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ON THE ANALYTIC REPRESENTATION OF DISTRIBUTIONS OF SEVERAL VARIABLES

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ABSTRACT. In this article we present one proof for the analytic representation of distributions of several variables. For simplicity we give detailed proof for the distributions of two variables.

1. INTRODUCTION

The basic functions in the analytic representation of distributions are the Cauchy and the Poisson kernel:

$$K(z-t) = \frac{1}{(2\pi i)^n} \prod_{j=1}^{n} \frac{1}{(t_j - z_j)}\,, \tag{1.1}$$

where $z = (z_1, , z_n)$, $z_j = x_j + iy_j$, and

$$P(z,t) = sgny \frac{(y_1 \dots y_n)}{\pi^n} \prod_{j=1}^{n} \frac{1}{|t_j - z_j|^2}\,,$$

where $y = (y_1, \dots, y_n)$ and $sgny = sgny_1 sgny_n$ respectively. If the analytic representation is defined by the Cauchy kernel, then the representation is also called Cauchy representation. For the Cauchy representation is specially adapted the spaces $O_{\alpha_1,\dots,\alpha_n}$, where $\alpha_1, \dots, \alpha_n$ are real numbers. The dual space for $O_{\alpha_1,\dots,\alpha_n}$ is denoted by $O'_{\alpha_1,\dots,\alpha_n}$. This spaces are given in [1].

It is clear, for example, that the Poisson kernel belongs to the space $O_(0, \dots, 0)$. Also if the distribution $T \in O'_{-1,\dots,-1}$ then the function

$$\hat{T}(z) = \frac{1}{(2\pi i)^n} \langle T, \prod_{j=1}^{n} \frac{1}{t_j - z_j} \rangle$$

where $z = (z_1, \dots, z_n), Imz_j \neq 0$ and $t = (t_1, \dots, t_n)$ is well defined in this domain of the n-dimensional complex space C^n. The function $\hat{T}(z)$ have the main role in the analytic

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representation of distributions of one or several variables. It is easy to prove that $\hat{T}(z)$ is analytic function of each complex variable z_j respectively, and Hartog's theorem says that this function is analytic as a function of n-complex variables in the domain $Imz_j \neq 0$ for $j = 1, , n$.

In the one-dimensional case $\hat{T}(x+iy) - \hat{T}(x-iy)$ converges in distribution sense to the distribution T, and this fact may suggest to consider the sum

$$(1.2)\quad \hat{T}(z_1,\dots,z_n)-\hat{T}(\bar{z}_1,\dots,z_n)+\dots+(-1)^k\hat{T}(z_1,\bar{z}_2,\dots,z_n)+\dots+(-1)^n\hat{T}(\bar{z}_1,\dots,\bar{z}_n)$$

where k is equal to the number of conjugates in $\hat{T}(z)$. See also the lemma in [1], p. 207. We will give a direct proof that the sum (1.2) converges to the distribution T. Another proof is given in [1]. For simplicity we consider the two-dimensional distributions.

2. Main result

Theorem 2.1. *If $T \in O'_{-1,-1}$ and if*

$$\hat{T}(z) = \frac{1}{(2\pi i)^2}\langle T, \frac{1}{(t_1-z_1)(t_2-z_2)}\rangle$$

where $z = (z_1, z_2)$, $t = (t_1, t_2)$, then

$$\hat{T}(x+i\varepsilon_1, y+i\varepsilon_2) - \hat{T}(x-i\varepsilon_1, y+i\varepsilon_2) - \hat{T}(x+i\varepsilon_1, y-i\varepsilon_2) + \hat{T}(x-i\varepsilon_1, y-i\varepsilon_2)$$

converges in distribution sense to the distribution T as $\varepsilon_1, \varepsilon_2 \to 0^+$.

Proof. Let $\varphi \in D(R^2)$ and first we consider

$$\langle \hat{T}(x+i\varepsilon_1, y+i\varepsilon_2), \varphi(x,y)\rangle = \int\int_{R^2} \hat{T}(x+i\varepsilon_1, y+i\varepsilon_2)\varphi(x,y)dxdy\,.$$

The integral on the right side is equal to the limits of the Riemann sums , i.e.

$$\lim_{n,m\to\infty}\langle T_t, \frac{1}{(2\pi i)^2}\sum_{j,k=1}^{n,m}\frac{\varphi(x_j,y_k)\triangle x_j\triangle y_k}{(t_1-x_j-i\varepsilon_1)(t_2-y_k-i\varepsilon_2)}\rangle\,.$$

