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AN EXTENSION OPERATOR AND LOEWNER CHAINS ON SOME REINHARDT DOMAINS IN $\mathbb{C}^n$

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Abstract. In this paper we are concerned with an extension operator $\Phi_{n,p,\alpha}$, $\alpha \geq 0$, $p \geq 1$, that provides a way of extending a locally biholomorphic mapping $f \in H(B^n)$ to a locally biholomorphic mapping $F \in H(\Omega_{n,p})$, where $\Omega_{n,p} = \{z = (z', z_{n+1}) \in \mathbb{C}^{n+1} : \|z'\|^2 + |z_{n+1}|^p < 1\}$. By using the method of Loewner chains, we prove that if $f \in S^0(B^n)$, then $\Phi_{n,p,\alpha}(f) \in S^0(\Omega_{n,p})$, for $p \geq \max\{2n\alpha, 1\}$ and $\alpha \in [0, 1/(n+1)]$. In particular, if $f \in S^*(B^n)$, then $\Phi_{n,p,\alpha}(f) \in S^*(\Omega_{n,p})$ and if f is spirallike of type $\beta \in (-\pi/2, \pi/2)$ on B^n, then $\Phi_{n,p,\alpha}(f)$ is also spirallike of type β on $\Omega_{n,p}$. Finally, we consider the preservation of ε-starlikeness under the operator $\Phi_{n,p,\alpha}$.

1. Introduction and preliminaries

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \dots, z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w}_j$ and the Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. The open ball $\{z \in \mathbb{C}^n : \|z\| < r\}$ is denoted by B_r^n and the unit ball B_1^n is denoted by B^n. In the case of one complex variable, B^1 is denoted by U.

Let $L(\mathbb{C}^n, \mathbb{C}^m)$ denote the space of linear continuous operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm, and let I_n be the identity of $L(\mathbb{C}^n, \mathbb{C}^n)$. If Ω is a domain in $\mathbb{C}^n$, we denote by $H(\Omega)$ the set of holomorphic mappings from Ω into $\mathbb{C}^n$. If $0 \in \Omega$, such a mapping f is said to be normalized if $f(0) = 0$ and $Df(0) = I_n$.

From now on, we assume that Ω is a domain in $\mathbb{C}^n$ that contains the origin. We say that $f \in H(\Omega)$ is locally biholomorphic on Ω if the complex Jacobian matrix $Df(z)$ is nonsingular at each $z \in \Omega$. Let $J_f(z) = \det Df(z)$. Let $\mathcal{L}S_n(\Omega)$ be the set of normalized locally biholomorphic mappings on Ω and let $S(\Omega)$ be the set of normalized biholomorphic mappings on Ω. A map $f \in S(\Omega)$ is said to be convex if its image is a convex domain in $\mathbb{C}^n$, and starlike if the image is a starlike domain with respect to the origin. We denote the classes of normalized convex and starlike mappings on Ω respectively by $K(\Omega)$ and $S^*(\Omega)$.

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In one variable we write $\mathcal{L}S_1(B^1) = \mathcal{L}S$, $S(B^1) = S$, $K(B^1) = K$ and $S^*(B^1) = S^*$. A mapping $f \in S(\Omega)$ is spirallike of type $\beta \in (-\pi/2, \pi/2)$ if the spiral $\exp(-e^{-i\beta}\tau)f(z)$ $(\tau \geq 0)$ is contained in $f(\Omega)$ for any $z \in \Omega$. We denote by $\hat{S}_\beta(\Omega)$ the class of normalized spirallike mappings of type β on Ω.

We next present the notion of ε-starlikeness due to Gong and Liu (see [3]). This notion interpolates between starlikeness and convexity as ε ranges from 0 to 1.

Definition 1.1. *Let $f : \Omega \to \mathbb{C}^n$ be a biholomorphic mapping such that $f(0) = 0$. We say that f is ε-starlike, $0 \leq \varepsilon \leq 1$, if $f(\Omega)$ is starlike with respect to each point in $\varepsilon f(\Omega)$, i.e.*

$$(1 - \lambda)f(z) + \lambda\varepsilon f(w) \in f(\Omega),\ \ \lambda \in [0, 1],\ \ z, w \in \Omega.$$

When $\varepsilon = 0$ we obtain the family of starlike mappings on Ω, and when $\varepsilon = 1$ we obtain the family of convex mappings on Ω. The analytical characterization of ε-starlikeness was given in [4].

