

SIMPLE CRITERIA FOR BOUNDED TURNING OF AN ANALYTIC FUNCTION

NIKOLA TUNESKI, MASLINA DARUS AND ELENA GELOVA

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ABSTRACT. Let f be an analytic function in the open unit disk and normalized such that $f(0) = f'(0) - 1 = 0$. In this work, by means of the theory of differential subordinations, we study the expression

$$\alpha \cdot f'(z) + \beta \cdot \frac{f(z)}{z}$$

and receive results over the modulus and the real parts of

$$f'(z) \quad \text{and} \quad \frac{f(z)}{z},$$

that lead to simple criteria for bounded turning of an analytic function.

1. INTRODUCTION AND PRELIMINARIES

Let \mathcal{A} denote the class of analytic functions in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ that are normalized such that $f(0) = f'(0) - 1 = 0$, i.e. $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$.

Function $f \in \mathcal{A}$ is in the class of *starlike functions*, S^* , if and only if

$$\operatorname{Re} \left[\frac{z f'(z)}{f(z)} \right] > 0, \quad z \in \mathbb{D}.$$

Such functions are univalent and their geometric characterization (which motivates the name of the class) is that they map the unit disk onto a starlike region, i.e. if $\omega \in f(\mathbb{D})$ then $t\omega \in f(\mathbb{D})$ for all $t \in [0, 1]$.

Another well known class of univalent functions is the *class of functions with bounded turning*,

$$R = \{f \in \mathcal{A} : \operatorname{Re} f'(z) > 0, z \in \mathbb{D}\}.$$

Here also, the name of class follows from its geometric characterization, i.e. from the fact that $\operatorname{Re} f'(z) > 0$ is equivalent with $|\arg f'(z)| < \pi/2$ and $\arg f'(z)$ is the angle of rotation of the image of a line segment from z under the mapping f .

More details on these classes can be found in [2]. One of the main results concerning them is due to Krzyż ([7]), claiming that S^* does not contain R and R does not contain S^* . This makes class R interesting and lots of research is dedicated to it. Some references in that direction are [6] – [9].

In this paper we will study the linear combination of two simple expressions, $f'(z)$ and $f(z)/z$, i.e. we will study the modulus and the real part of

$$(1.1) \quad \alpha \cdot f'(z) + \beta \cdot \frac{f(z)}{z}$$

and receive criteria for a function $f \in \mathcal{A}$ to be of bounded turning. For that purpose we will use a method from the theory of differential subordinations. Valuable references on this topic are [1] and [3].

First we introduce subordination. Let $f, g \in \mathcal{A}$. Then we say that $f(z)$ is *subordinate* to $g(z)$, and write $f(z) \prec g(z)$, if there exists a function $\omega(z)$, analytic in the unit disc \mathbb{D} , such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$ for all $z \in \mathbb{D}$. Specially, if $g(z)$ is univalent in \mathbb{D} then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

For obtaining the main result we will use the method of differential subordinations. The general theory of differential subordinations, as well as the theory of first-order differential subordinations, was introduced by Miller and Mocanu in [4] and [5]. Namely, if $\phi : \mathbb{C}^2 \rightarrow \mathbb{C}$ (where \mathbb{C} is the complex plane) is analytic in a domain D , if $h(z)$ is univalent in \mathbb{D} , and if $p(z)$ is analytic in \mathbb{D} with $(p(z), zp'(z)) \in D$ when $z \in \mathbb{D}$, then $p(z)$ is said to satisfy a first-order differential subordination if

$$(1.2) \quad \phi(p(z), zp'(z)) \prec h(z).$$

The univalent function $q(z)$ is said to be a *dominant* of the differential subordination (1.2) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.2). If $\tilde{q}(z)$ is a dominant of (1.2) and $\tilde{q}(z) \prec q(z)$ for all dominants of (1.2), then we say that $\tilde{q}(z)$ is the *best dominant* of the differential subordination (1.2).

From the theory of first-order differential subordinations we will make use of the following lemma.

Lemma 1.1 ([5]). *Let $q(z)$ be univalent in the unit disk \mathbb{D} , and let $\theta(\omega)$ and $\phi(\omega)$ be analytic in a domain D containing $q(\mathbb{D})$, with $\phi(\omega) \neq 0$ when $\omega \in q(\mathbb{D})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$, and suppose that*

i) $Q(z)$ is starlike in the unit disk \mathbb{D} ; and

ii) $\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \left\{ \frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)} \right\} > 0$, $z \in \mathbb{D}$.

