ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **1** (2012), no.2, 81–85
ISSN 1857-8365, UDC: 517.546

ON PROPERTIES OF THE CLASS OF SEMI-TYPICALLY REAL FUNCTIONS

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Presented at the 8^{th} International Symposium GEOMETRIC FUNCTION THEORY AND APPLICATIONS, 27-31 August 2012, Ohrid, Republic of Macedonia.

Abstract. In this paper we present some properties of the class of semi-typically real functions denoted by $\mathcal{T}$ and defined as follows

$$\mathcal{T} = \{F \in \mathcal{A} : F(z) > 0 \Longleftrightarrow z \in (0,1)\}.$$

We denote by $\mathcal{A}$ the set of all functions that are analytic in the unit disc $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. The class $\mathcal{T}$ is related to the well-known class of typically real functions T i.e. the subclass of $\mathcal{A}$, consisting of functions f which satisfy the condition $\mathfrak{Im} z\, \mathfrak{Im} f(z) \geq 0$, $z \in \Delta$. We investigate relations between the class $\mathcal{T}$ and the class T and also between the class $\mathcal{T}$ and the class of odd typically real functions $\mathrm{T}^{(2)}$. Moreover, for the class $\mathcal{T}$ we determine sets of local univalence and of typical reality, radii of local univalence and of typical reality, sets of variability of initial coefficients and estimation of coefficients. We compare these results with well-known results obtained in the class T.

1. Introduction

Suppose that $\mathcal{A}$ is the family of all functions that are analytic in the unit disc

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}$$

and normalized by $f(0) = f'(0) - 1 = 0$. Let T denote the well-known class of all typically real functions, i.e. the subclass of $\mathcal{A}$ consisting of functions f which satisfy the condition $\mathfrak{Im}\, z\, \mathfrak{Im}\, f(z) \geq 0$, $z \in \Delta$. From this definition we conclude that

$$\mathrm{T} = \{f \in \mathcal{A} : f(z) \in \mathbb{R} \Longleftrightarrow z \in (-1,1)\}.$$

Let us define the class of semi-typically real functions in the following way

$$\mathcal{T} = \{F \in \mathcal{A} : F(z) > 0 \Longleftrightarrow z \in (0,1)\}.$$

In this paper we compare well-known results obtained in the class T with our results obtained in the class $\mathcal{T}$.

2010 *Mathematics Subject Classification.* 30C45, 30C50.

Key words and phrases. typically real functions, domain of local univalence, domain of univalence, set of variability of coefficients.

2. Main properties

At the beginning we present main properties of the class of semi-typically real functions. In Theorem 2.1 we have the relationship between the class $\mathcal{T}$ and the class of typically real odd functions $\mathrm{T}^{(2)}$ (where $\mathrm{T}^{(2)} = \{f \in \mathrm{T} : f(z) = -f(-z), z \in \Delta\}$). Theorem 2.2 gives us the relationship between the classes $\mathcal{T}$ and T. The proofs of these theorems one can find in [3].

Theorem 2.1.

$$F \in \mathcal{T} \Longleftrightarrow F(z^2) = f^2(z) \textit{ for some } f \in \mathrm{T}^{(2)}.$$

Theorem 2.2.

$$F \in \mathcal{T} \Longleftrightarrow F(z) = (1+z)^2 f^2(z)/z \textit{ for some } f \in \mathrm{T}.$$

3. Problems of univalence

In this section we discuss problems of univalence in the class $\mathcal{T}$.