For $n \geq 1$, set $z' = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $z = (z', z_{n+1}) \in \mathbb{C}^{n+1}$. Let $p \geq 1$ and

$$\Omega_{n,p} = \{z = (z', z_{n+1}) \in \mathbb{C}^{n+1} : \|z'\|^2 + |z_{n+1}|^p < 1\}.$$

The complex ellipsoid $\Omega_{n,p}$ is a balanced convex domain in $\mathbb{C}^{n+1}$ (see e.g. [10]) and $B^n \times \{0\} \subset \Omega_{n,p} \subset B^n \times U$.

We next refer to the notions of subordination and Loewner chains.

Let $f, g \in H(\Omega_{n,p})$. We say that f is subordinate to g (and write $f \prec g$) if there is a Schwarz mapping v (i.e. $v \in H(\Omega_{n,p})$ and $\|v(z)\| \leq \|z\|$, $z \in \Omega_{n,p}$) such that $f(z) = g(v(z))$, $z \in \Omega_{n,p}$.

Definition 1.2. *A mapping $f : \Omega_{n,p} \times [0, \infty) \to \mathbb{C}^n$ is called a Loewner chain if $f(\cdot, t)$ is biholomorphic on $\Omega_{n,p}$, $f(0, t) = 0$, $Df(0, t) = e^t I_n$ for $t \geq 0$, and $f(z, s) \prec f(z, t)$ whenever $0 \leq s \leq t < \infty$ and $z \in \Omega_{n,p}$.*

The requirement $f(z, s) \prec f(z, t)$ is equivalent to the condition that there is a unique biholomorphic Schwarz mapping $v = v(z, s, t)$ called the transition mapping associated to $f(z, t)$ such that

$$f(z, s) = f(v(z, s, t), t),\ \ z \in \Omega_{n,p},\ \ t \geq s \geq 0.$$

Various results concerning Loewner chains can be found in [6].

Remark 1.1. *Certain subclasses of $S(\Omega_{n,p})$ can be characterized in terms of Loewner chains. In particular, f is starlike if and only if $f(z, t) = e^t f(z)$ is a Loewner chain. Also, f is spirallike of type β if and only if $f(z, t) = e^{(1-ia)t} f(e^{iat} z)$ is a Loewner chain, where $a = \tan\beta$ (see e.g. [9]).*

The notion of parametric representation is related to that of a Loewner chain (see e.g. [9]).

Definition 1.3. *A normalized mapping $f \in H(\Omega_{n,p})$ has parametric representation if there exists a Loewner chain $f(z, t)$ such that $\{e^{-t} f(\cdot, t)\}_{t \geq 0}$ is a normal family on $\Omega_{n,p}$ and $f(z) = f(z, 0)$, $z \in \Omega_{n,p}$.*

Let $S^0(\Omega_{n,p})$ be the set of mappings which have parametric representation on $\Omega_{n,p}$.

A key role in our discussion is played by the following Schwarz-type lemma for the Jacobian determinant of a holomorphic mapping from B^n into B^n (see [13]):

Lemma 1.1. *Let $\psi \in H(B^n)$ be such that $\psi(B^n) \subseteq B^n$. Then*

$$|J_\psi(z)| \leq \left[\frac{1-\|\psi(z)\|^2}{1-\|z\|^2}\right]^{\frac{n+1}{2}}, \ z \in B^n. \tag{1.1}$$

Definition 1.4. *Let $\alpha \geq 0$. The extension operator $\Phi_{n,p,\alpha} : \mathcal{L}S_n(B^n) \to \mathcal{L}S_{n+1}(\Omega_{n,p})$ is defined by*

$$\Phi_{n,p,\alpha}(f)(z) = (f(z'), z_{n+1}[J_f(z')]^{\frac{2\alpha}{p}}), \ z = (z', z_{n+1}) \in \Omega_{n,p}.$$

We choose the branch of the power function such that $[J_f(z')]^{\frac{2\alpha}{p}}\big|_{z'=0} = 1$.