If $p(z)$ is analytic in \mathbb{D} , with $p(0) = q(0)$, $p(\mathbb{D}) \subseteq D$ and

$$(1.3) \quad \theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z)$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominant of (1.3).

Now, using Lemma 1.1 we will prove the following result.

Lemma 1.2. *Let $f \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ be such that $\alpha + \beta = 0$ or $\alpha + \beta = 1$. Also, let $q(z)$ be univalent in the unit disk \mathbb{D} satisfying $q(0) = 0$ and*

$$(1.4) \quad \operatorname{Re} \left[1 + \frac{zq''(z)}{q'(z)} \right] > 0, \quad z \in \mathbb{D}.$$

Additionally, $\operatorname{Re} \frac{1}{\alpha} > -1$ and

$$(1.5) \quad \operatorname{Re} \left[1 + \frac{zq''(z)}{q'(z)} \right] > -\operatorname{Re} \frac{1}{\alpha}, \quad z \in \mathbb{D},$$

in the case when $\alpha + \beta = 1$. If

$$(1.6) \quad \alpha \cdot f'(z) + \beta \cdot \frac{f(z)}{z} \prec (\alpha + \beta) \cdot [q(z) + 1] + \alpha z q'(z) \equiv h(z)$$

then $\frac{f(z)}{z} - 1 \prec q(z)$, and $q(z)$ is the best dominant of (1.6).

Proof. Functions $\theta(\omega) = (\alpha + \beta) \cdot (\omega + 1)$ and $\phi(\omega) = \alpha$ are analytic in a domain $D = \mathbb{C}$ which contains $q(\mathbb{D})$ and $\phi(\omega) \neq 0$ when $\omega \in q(\mathbb{D})$. Further, $Q(z) = zq'(z)\phi(q(z)) = \alpha zq'(z)$ is starlike since

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} = \operatorname{Re} \left[1 + \frac{zq''(z)}{q'(z)} \right] > 0, \quad z \in \mathbb{D},$$

and for the function $h(z) = \theta(q(z)) + Q(z) = Q(z)$ we have

$$\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \left[1 + \frac{\alpha + \beta}{\alpha} + \frac{zq''(z)}{q'(z)} \right] > 0, \quad z \in \mathbb{D},$$

for $\alpha + \beta = 0$ due to (1.4) and for $\alpha + \beta = 1$ due to (1.5).

Now, let choose $p(z) = \frac{f(z)}{z} - 1$ which is analytic in \mathbb{D} , $p(0) = q(0) = 0$ and $p(\mathbb{D}) \subseteq D = \mathbb{C}$. Finally, having in mind that subordinations (1.3) and (1.6) are equivalent, from Lemma 1.1 we receive the conclusions of Lemma 1.2. \square

2. RESULTS OVER THE MODULUS OF (1.1)

In this section we will study the modulus of (1.1) and receive conclusions that will lead to criteria for a function f to be in the class R .

Theorem 2.1. *Let $f \in \mathcal{A}$, $\mu > 0$ and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ be such that $\alpha + \beta = 0$ or $\alpha + \beta = 1$. Also, let $\operatorname{Re} \frac{1}{\alpha} > -1$ in the case when $\alpha + \beta = 1$. If*

$$(2.1) \quad \left| \alpha \cdot f'(z) + \beta \cdot \frac{f(z)}{z} - (\alpha + \beta) \right| < \delta \equiv \begin{cases} \mu \cdot |\alpha|, & \alpha + \beta = 0 \\ \mu \cdot |1 + \alpha|, & \alpha + \beta = 1 \end{cases},$$

for all $z \in \mathbb{D}$, then

$$(2.2) \quad \left| \frac{f(z)}{z} - 1 \right| < \mu, \quad z \in \mathbb{D}.$$

This implication is sharp, i.e., in the inequality (2.2), μ can not be replaced by a smaller number so that the implication holds. Also,

$$|f'(z) - 1| < \lambda \equiv \begin{cases} 2\mu, & \alpha + \beta = 0 \\ \mu \cdot \left(\left| 1 + \frac{1}{\alpha} \right| + \left| 1 - \frac{1}{\alpha} \right| \right), & \alpha + \beta = 1 \end{cases}, \quad z \in \mathbb{D}.$$

This implication is also sharp, i.e., λ can not be replaced by a smaller number so that the implication holds, if

- (i) $\alpha + \beta = 0$; or
- (ii) $\alpha + \beta = 1$ and $\left| 1 + \frac{1}{\alpha} \right| + \left| 1 - \frac{1}{\alpha} \right| = 2$.