Definition 3.1. *A domain $G \subset \Delta$ is called the domain of local univalence for the class $A \subset \mathcal{A}$, if:*

(i) *for all functions $f \in A$ and for all $z \in G$ we have $f'(z) \neq 0$,*
(ii) *for all $z \in \Delta \setminus G$ there exists a function $f \in A$ such that $f'(z) = 0$.*

The domain of local univalence for the class A is unique and we denote it by $G_{LU}(A)$. We know the domains of local univalence for the classes T and $\mathrm{T}^{(2)}$, namely

$$G_{LU}(\mathrm{T}) = \left\{z \in \Delta : |z^2+1| > 2|z|\right\}, \tag{3.1}$$

$$G_{LU}\left(\mathrm{T}^{(2)}\right) = \left\{z \in \Delta : |3z^4 + 2z^2 + 3| > 8|z|^2\right\} \setminus \left\{\pm ir : r \geq \sqrt{2} - 1\right\}. \tag{3.2}$$

The first one is the well-known Golusin's lens (see [1]) and the second one was described by Koczan and Zaprawa in [5]. From (3.2) and Theorem 2.1 we get the domain of local univalence for the class $\mathcal{T}$.

Theorem 3.1.

$$G_{LU}(\mathcal{T}) = \left\{z \in \Delta : |3z^2 + 2z + 3| > 8|z|\right\} \setminus \left\{z \in \mathbb{R} : z \leq -\left(\sqrt{2}-1\right)^2\right\}.$$

We have $G_{LU}(\mathcal{T}) \subset G_{LU}(\mathrm{T}) \subset G_{LU}\left(\mathrm{T}^{(2)}\right)$. Figure 1 (a) presents the domains of local univalence for the classes $\mathcal{T}$ (solid line), T (dashed line) and $\mathrm{T}^{(2)}$ (dotted line).

Definition 3.2. *A domain $G \subset \Delta$ is called the domain of univalence for the class $A \subset \mathcal{A}$, if:*

(i) *all functions belonging to A are univalent in G,*
(ii) *for every domain H such that $G \subset H \subset \Delta$ and $G \neq H$ there exists a function in A that is not univalent in H.*

We know that the domain of univalence for the class T is the Golusin's lens. Furthermore, there are infinitely many domains of univalence for the class $\mathrm{T}^{(2)}$ and one of them is the Golusin's lens (see [4]). From these facts and from Theorem 2.1 we can determine one of the domains of univalence for the class $\mathcal{T}$ (see also Figure 1 (b)).

Theorem 3.2. *A set* $\{z \in \Delta : |z+1|^2 > 4|z|\}$ *is the domain of univalence for the class* $\mathcal{T}$.

Now let A be a class of functions with real coefficients.

Definition 3.3. *A set* $G \subset \Delta$ *is called the set of typical reality for the class* $A \subset \mathcal{A}$, *if:*

(i) $\mathfrak{Im}\, z\, \mathfrak{Im}\, f(z) \geq 0$ *for all* $f \in A$ *and all* $z \in G$,
(ii) *for all* $z \in \Delta \setminus G$ *there exists a function* $f \in A$ *such that* $\mathfrak{Im}\, z\, \mathfrak{Im}\, f(z) < 0$.

The interior of this set is called the domain of typical reality for the class A, whenever this interior is a domain. The set of typical reality for the class A we denote by $G_{TR}(A)$. Figure 1 (c) shows the sets $G_{TR}(\mathcal{T})$ (solid line) and $G_{TR}(\mathrm{T})$ (dashed line).

Theorem 3.3.

$$G_{TR}(\mathcal{T}) = \{z \in \Delta : |z+1|^2 \geq 4|z|\} \cup (-1, 1).$$

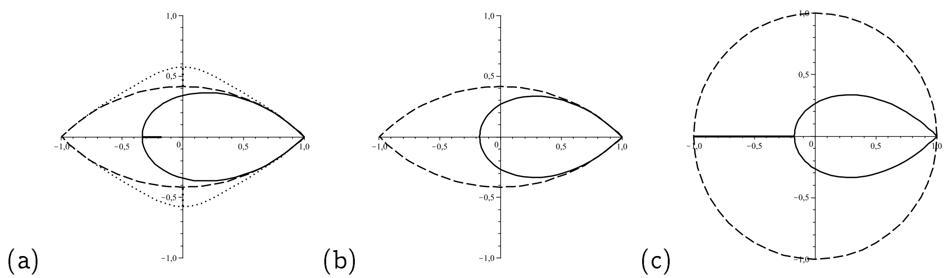
The proofs of Theorems 3.1-3.3 one can find in [3].