If $\alpha = 1/(n+1)$, the operator $\Phi_{n,p,1/(n+1)}$ was studied in [9]. In the case $p = 2$, $\Omega_{n,2} = B^{n+1}$ and the operator $\Phi_{n,2,\alpha}$ was recently studied in [1]. If $p = 2$ and $\alpha = 1/(n+1)$, the operator $\Phi_{n,2,1/(n+1)}$ reduces to the Pfaltzgraff-Suffridge extension operator (see [11]). If $n = 1$, $p = 2$ and $\alpha = 1/2$, then $\Phi_{1,2,1/2}$ reduces to the well-known Roper-Suffridge extension operator. For $n \geq 2$, the Roper-Suffridge extension operator $\Psi_n : \mathcal{L}S \to \mathcal{L}S_n(B^n)$ is defined by (see [12])

$$\Psi_n(f)(z) = (f(z_1), \tilde{z}\sqrt{f'(z_1)}), \ z = (z_1, \tilde{z}) \in B^n.$$

We choose the branch of the power function such that $\sqrt{f'(z_1)}\big|_{z_1=0} = 1$.

Roper and Suffridge proved that if f is convex on U then $\Psi_n(f)$ is also convex on B^n. Graham and Kohr proved that if f is starlike on U then so is $\Psi_n(f)$ on B^n. Graham, Kohr and Kohr [7] proved that if f has parametric representation on the unit disc, then $\Psi_n(f)$ has the same property on B^n. A general class of extension operators was studied in [2], [5]. For other extension operators, see [6] and the references therein.

In this paper we prove that if $f \in S^0(B^n)$, then $\Phi_{n,p,\alpha}(f) \in S^0(\Omega_{n,p})$, for $p \geq \max\{2n\alpha, 1\}$ and $\alpha \in [0, 1/(n+1)]$. In particular, we obtain various consequences related to the preservation of starlikeness and spirallikeness of type β under $\Phi_{n,p,\alpha}$. Finally, we consider the preservation of ε-starlikeness under the operator $\Phi_{n,p,\alpha}$.

2. Parametric representation and the operator $\Phi_{n,p,\alpha}$

We begin this section with the following main result. If $\alpha = 1/(n+1)$, see [9] and if $p = 2$, see [1] (see also [8] for $\alpha = 1/(n+1)$ and $p = 2$).

Theorem 2.1. *Assume $f \in S^0(B^n)$, $p \geq \max\{2n\alpha, 1\}$ and $\alpha \in [0, 1/(n+1)]$. Then $\Phi_{n,p,\alpha}(f) \in S^0(\Omega_{n,p})$.*

Proof. Since $f \in S^0(B^n)$ there exists a Loewner chain $f(z', t)$ such that $\{e^{-t}f(z', t)\}_{t\geq 0}$ is a normal family on B^n and $f(z') = f(z', 0)$ for $z' \in B^n$. Let $F = \Phi_{n,p,\alpha}(f)$. Then it is easy to see that $F \in S(\Omega_{n,p})$. Let $v = v(z', s, t)$ be the transition mapping associated to $f(z', t)$. Then

$$f(z', s) = f(v(z', s, t), t), \ z' \in B^n, \ 0 \leq s \leq t < \infty. \tag{2.1}$$

Let $f_t(z') = f(z', t)$ for $z' \in B^n$ and $t \geq 0$, and let $v_{s,t}(z') = v(z', s, t)$, $z' \in B^n$, $t \geq s \geq 0$. Also let $F : \Omega_{n,p} \times [0, \infty) \to \mathbb{C}^{n+1}$ be given by

$$F(z, t) = (f(z', t), z_{n+1}e^{t\left(1-\frac{2n\alpha}{p}\right)}[J_{f_t}(z')]^{\frac{2\alpha}{p}}) \tag{2.2}$$

for $z = (z', z_{n+1}) \in \Omega_{n,p}$ and $t \geq 0$. We choose the branch of the power function such that $[J_{f_t}(z')]^{\frac{2\alpha}{p}}\big|_{z'=0} = e^{\frac{2n\alpha t}{p}}$.