Additionally, if $\mu \leq \frac{1}{2}$ for $\alpha + \beta = 0$ or $|1 + \frac{1}{\alpha}| + |1 - \frac{1}{\alpha}| \leq \frac{1}{\mu}$ for $\alpha + \beta = 1$ then $f \in R$.

Proof. Choosing $q(z) = \mu z$ we have $1 + \frac{z q''(z)}{q'(z)} = 1$, meaning that (1.4) and (1.5) form Lemma 1.2 hold. Further, for the function $h(z)$ defined in (1.6) we have

$$h(z) = \alpha + \beta + \mu z(2\alpha + \beta),$$

meaning that subordination (1.6) is equivalent to

$$\left| \alpha \cdot f'(z) + \beta \cdot \frac{f(z)}{z} - (\alpha + \beta) \right| < \mu \cdot |2\alpha + \beta| = \delta, \quad z \in \mathbb{D},$$

i.e. to (2.1). Therefore, (2.2) follows directly from Lemma 1.2 and the definition of subordination.

Further, for all $z \in \mathbb{D}$,

$$\left| \alpha \cdot f'(z) + \beta \cdot \frac{f(z)}{z} - (\alpha + \beta) \right| = \left| \alpha \cdot [f'(z) - 1] + \beta \cdot \left[\frac{f(z)}{z} - 1 \right] \right|$$

and

$$\begin{aligned} |\alpha| \cdot |f'(z) - 1| &\leq \left| \alpha \cdot [f'(z) - 1] + \beta \cdot \left[\frac{f(z)}{z} - 1 \right] \right| + \left| \beta \cdot \left[\frac{f(z)}{z} - 1 \right] \right| \\ &< \delta + |\beta| \cdot \mu = |\alpha| \cdot \lambda, \end{aligned}$$

since $|w_1| \leq |w_1 + w_2| + |w_2|$. Therefore, the implication of this corollary holds.

Both implication are sharp as the function $f_*(z) = z + \mu z^2$ shows, since

$$\left| \alpha \cdot f'_*(z) + \beta \cdot \frac{f_*(z)}{z} - (\alpha + \beta) \right| = \mu \cdot |2\alpha + \beta| \cdot |z| = \delta \cdot |z|, \quad z \in \mathbb{D},$$

$$\left| \frac{f_*(z)}{z} - 1 \right| = \mu \cdot |z|, \quad z \in \mathbb{D},$$

$$|f'_*(z) - 1| = 2 \cdot \mu \cdot |z|, \quad z \in \mathbb{D},$$

and $2\mu = \lambda$ if (i) or (ii) hold. □

3. RESULTS OVER THE REAL PART OF (1.1)

In this section we will study the real part of the expression (1.1) and receive criteria over it that will embed a function $f \in \mathcal{A}$ in the class R .

Theorem 3.1. *Let $f \in \mathcal{A}$, $\mu > 0$ and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ be such that $\alpha + \beta = 0$ or $\alpha + \beta = 1$. Also, let $\operatorname{Re} \frac{1}{\alpha} > 0$ in the case when $\alpha + \beta = 1$. If*

$$(3.1) \quad \alpha \cdot f'(z) + \beta \cdot \frac{f(z)}{z} \prec (\alpha + \beta) \left(1 + \frac{2\mu z}{1-z} \right) + \frac{2\alpha\mu z}{(1-z)^2} \equiv h_2(z)$$

then

$$(3.2) \quad \operatorname{Re} \left[\frac{f(z)}{z} \right] > 1 - \mu, \quad z \in \mathbb{D}.$$

This implication is sharp, i.e., in the inequality (3.2), μ can not be replaced by a bigger number so that the implication holds.