Moreover, we obtain the radii of local univalence, of univalence and of typical reality in the class $\mathcal{T}$, namely

$$r_{LU}(\mathcal{T}) = r_U(\mathcal{T}) = r_{TR}(\mathcal{T}) = (\sqrt{2}-1)^2.$$

These radii for the class T are the following: $r_{LU}(\mathrm{T}) = r_U(\mathrm{T}) = \sqrt{2}-1$ and $r_{TR}(\mathrm{T}) = 1$. We say that a number $r_U(A)$ ($r_{LU}(A)$) is called the radius of univalence (the radius of local univalence) in the class A, if it is the maximum of numbers $r \in (0,1]$, such that every function $f \in A$ is univalent (locally univalent) in the disc $|z| < r$. We say that a number $r_{TR}(A)$ is called the radius of typical reality in the class A, if it is the radius of the largest disc $|z| < r$ included in the domain of typical reality.

FIGURE 1. (a) The boundary of the domain of local univalence for $\mathcal{T}$ (solid line), for T (dashed line) and for $\mathrm{T}^{(2)}$ (dotted line). (b) The boundary of the domain of univalence for $\mathcal{T}$ (solid line) and for T (dashed line). (c) The boundary of the set of typical reality for $\mathcal{T}$ (solid line) and for T (dashed line).



4. Coefficients

In this section we analyse coefficient problems for semi-typically real functions. Let $A_{i,j}(\mathrm{A}) = \{(a_i(f), a_j(f)) : f \in \mathrm{A}\}$ for $\mathrm{A} \subset \mathcal{A}$. We determine the set $A_{2,3}(\mathcal{T})$ of variability of the second and the third coefficients.

From Theorem 2.2 we obtain

$$F(z) = \frac{(1+z)^2 f^2(z)}{z} = \frac{(1+z)^2 \left(z + a_2 z^2 + a_3 z^3 + \dots\right)^2}{z}$$

and, consequently,

$$F(z) = z + z^2(2 + 2a_2) + z^3(a_2^2 + 4a_2 + 2a_3 + 1) + \dots .$$

Writing $F \in \mathcal{T}$ in the form $F(z) = z + A_2 z^2 + A_3 z^3 + \dots \in \mathcal{T}$ we have:

$$A_2 = 2 + 2a_2 \text{ and } A_3 = a_2^2 + 4a_2 + 2a_3 + 1.$$

Deriving a_2 from the first expression and we putting it into the second one, we have $A_3 = A_2^2/4 + A_2 - 2 + 2a_3$. Taking into account the well-known set of variability of coefficients in the class T, namely

$$A_{2,3}(\mathrm{T}) = \{(x, y) : x^2 - 1 \leq y \leq 3\},$$

we obtain the following inequalities:

$$-2 \leq A_2 \leq 6,$$
$$\frac{3}{4}A_2^2 - A_2 - 2 \leq A_3 \leq \frac{1}{4}A_2^2 + A_2 + 4$$

and the set of variability of the second and the third coefficients in the class $\mathcal{T}$.

Theorem 4.1.

$$A_{2,3}(\mathcal{T}) = \left\{(x, y) : \frac{3}{4}x^2 - x - 2 \leq y \leq \frac{1}{4}x^2 + x + 4\right\}.$$

This set is not convex, because the class $\mathcal{T}$ is not convex. Figure 2 presents the sets $A_{2,3}(\mathcal{T})$ (solid line) and $A_{2,3}(\mathrm{T})$ (dashed line).