Let us prove that $F(z,t)$ is a Loewner chain. Indeed, since $f(\cdot,t)$ is biholomorphic on B^n, $f(0,t) = 0$ and $Df(0,t) = e^t I_n$, it is not difficult to see that $F(\cdot,t)$ is biholomorphic on $\Omega_{n,p}$, $F(0,t) = 0$ and $DF(0,t) = e^t I_{n+1}$.

Let $V_{s,t} : \Omega_{n,p} \to \mathbb{C}^{n+1}$ be given by $V_{s,t}(z) = V(z,s,t)$ where

$$V(z,s,t) = (v(z',s,t), z_{n+1} e^{(s-t)\left(1-\frac{2n\alpha}{p}\right)}[J_{v_{s,t}}(z')]^{\frac{2\alpha}{p}})$$

for $z = (z', z_{n+1}) \in \Omega_{n,p}$ and $t \geq s \geq 0$. We choose the branch of the power function such that $[J_{v_{s,t}}(z')]^{\frac{2\alpha}{p}}\big|_{z'=0} = e^{\frac{2n\alpha(s-t)}{p}}$. Then $V_{s,t}$ is biholomorphic on $\Omega_{n,p}$, $V_{s,t}(0) = 0$, $DV_{s,t}(0) = e^{s-t} I_{n+1}$ and $V_{s,t}(\Omega_{n,p}) \subset \Omega_{n,p}$. Indeed, fix $z \in \Omega_{n,p}$ and let $w = V_{s,t}(z)$. We have to prove that $\|w'\|^2 + |w_{n+1}|^p < 1$. By Lemma 1.1 and the fact that $p \geq \max\{2n\alpha, 1\}$, $\alpha \in [0, 1/(n+1)]$, we obtain

$$\begin{aligned}
\|w'\|^2 + |w_{n+1}|^p &= \|v_{s,t}(z')\|^2 + e^{(s-t)(p-2n\alpha)}|z_{n+1}|^p |J_{v_{s,t}}(z')|^{2\alpha} \\
&\leq \|v_{s,t}(z')\|^2 + |z_{n+1}|^p \left[\frac{1-\|v_{s,t}(z')\|^2}{1-\|z'\|^2}\right]^{(n+1)\alpha} \\
&\leq \|v_{s,t}(z')\|^2 + \frac{|z_{n+1}|^p}{1-\|z'\|^2}(1-\|v_{s,t}(z')\|^2) \\
&< \|v_{s,t}(z')\|^2 + 1 - \|v_{s,t}(z')\|^2 = 1,
\end{aligned}$$

for $z = (z', z_{n+1}) \in \Omega_{n,p}$ and $t \geq s \geq 0$. Hence $w = (w', w_{n+1}) \in \Omega_{n,p}$ and thus $V_{s,t}(\Omega_{n,p}) \subset \Omega_{n,p}$, as claimed.

Further, taking into account (2.1), we easily deduce that $F(z,s) = F(V(z,s,t),t)$ for $z \in \Omega_{n,p}$ and $t \geq s \geq 0$. Indeed,

$$\begin{aligned}
F(V(z,s,t),t) &= (f(v(z',s,t),t), z_{n+1} e^{(s-t)\left(1-\frac{2n\alpha}{p}\right)} e^{t\left(1-\frac{2n\alpha}{p}\right)}[J_{v_{s,t}}(z')]^{\frac{2\alpha}{p}}[J_{f_t}(v_{s,t}(z'))]^{\frac{2\alpha}{p}}) \\
&= (f(z',s), z_{n+1} e^{s\left(1-\frac{2n\alpha}{p}\right)}[J_{f_s}(z')]^{\frac{2\alpha}{p}}) = F(z,s),
\end{aligned}$$

for all $z \in \Omega_{n,p}$ and $t \geq s \geq 0$. Here we have used (2.1) and the fact that

$$J_{f_s}(z') = J_{f_t}(v_{s,t}(z'))J_{v_{s,t}}(z'),\ z' \in B^n,\ t \geq s \geq 0.$$

Therefore we have proved that $F(z,t)$ is a Loewner chain.