Now we consider the sequence of functions $(\psi_{n,m}(t))$, where

$$(2.1)\qquad \psi_{n,m}(t) = \frac{1}{(2\pi i)^2}\sum_{j,k=1}^{n,m}\frac{\varphi(x_j,y_k)\triangle x_j\triangle y_k}{(t_1-x_j-i\varepsilon_1)(t_2-y_k-i\varepsilon_2)}\,.$$

Since the function φ has compact support there exists square $[-L, L]\times[-L, L], L > 0$ in which the support of φ is contained. Let $M = max|\varphi(x,y)|$. This implies that the terms of the sum in (2.1) are non zero only if they belong in the square. From above it follows that

$$|\psi_{n,m}(t)| \leq \frac{1}{4\pi^2}\sum_{j,k=1}^{n,m}\frac{M\triangle x_j\triangle y_k}{\varepsilon_1\varepsilon_2} = \frac{4ML^2}{4\pi^2\varepsilon_1\varepsilon_2}\,,$$

for any partition of R^2 with elementary squares.

Thus we conclude that the sequence $(\psi_{n,m}(t))$ is uniformly bounded. Now we will prove that the sequence is uniform continuous on every compact set. Let (t'_1, t'_2) and (t''_1, t''_2) are in R^2 and consider

$$|\psi_{n,m}(t'_1, t'_2) - \psi_{n,m}(t''_1, t''_2)|\ ,$$

It is easy to see that $(\psi_{n,m}(t))$ is uniformly is continuous, since for fixed (t_1, t_2) the sequence $\psi_{n,m}(t_1, t_2)$ converge to the integral

$$\frac{1}{(2\pi i)^2} \int\int_{R^2} \frac{\varphi(x,y)dxdy}{(t_1 - x - i\varepsilon_1)(t_2 - y - i\varepsilon_2)} \tag{2.2}$$

and the Arzela-Ascoli theorem says that the convergence is in the space $O_{-1,-1}$. We have

$$\langle \hat{T}(x + i\varepsilon_1, y + i\varepsilon_2), \varphi(x,y)\rangle = \langle T, \frac{1}{(2\pi i)^2} \int\int_{R^2} \frac{\varphi(x,y)dxdy}{(t_1 - x - i\varepsilon_1)(t_2 - y - i\varepsilon_2)}\rangle \tag{2.3}$$

Similarly we have:

$$\langle -\hat{T}(x + i\varepsilon_1, y - i\varepsilon_2), \varphi(x,y)\rangle = \langle T, \frac{-1}{(2\pi i)^2} \int\int_{R^2} \frac{\varphi(x,y)dxdy}{(t_1 - x - i\varepsilon_1)(t_2 - y + i\varepsilon_2)}\rangle\,, \tag{2.4}$$

$$\langle -\hat{T}(x - i\varepsilon_1, y + i\varepsilon_2), \varphi(x,y)\rangle = \langle T, \frac{-1}{(2\pi i)^2} \int\int_{R^2} \frac{\varphi(x,y)dxdy}{(t_1 - x + i\varepsilon_1)(t_2 - y - i\varepsilon_2)}\rangle\,, \tag{2.5}$$

$$\langle \hat{T}(x - i\varepsilon_1, y - i\varepsilon_2), \varphi(x,y)\rangle = \langle T, \frac{1}{(2\pi i)^2} \int\int_{R^2} \frac{\varphi(x,y)dxdy}{(t_1 - x + i\varepsilon_1)(t_2 - y + i\varepsilon_2)}\rangle\,. \tag{2.6}$$

By adding (2.3), (2.4), (2.5) and (2.6) we obtain:

$$\begin{aligned}
&\langle \hat{T}(x + i\varepsilon_1, y + i\varepsilon_2) - \hat{T}(x - i\varepsilon_1, y + i\varepsilon_2) - \hat{T}(x + i\varepsilon_1, y - i\varepsilon_2) + \hat{T}(x - i\varepsilon_1, y - i\varepsilon_2)\rangle = \\
&= \langle T, \frac{1}{(2\pi i)^2} \int\int_{R^2} \varphi(x,y) \left[\frac{1}{(t_1 - x - i\varepsilon_1)(t_2 - y - i\varepsilon_2)} - \frac{1}{(t_1 - x - i\varepsilon_1)(t_2 - y + i\varepsilon_2)} - \right. \\
&\qquad \left. - \frac{1}{(t_1 - x + i\varepsilon_1)(t_2 - y - i\varepsilon_2)} + \frac{1}{(t_1 - x + i\varepsilon_1)(t_2 - y + i\varepsilon_2)}\right] dxdy\rangle\,.
\end{aligned}$$

