|\gamma_n(z, z_0)| \leq A \quad \text{for } n > N_1. \tag{3.7}$$

Further, setting

$$e_n(z, z_0) = \frac{\widetilde{E}_n(z)}{\widetilde{E}_n(z_0)} - \frac{\widetilde{E}_{n+1}(z)}{\widetilde{E}_{n+1}(z_0)} = \frac{z^n}{z_0^n} \times \left(\frac{1 + \theta_n(z)}{1 + \theta_n(z_0)} - \frac{z}{z_0} \times \frac{1 + \theta_{n+1}(z)}{1 + \theta_{n+1}(z_0)} \right)$$

and observing that $e_n(z_0, z_0) = 0$, we apply the Schwartz lemma for $e_n(z, z_0)$. So, we get that there exists a constant C:

$$|e_n(z, z_0)| = |\widetilde{E}_n(z)/\widetilde{E}_n(z_0) - \widetilde{E}_{n+1}(z)/\widetilde{E}_{n+1}(z_0)| \leq C|z - z_0||z/z_0|^n,$$

whence, and in accordance to (3.4):

$$\sum_{n=k+1}^{k+p+1} |e_n(z, z_0)| \leq \sum_{n=0}^{\infty} C|z - z_0||z/z_0|^n = C|z_0| \times \frac{|z - z_0|}{|z| - |z_0|} < \frac{2C|z_0|}{\cos\varphi}. \tag{3.8}$$

Let ε be an arbitrary positive number. Taking in view the third of the relations (3.5), we can confirm that there exists a positive number $N_2 > N_0$ so large that

$$|S_n(z_0) - s| < \min\left(\frac{\varepsilon}{3A}, \frac{\varepsilon \cos\varphi}{6C|z_0|}\right) \quad \text{for } n > N_2. \tag{3.9}$$

Now, let $N = N(\varepsilon) = \max(N_1, N_2)$ and $k > N$. Therefore (3.6) - (3.9) give

$$|S_{k+p}(z) - S_k(z)| < \frac{2\varepsilon}{3} + \frac{\varepsilon\cos\varphi}{6C|z_0|}\sum_{n=k+1}^{k+p+1}|e_n(z,z_0)| < \frac{2\varepsilon}{3} + \frac{\varepsilon\cos\varphi}{6C|z_0|}\frac{2C|z_0|}{\cos\varphi} = \varepsilon$$

that completes the proof of the theorem for the considered series. The proofs for the other two series go by analogy. □

4. Fatou type theorems

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers with $\limsup_{n\to\infty}(\,|a_n|\,)^{1/n} = R$, and $\widetilde{f}(z)$ be the sum of the power series $\sum_{n=0}^{\infty} a_n z^n$ on the open disk $U(0;R) = \{z : z \in \mathbb{C},\ |z| < R\}$, i.e.

$$\widetilde{f}(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in U(0;R).$$

Definition 4.1. *A point $z_0 \in \partial U(0;R)$ is called regular for the function $\widetilde{f}$ if there exist a neighbourhood $U(z_0;\rho)$ and a function $\widetilde{f}^*_{z_0} \in \mathcal{H}(U(z_0;\rho))$ (the space of complex-valued functions, holomorphic in the set $U(z_0;\rho)$), such that $\widetilde{f}^*_{z_0}(z) = \widetilde{f}(z)$ for $z \in U(z_0;\rho) \cap U(0;R)$.*

By this definition it follows that the set of regular points of the power series is an open subset of the circle $C(0;R) = \partial U(0;R)$ with respect to the relative topology on $\partial U(0;R)$, i.e. the topology induced by that of $\mathbb{C}$.

In general, there is no relation between the convergence (divergence) of a power series at points on the boundary of its disk of convergence and the regularity (singularity) of its sum of such points. For example, the power series $\sum_{n=0}^{\infty} z^n$ is divergent at each point of the circle $C(0;1)$ regardless of the fact that all the points of this circle, except $z = 1$, are regular for its sum. The series $\sum_{n=1}^{\infty} n^{-2} z^n$ is (absolutely) convergent at each point of the circle $C(0;1)$, but nevertheless one of them, namely $z = 1$, is a singular (i.e. not regular) for its sum. But under additional conditions on the sequence $\{a_n\}_{n=0}^{\infty}$, such a relation do exists (see for details [3], vol.1, ch. 3, §7, 7.3, p.357).

Proposition refereing to the above discussed properties holds also for series in the Laguerre and Hermite systems (see e.g.[7]). Here we give such a type of theorem for the Mittag-Leffler systems as follows.

Theorem 4.1. *(of Fatou type) Let* $\{a_n\}_{n=0}^{\infty}$ *be a sequence of complex numbers satisfying the condition* $\limsup\limits_{n\to\infty} (\,|a_n|\,)^{1/n} = 1$ *and* $f(z)$, $g(z;\beta)$, $h(z;\alpha)$ *be the sums respectively of the first, second and third of the series (3.2) on the disk* $D = \{z : z \in \mathbb{C},\ |z| < 1\}$, *i.e.*

$$f(z) = \sum_{n=0}^{\infty} a_n \widetilde{E}_n(z),\ g(z;\ \beta) = \sum_{n=0}^{\infty} a_n \widetilde{E}_{n,\,\beta}(z),\ h(z;\ \alpha) = \sum_{n=0}^{\infty} a_n \widetilde{E}_{\alpha,\,n}(z);\ z \in D.$$

Let σ *be an arbitrary arc of the unit circle* $|z| = 1$ *with all its points (including the ends) regular to the function* f *(resp.* g *or* h*). Let* $\lim\limits_{n\to\infty} a_n = 0$ *and* $\widetilde{E}_n(z) \neq 0$ *(respectively* $\widetilde{E}_{n,\,\beta}(z) \neq 0$, $\widetilde{E}_{\alpha,\,n}(z) \neq 0$*) for* $z \in \sigma$. *Then the first (resp. second or third) of the series (3.2) converges, even uniformly, on the arc* σ.

The proof will be given elsewhere.

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