Now we determine the upper estimation of n-th coefficient of the function $F(z) = z + A_2 z^2 + A_3 z^3 + \dots \in \mathcal{T}$. For $f(z) = \sum_{k=1}^{\infty} a_k z^k$ we have $f^2(z) = \sum_{n=2}^{\infty} b_n z^n$, where $b_n = \sum_{k=1}^{n-1} a_k a_{n-k}$. Hence from Theorem 2.2 we obtain

$$F(z) = \frac{(1+z)^2}{z} \sum_{n=2}^{\infty} b_n z^n = \sum_{n=1}^{\infty} b_{n+2} z^n + 2\sum_{n=1}^{\infty} b_{n+2} z^{n+1} + \sum_{n=1}^{\infty} b_{n+2} z^{n+2} =$$
$$= b_2 z + (b_3 + 2b_2) z^2 + \sum_{n=3}^{\infty} (b_{n+1} + 2b_n + b_{n-1}) z^n.$$

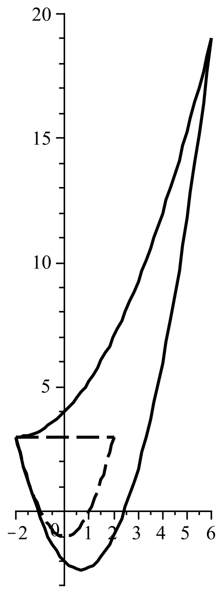
For $n \geq 3$ we have

$$A_n = b_{n+1} + 2b_n + b_{n-1} = \sum_{k=1}^{n} a_k a_{n+1-k} + \sum_{k=1}^{n-1} 2a_k a_{n-k} + \sum_{k=1}^{n-2} a_k a_{n-1-k}.$$

From the well-known inequality for typically real functions $a_n \leq n$ we obtain

$$A_n \leq \sum_{k=1}^{n} k(n+1-k) + \sum_{k=1}^{n-1} 2k(n-k) + \sum_{k=1}^{n-2} k(n-1-k) = \frac{(2n^2+1)n}{3}.$$

FIGURE 2. The boundary of the sets of variability of coefficients $A_{2,3}(\mathcal{T})$ (solid line) and $A_{2,3}(\mathrm{T})$ (dashed line).



Theorem 4.2. *If $F \in \mathcal{T}$ and $F(z) = z + A_2z^2 + A_3z^3 + \dots$, then $A_n \le (2n^2+1)n/3$.*

The equality sign in the above inequality is achieved for the function $F_0(z) = \frac{z(1+z)^2}{(1-z)^4}$. Indeed,

$$F_0(z) = \left(\frac{1+z}{1-z}\right)^2 \frac{z}{(1-z)^2} = \left(1+2z+2z^2+2z^3+\dots\right)^2 \left(z+2z^2+3z^3+\dots\right) =$$

$$= \sum_{n=1}^{\infty} nz^n + 4\left(\sum_{n=1}^{\infty} nz^n\right)^2 = \sum_{n=1}^{\infty} \frac{(2n^2+1)n}{3}\, z^n.$$

REFERENCES

[1] G. GOLUSIN: *On typically real functions*, Mat. Sb. **27**(69), (1950), 201–218.
[2] A.W. GOODMAN: *Univalent functions*, Mariner Publishing Coompany, Inc., Tampa, Florida, 1983.
[3] L. KOCZAN, K. TRĄBKA-WIĘCŁAW: *On semi-typically real functions*, Ann. Univ. Mariae Curie-Sklodowska, Sect. A, Vol. **LXIII** (2009), 139–148.
[4] L. KOCZAN, P. ZAPRAWA: *Domains of univalence for typically real odd functions*, Complex Variables, Vol. **48**(1) (2003), 1–17.
[5] L. KOCZAN, P. ZAPRAWA: *On typically real functions with n-fold symmetry*, Ann. Univ. Mariae Curie-Sklodowska, Sect. A, Vol. **L II.2**(11) (1998), 103–112.

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