Proof. The implication of this theorem follows directly from Lemma 1.2 for $q(z) = \frac{2\mu z}{1-z}$. Condition $\operatorname{Re} \frac{1}{\alpha} > 0$ stands in stead of $\operatorname{Re} \frac{1}{\alpha} > -1$ in order (1.5) to hold. The result is sharp due to the function $f_*(z) = z + z \cdot q(z)$ such that

$$\alpha \cdot f'_*(z) + \beta \cdot \frac{f_*(z)}{z} = (\alpha + \beta) \left(1 + \frac{2\mu z}{1-z} \right) + \frac{2\alpha\mu z}{(1-z)^2}$$

and $\operatorname{Re} \frac{f(z)}{z} = 1 - \mu$ for $z = -1$. □

In the case when $\alpha + \beta = 1$ we receive the following corollary.

Corollary 3.1. *Let $f \in \mathcal{A}$, $\alpha > 0$ and $\mu > 0$. If*

$$(3.3) \quad \operatorname{Re} \left[\alpha \cdot f'(z) + (1 - \alpha) \cdot \frac{f(z)}{z} \right] > 1 - \mu \cdot \left(1 + \frac{\alpha}{2} \right), \quad z \in \mathbb{D},$$

then

$$\operatorname{Re} \left[\frac{f(z)}{z} \right] > 1 - \mu, \quad z \in \mathbb{D}.$$

If, additionally,

- (i) $\alpha > 1$ and $\mu \leq 1$; or
- (ii) $\alpha < 1$ and $\mu \geq 1$;

then

$$(3.4) \quad \operatorname{Re} f'(z) > 1 - \frac{3}{2} \cdot \mu, \quad z \in \mathbb{D}.$$

These results are sharp.

Proof. Let $\alpha + \beta = 1$. So, for the function h_2 defined in (3.1) we have

$$h_2(z) = 1 + \frac{2\mu z}{1-z} + \frac{2\alpha\mu z}{(1-z)^2},$$

$h_2(0) = 1$ and

$$h_2(e^{i\theta}) = 1 - \frac{\mu\alpha}{2}(1 + t^2) - \mu + \mu ti,$$

where $t = \operatorname{ctg}(\theta/2)$. Therefore,

$$X = \operatorname{Re} h(e^{i\theta}) = 1 - \mu \left(\frac{\alpha}{2} + 1 \right) - \frac{\alpha}{2\mu} \cdot Y^2,$$

where

$$Y = \operatorname{Im} h(e^{i\theta}) = \mu t$$

attains all real numbers. This leads to

$$h_2(e^{i\theta}) = \left\{ x + iy : x = 1 - \mu \left(1 + \frac{\alpha}{2} \right) - \frac{\alpha}{2\mu} \cdot y^2, y \in \mathbb{R} \right\}.$$

From here, having in mind the definition of subordination, the inequality (3.3) and the fact that

$$\left\{ x + iy : x > 1 - \mu \left(1 + \frac{\alpha}{2} \right), y \in \mathbb{R} \right\} \subseteq h_2(\mathbb{D}),$$

we receive subordination (3.1). Therefore, from Theorem 3.1 follows

$$\operatorname{Re} \left[\frac{f(z)}{z} \right] > 1 - \mu, \quad z \in \mathbb{D}.$$

Further, in the case when (i) or (ii) holds we have

$$\begin{aligned} \operatorname{Re} f'(z) &= \frac{1}{\alpha} \cdot \left\{ \operatorname{Re} \left[\alpha \cdot f'(z) + (1-\alpha) \cdot \frac{f(z)}{z} \right] - (1-\alpha) \cdot \operatorname{Re} \left[\frac{f(z)}{z} \right] \right\} \\ &> \frac{1}{\alpha} \cdot \left[1 - \mu \left(1 + \frac{\alpha}{2} \right) - (1-\alpha)(1-\mu) \right] = 1 - \frac{3}{2} \cdot \mu, \end{aligned}$$

for all $z \in \mathbb{D}$.