Now the right side we write in the form :

$$\langle T, \frac{1}{(2\pi i)^2} \int\int_{R^2} \frac{(2i)^2\varepsilon_1\varepsilon_2\varphi(x,y)dxdy}{[(t_1 - x)^2 + \varepsilon_1^2][(t_2 - y)^2 + \varepsilon_2^2]}\rangle\,.$$

Since φ has compact support the integral is

$$I = \langle T, \frac{1}{\pi^2} \int_{-a}^{a}\int_{-a}^{a} \frac{\varepsilon_1\varepsilon_2\varphi(x,y)dxdy}{[(t_1 - x)^2 + \varepsilon_1^2][(t_2 - y)^2 + \varepsilon_2^2]}\rangle\,, \tag{2.7}$$

for some $a > 0$.

The integral in (2.7) is equal to:

$$\int_{-a}^{a}\int_{-a}^{a} \frac{\varepsilon_1\varepsilon_2[\varphi(x,y) - \varphi(t_1,t_2)]dxdy}{[(t_1 - x)^2 + \varepsilon_1^2][(t_2 - y)^2 + \varepsilon_2^2]} + \varphi(t_1,t_2)\int_{-a}^{a}\int_{-a}^{a} \frac{\varepsilon_1\varepsilon_2 dxdy}{[(t_1 - x)^2 + \varepsilon_1^2][(t_2 - y)^2 + \varepsilon_2^2]}\,.$$

Now we consider the integrals I_1 and I_2 where

$$I_1 = \int_{-a}^{a}\int_{-a}^{a} \frac{\varepsilon_1\varepsilon_2[\varphi(x,y) - \varphi(t_1,t_2)]dxdy}{[(t_1 - x)^2 + \varepsilon_1^2][(t_2 - y)^2 + \varepsilon_2^2]}\,,$$

and

$$I_2 = \int_{-a}^{a}\int_{-a}^{a} \frac{\varepsilon_1\varepsilon_2 dxdy}{[(t_1 - x)^2 + \varepsilon_1^2][(t_2 - y)^2 + \varepsilon_2^2]}\,.$$

By the mean value theorem we have

$$\begin{aligned} I_1 &= \int_{-a}^{a}\int_{-a}^{a} \frac{\varepsilon_1\varepsilon_2(x-t_1)\frac{\partial\varphi}{\partial x}[t_1+\theta(x-t_1),t_2+\theta(y-t_2)]dxdy}{[(t_1-x)^2+\varepsilon_1^2][(t_2-y)^2+\varepsilon_2^2]} + \\ &+ \int_{-a}^{a}\int_{-a}^{a} \frac{\varepsilon_1\varepsilon_2(y-t_2)\frac{\partial\varphi}{\partial x}[t_1+\theta(x-t_1),t_2+\theta(y-t_2)]dxdy}{[(t_1-x)^2+\varepsilon_1^2][(t_2-y)^2+\varepsilon_2^2]}, \end{aligned}$$

where $0<\theta<1$.

Further we consider two parts of I_1 namely:

$$I_{1,1} = \int_{-a}^{a}\int_{-a}^{a} \frac{\varepsilon_1\varepsilon_2(x-t_1)\frac{\partial\varphi}{\partial x}[t_1+\theta(x-t_1),t_2+\theta(y-t_2)]dxdy}{[(t_1-x)^2+\varepsilon_1^2][(t_2-y)^2+\varepsilon_2^2]},$$

and

$$I_{1,2} = \int_{-a}^{a}\int_{-a}^{a} \frac{\varepsilon_1\varepsilon_2(y-t_2)\frac{\partial\varphi}{\partial x}[t_1+\theta(x-t_1),t_2+\theta(y-t_2)]dxdy}{[(t_1-x)^2+\varepsilon_1^2][(t_2-y)^2+\varepsilon_2^2]}.$$

We use the Taylor formula with respect to x in t_1 and we have

$$\frac{\partial\varphi}{\partial x}[t_1+\theta(x-t_1),t_2+\theta(y-t_2)] =$$

$$= \frac{\partial\varphi}{\partial x}[t_1,t_2+\theta(y-t_2)] + (x-t_1)\frac{\partial^2\varphi}{\partial x^2}[t_1+\theta(c-t_1),t_2+\theta(y-t_2)]\theta.$$