It remains to prove that $\{e^{-t}F(z,t)\}_{t\geq 0}$ is a normal family. Since $\{e^{-t}f(z',t)\}_{t\geq 0}$ is a normal family on B^n, there exists a sequence $\{t_k\}_{k\in\mathbb{N}}$ such that $0 < t_k \to \infty$ and $e^{-t_k}f(z',t_k) \to g(z')$ locally uniformly on B^n as $k \to \infty$, where g is a holomorphic mapping on B^n. Since $g(0) = 0$ and $Dg(0) = I_n$, we deduce in view of Hurwitz's theorem for holomorphic mappings that $g \in S(B^n)$. Further, applying Vitali's theorem in several complex variables, we deduce that $\Phi_{n,p,\alpha}(e^{-t_k}f(z,t_k)) = e^{-t_k}F(z,t_k) \to \Phi_{n,p,\alpha}(g)$ locally uniformly on $\Omega_{n,p}$ as $k \to \infty$. This completes the proof. $\square$

Taking into account Theorem 2.1 and Remark 1.1, we may prove that the operator $\Phi_{n,p,\alpha}$ preserves the notions of starlikeness and spirallikeness of type β. We omit the proofs of the next corollaries (see [1] in the case $p = 2$). Corollary 2.1 was obtained in [9] in the case $\alpha = 1/(n+1)$ (see also [8] for $\alpha = 1/(n+1)$ and $p = 2$).

Corollary 2.1. *Assume* $f \in S^*(B^n)$, $p \geq \max\{2n\alpha, 1\}$, $\alpha \in [0, 1/(n+1)]$. *Then* $F = \Phi_{n,p,\alpha}(f) \in S^*(\Omega_{n,p})$.

Corollary 2.2. *Assume* $f \in \hat{S}_\beta(B^n)$, *where* $\beta \in (-\pi/2, \pi/2)$. *Then* $F = \Phi_{n,p,\alpha}(f) \in \hat{S}_\beta(\Omega_{n,p})$ *for* $p \geq \max\{2n\alpha, 1\}$ *and* $\alpha \in [0, 1/(n+1)]$.

3. ε-STARLIKENESS AND THE OPERATOR $\Phi_{n,p,\alpha}$

We next discuss the case of ε-starlike mappings associated with the operator $\Phi_{n,p,\alpha}$, for $\alpha \in \left[\frac{1}{n+1}, \frac{p}{2n}\right]$. To this end, for $a \in (0, 1]$, let

$$\Omega_{a,n,p,\alpha} = \{z = (z', z_{n+1}) \in \mathbb{C}^{n+1} : |z_{n+1}|^p < a^{2n\alpha}(1 - \|z'\|^2)^{(n+1)\alpha}\}.$$

Then $B^n \times \{0\} \subset \Omega_{a,n,p,\alpha} \subseteq \Omega_{n,p}$. For $a = 1$ and $\alpha = \frac{1}{n+1}$, we obtain $\Omega_{1,n,p,\frac{1}{n+1}} = \Omega_{n,p}$.

We have obtained the following result regarding ε-starlikeness, which when $\varepsilon = 1$ gives a partial answer to the question of whether $\Phi_{n,p,\alpha}$ preserves convexity. If $p = 2$, see [1].

Theorem 3.1. *Let* $\varepsilon \in [0, 1]$ *and* $f : B^n \to \mathbb{C}^n$ *be a normalized* ε*-starlike mapping. Also let* $F = \Phi_{n,p,\alpha}(f)$, *where* $\alpha \in \left[\frac{1}{n+1}, \frac{p}{2n}\right]$ *and let* $a_1, a_2 > 0$ *be such that* $a_1 + a_2 \leq 1$. *Then*

$$(1-\lambda)F(z) + \lambda\varepsilon F(w) \in F(\Omega_{a_1+a_2,n,p,\alpha}), \ z \in \Omega_{a_1,n,p,\alpha}, \ w \in \Omega_{a_2,n,p,\alpha}, \ \lambda \in [0, 1].$$