The results are sharp due to the function $f_*(z) = z + \frac{2\mu z^2}{1-z}$ such that $f_*(z)/z = 1 + \frac{2\mu z}{1-z} \equiv g(z)$, $g(\mathbb{D}) = \{x + iy : x > 1 - \mu, y \in \mathbb{R}\}$,

$$\alpha \cdot f'_*(z) + (1-\alpha) \cdot \frac{f_*(z)}{z} = h_2(z)$$

and

$$\operatorname{Re} f'_*(z) = \operatorname{Re} h_2(z) = 1 - \frac{3}{2} \cdot \mu \quad \text{for } z = -1.$$

□

In a similar way as in Corollary 3.1, for the case $\alpha = -\beta = 1$ we receive

Corollary 3.2. *Let $f \in \mathcal{A}$ and $\mu > 0$. If*

$$(3.5) \quad \operatorname{Re} \left[f'(z) - \frac{f(z)}{z} \right] > -\frac{\mu}{2}, \quad z \in \mathbb{D},$$

then $\operatorname{Re} \left[\frac{f(z)}{z} \right] > 1 - \mu$, $z \in \mathbb{D}$, and $\operatorname{Re} f'(z) > 1 - \frac{3}{2} \cdot \mu$, $z \in \mathbb{D}$. If, additionally, $\mu \leq \frac{2}{3}$, then $\operatorname{Re} f'(z) > 0$, $z \in \mathbb{D}$, i.e. $f \in \mathcal{R}$. Both implications are sharp.

4. EXAMPLES

The following example exhibits some concrete conclusions that can be obtained from the results of the previous sections by specifying the values α , β and μ .

Example 4.1. *Let $f \in \mathcal{A}$.*

- (i) *If $\left| f'(z) - \frac{f(z)}{z} \right| < \frac{1}{2}$ ($z \in \mathbb{D}$) then $|f'(z) - 1| < 1$ ($z \in \mathbb{D}$) and $f \in \mathcal{R}$. ($\alpha = -\beta = 1$ and $\mu = \frac{1}{2}$ in Theorem 2.1);*
- (ii) *If $\left| f'(z) + \frac{f(z)}{z} - 2 \right| < 1$ ($z \in \mathbb{D}$) then $|f'(z) - 1| < 1$ ($z \in \mathbb{D}$) and $f \in \mathcal{R}$. ($\alpha = \beta = \frac{1}{2}$ and $\mu = \frac{1}{4}$ in Theorem 2.1);*
- (iii) *If $\alpha > 0$ and $\operatorname{Re} \left[\alpha \cdot f'(z) + (1-\alpha) \cdot \frac{f(z)}{z} \right] > -\frac{\alpha}{2}$ ($z \in \mathbb{D}$) then $\operatorname{Re} \left[\frac{f(z)}{z} \right] > 0$ ($z \in \mathbb{D}$) and $\operatorname{Re} f'(z) > -1/2$ ($z \in \mathbb{D}$). ($\mu = 1$ in Corollary 3.1);*
- (iv) *If $\operatorname{Re} \left[f'(z) + \frac{f(z)}{z} \right] > -\frac{1}{2}$ ($z \in \mathbb{D}$) then $\operatorname{Re} \left[\frac{f(z)}{z} \right] > 0$ ($z \in \mathbb{D}$) and $\operatorname{Re} f'(z) > -1/2$ ($z \in \mathbb{D}$). ($\alpha = 1/2$ and $\mu = 1$ in Corollary 3.1);*
- (v) *If $\operatorname{Re} \left[f'(z) - \frac{f(z)}{z} \right] > -\frac{1}{3}$ ($z \in \mathbb{D}$) then $\operatorname{Re} f'(z) > 0$ ($z \in \mathbb{D}$) and $f \in \mathcal{R}$. ($\mu = \frac{2}{3}$ in Corollary 3.2);*

Remark 4.1. *It is worth noting that in part (iii) of the previous example, the conclusion does not depend on α .*

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FACULTY OF MECHANICAL ENGINEERING
 SS. CYRIL AND METHODIUS UNIVERSITY IN SKOPJE
 KARPOŠ II B.B., 1000 SKOPJE, REPUBLIC OF MACEDONIA
E-mail address: nikola.tuneski@mf.edu.mk

SCHOOL OF MATHEMATICAL SCIENCES
 FACULTY OF SCIENCE AND TECHNOLOGY
 UNIVERSITY KEBANGSAAN MALAYSIA
 BANGI 43600, SELANGOR DARUL EHSAN, MALAYSIA
E-mail address: maslina@ukm.my

UNIVERSITY GOCE DELCEV
 FACULTY OF COMPUTER SCIENCE, DEPARTMENT OF MATHEMATICS
 ŠTIP, REPUBLIC OF MACEDONIA
E-mail address: gelovae@yahoo.com