The first term of $I_{1,1}$ is

$$A_1 = \int_{-a}^{a} \frac{\varepsilon_1(x-t_1)}{(t_1-x)^2+\varepsilon_1^2}dx \int_{-a}^{a} \frac{\frac{\partial\varphi}{\partial x}[t_1,t_2+\theta(y-t_2)]\varepsilon_2}{(t_2-y)^2+\varepsilon_2^2}dy.$$

Put

$$\left|\frac{\partial^2\varphi}{\partial x^2}\right|, \left|\frac{\partial^2\varphi}{\partial y^2}\right| \le M,$$

then we have

$$\begin{aligned} |A_1| &\le M\left|\int_{-a}^{a} \frac{\varepsilon_1(x-t_1)}{(t_1-x)^2+\varepsilon_1^2}dx\right|\left|\int_{-a}^{a} \frac{\varepsilon_2}{(t_2-y)^2+\varepsilon_2^2}dy\right| = \\ &= \frac{1}{2}\varepsilon_1|\ln[(t_1-x)^2+\varepsilon_1^2]M\arctan\frac{y-t_2}{\varepsilon_2}. \end{aligned}$$

From that it is easy to see that $A_1\to 0$ as $\varepsilon_1,\varepsilon_2\to 0$. The second part

$$A_2 = \int_{-a}^{a}\int_{-a}^{a} \frac{\varepsilon_1\varepsilon_2(x-t_1)^2\frac{\partial^2\varphi}{\partial x^2}[t_1+\theta(x-t_1),t_2+\theta(y-t_2)]\theta dxdy}{[(t_1-x)^2+\varepsilon_1^2][(t_2-y)^2+\varepsilon_2^2]}.$$

It is obvious that

$$|A_2| \le \int_{-a}^{a}\int_{-a}^{a} \frac{\varepsilon_1\varepsilon_2 M}{(t_2-y)^2+\varepsilon_y^2}dxdy,$$

and $A_2\to 0$ as $\varepsilon_1,\varepsilon_2\to 0$.

Similarly we can conclude that $I_{1,2}\to 0$. Now we consider the integral I_2.

$$I_2 = \varphi(t_1,t_2)\int_{-a}^{a}\int_{-a}^{a} \frac{\varepsilon_1\varepsilon_2 dxdy}{[(t_1-x)^2+\varepsilon_1^2][(t_2-y)^2+\varepsilon_2^2]} = \varphi(t_1,t_2)\arctan\frac{x-t_1}{\varepsilon_1}\arctan\frac{y-t_2}{\varepsilon_2}.$$

If $t=(t_1,t_2)$ belongs to a compact set, then we can choose a such that $a-t_1>0, a-t_2>0$ and $-a-t_1<0, -a-t_2<0$. In this case if $\varepsilon_1,\varepsilon_2\to 0$ then the last expression tends to $\varepsilon(t_1,t_2)\pi^2$. In this way we have proved that the integrals in (2.7) converge to the function $\varepsilon(t_1, t_2)$ in the space $O_{-1,-1}$. With this we proved that the assertion in (1.2) is true. Of course this method may apply in general case when we have distributions of n variables.

$\square$

Example 2.1. *If δ is the Dirac distribution in the space $D(R^2)$, then*

$$\hat{\delta}(z_1, z_2) = \langle \delta_{(t_1,t_2)}, \frac{1}{(2\pi i)^2(t_1 - z_1)(t_2 - z_2)} \rangle = \frac{1}{(2\pi i)^2 z_1 z_2}.$$

Here we have four functions

$$\frac{1}{(2\pi i)^2(x_1 + iy_1)(x_2 + iy_2)}, \frac{-1}{(2\pi i)^2(x_1 - iy_1)(x_2 + iy_2)},$$
$$\frac{1}{(2\pi i)^2(x_1 + iy_1)(x_2 - iy_2)}, \frac{-1}{(2\pi i)^2(x_1 - iy_1)(x_2 - iy_2)},$$

where $y_1, y_2 > 0$. Evidently that $\hat{\delta}(z_1, z_2)$ is not analytic if $z_1 = 0$ or $z_2 = 0$, but the support of δ is the point (0,0). This fact is different from the corresponding of one-dimensional δ. The different is due to the properties of the analytic functions of several complex variables see ([1] *p. 208).*

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