Proof. Since f is biholomorphic on B^n, it follows that $F = \Phi_{n,p,\alpha}(f)$ is also biholomorphic on $\Omega_{n,p}$. Fix $\lambda \in [0, 1]$ and let $z \in \Omega_{a_1,n,p,\alpha}$, $w \in \Omega_{a_2,n,p,\alpha}$. We want to find a point $u = (u', u_{n+1}) \in \Omega_{a_1+a_2,n,p,\alpha}$ such that

$$(1-\lambda)F(z) + \lambda\varepsilon F(w) = F(u),$$

i.e. $f(u') = (1-\lambda)f(z') + \lambda\varepsilon f(w')$ and

$$u_{n+1}[J_f(u')]^{\frac{2\alpha}{p}} = (1-\lambda)z_{n+1}[J_f(z')]^{\frac{2\alpha}{p}} + \lambda\varepsilon w_{n+1}[J_f(w')]^{\frac{2\alpha}{p}}.$$

If $\lambda = 0$, let $u = z$. If $\lambda = 1$, then using the fact that f is ε-starlike and the equality $\varepsilon F(w) = F(u)$, we easily deduce that $u = (u', u_{n+1}) \in \Omega_{a_2,n,p,\alpha} \subseteq \Omega_{a_1+a_2,n,p,\alpha}$. Hence, it suffices to assume that $\lambda \in (0, 1)$. Since f is ε-starlike, we obtain that

$$u' = f^{-1}((1-\lambda)f(z') + \lambda\varepsilon f(w')).$$

Then $u' = u'(z', w')$ can be viewed as a mapping from $B^n \times B^n$ into B^n. Let

$$u_{n+1} = (1-\lambda)z_{n+1}\left[\frac{J_f(z')}{J_f(u')}\right]^{\frac{2\alpha}{p}} + \lambda\varepsilon w_{n+1}\left[\frac{J_f(w')}{J_f(u')}\right]^{\frac{2\alpha}{p}}.$$

We prove that $u = (u', u_{n+1}) \in \Omega_{a_1+a_2,n,p,\alpha}$. It is obvious that

$$\frac{\partial u'}{\partial z'} = (1-\lambda)[Df(u')]^{-1}Df(z') \text{ and } \frac{\partial u'}{\partial w'} = \lambda\varepsilon[Df(u')]^{-1}Df(w').$$

Hence

$$u_{n+1} = (1-\lambda)^{\frac{p-2n\alpha}{p}} z_{n+1}[J_{u'_{z'}}]^{\frac{2\alpha}{p}} + (\lambda\varepsilon)^{\frac{p-2n\alpha}{p}} w_{n+1}[J_{u'_{w'}}]^{\frac{2\alpha}{p}}.$$

Next, let $1/\tilde{p} = 2n\alpha/p$. Since $p \geq 2n\alpha$, it follows that $\tilde{p} \geq 1$. Also let $1/\tilde{q} = 1 - 1/\tilde{p}$. Using Lemma 1.1 in the previous equation, we obtain

$$\begin{aligned} &|u_{n+1}| \\ &\leq (1-\lambda)^{1/\tilde{q}}|z_{n+1}|\left[\frac{1-\|u'(z',w')\|^2}{1-\|z'\|^2}\right]^{\frac{(n+1)\alpha}{p}} + (\lambda\varepsilon)^{1/\tilde{q}}|w_{n+1}|\left[\frac{1-\|u'(z',w')\|^2}{1-\|w'\|^2}\right]^{\frac{(n+1)\alpha}{p}} \\ &= (1-\|u'\|^2)^{\frac{(n+1)\alpha}{p}}\left\{(1-\lambda)^{1/\tilde{q}}\left[\frac{|z_{n+1}|^{\frac{p}{(n+1)\alpha}}}{1-\|z'\|^2}\right]^{\frac{(n+1)\alpha}{p}} + (\lambda\varepsilon)^{1/\tilde{q}}\left[\frac{|w_{n+1}|^{\frac{p}{(n+1)\alpha}}}{1-\|w'\|^2}\right]^{\frac{(n+1)\alpha}{p}}\right\}. \end{aligned}$$

We have two cases:

First case (compare with Corollary 2.1). If $\varepsilon = 0$ (i.e. f is starlike), then we obtain that

$$|u_{n+1}| \leq (1-\|u'\|^2)^{\frac{(n+1)\alpha}{p}}(1-\lambda)^{1/\tilde{q}}\frac{|z_{n+1}|}{(1-\|z'\|^2)^{\frac{(n+1)\alpha}{p}}} < a_1^{\frac{2n\alpha}{p}}(1-\|u'\|^2)^{\frac{(n+1)\alpha}{p}}.$$

Here we have used the fact that $z = (z', z_{n+1}) \in \Omega_{a_1,n,p,\alpha}$. Hence $|u_{n+1}|^p < a_1^{2n\alpha}(1 - \|u'\|^2)^{(n+1)\alpha}$, i.e. $u = (u', u_{n+1}) \in \Omega_{a_1,n,p,\alpha}$. On the other hand, since $\Omega_{a_1,n,p,\alpha} \subseteq \Omega_{a_1+a_2,n,p,\alpha}$, we deduce that $u = (u', u_{n+1}) \in \Omega_{a_1+a_2,n,p,\alpha}$, as desired.

Second case. For $\varepsilon \in (0, 1]$, using Hölder's inequality we obtain

$$\begin{aligned} &|u_{n+1}| \\ &\leq (1-\|u'\|^2)^{\frac{(n+1)\alpha}{p}}(1-\lambda+\lambda\varepsilon)^{1/\tilde{q}}\left\{\left[\frac{|z_{n+1}|^{\frac{p}{(n+1)\alpha}}}{1-\|z'\|^2}\right]^{\frac{n+1}{2n}} + \left[\frac{|w_{n+1}|^{\frac{p}{(n+1)\alpha}}}{1-\|w'\|^2}\right]^{\frac{n+1}{2n}}\right\}^{1/\tilde{p}} \\ &< (1-\|u'\|^2)^{\frac{(n+1)\alpha}{p}}(a_1+a_2)^{1/\tilde{p}}. \end{aligned}$$

Therefore, we have proved that $|u_{n+1}|^p < (a_1 + a_2)^{2n\alpha}(1 - \|u'\|^2)^{(n+1)\alpha}$, i.e. $u = (u', u_{n+1}) \in \Omega_{a_1+a_2,n,p,\alpha}$. This completes the proof. □

Taking $\varepsilon = 1$ in Theorem 3.1, we obtain the following convexity result for the operator $\Phi_{n,p,\alpha}$. If $p = 2$, see [1].

Corollary 3.1. *If $f \in K(B^n)$ and $F = \Phi_{n,p,\alpha}(f)$, where $\alpha \in \left[\frac{1}{n+1}, \frac{p}{2n}\right]$, then $(1 - \lambda)F(z) + \lambda F(w) \in F(\Omega_{a_1+a_2,n,p,\alpha})$, $z \in \Omega_{a_1,n,p,\alpha}$, $w \in \Omega_{a_2,n,p,\alpha}$, $\lambda \in [0, 1]$, where $a_1, a_2 > 0$, $a_1 + a_2 \leq 1$.*

Taking $\alpha = \frac{1}{n+1}$ and $a_1 = 1 - a_2$ in Corollary 3.1 and using the fact that $\Omega_{1,n,p,\frac{1}{n+1}} = \Omega_{n,p}$, we obtain the following corollary. In the case $\varepsilon = 1$, see [9] and when $p = 2$, see [1].

Corollary 3.2. *If f is a normalized ε-starlike mapping on B^n, $\varepsilon \in [0, 1]$, $a \in (0, 1)$ and $F = \Phi_{n,p,\frac{1}{n+1}}(f)$, $p \geq 2n/(n + 1)$, then*

$$(1-\lambda)F(z) + \lambda\varepsilon F(w) \in F(\Omega_{n,p}),\ z \in \Omega_{a,n,p,\frac{1}{n+1}},\ w \in \Omega_{1-a,n,p,\frac{1}{n+1}},\ \lambda \in [0,1].$$

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