ADV MATH SCI JOURNAL

Advances in Mathematics: Scientific Journal **1** (2012), no.2, 95–100
ISSN 1857-8365, UDC: 517.546:517.57

COEFFICIENT CONDITIONS FOR HARMONIC CLOSE-TO-CONVEX FUNCTIONS

TOSHIO HAYAMI

Presented at the 8^{th} International Symposium GEOMETRIC FUNCTION THEORY AND APPLICATIONS, 27-31 August 2012, Ohrid, Republic of Macedonia.

ABSTRACT. New sufficient conditions, concerned with the coefficients of harmonic functions $f(z)=h(z)+\overline{g(z)}$ in the open unit disk normalized by $f(0)=h(0)=h'(0)-1=0$, for $f(z)$ to be harmonic close-to-convex functions are discussed. Furthermore, several illustrative examples of the obtained theorems are enumerated.

1. INTRODUCTION

Let $\mathbb{D}$ be a simply connected domain in the complex plane $\mathbb{C}$. A continuous complex-valued function $f(z)=u(x,y)+iv(x,y)$ in $\mathbb{D}$ ($z=x+iy\in\mathbb{D}$), we say that $f(z)$ is harmonic in $\mathbb{D}$ if both $u(x,y)$ and $v(x,y)$ are real harmonic in $\mathbb{D}$, that is, $u(x,y)$ and $v(x,y)$ satisfy the Laplace's equations

$$\Delta u=u_{xx}+u_{yy}=0 \qquad \text{and} \qquad \Delta v=v_{xx}+v_{yy}=0 \qquad (z=x+iy\in\mathbb{D}).$$

A harmonic function $f(z)$ in $\mathbb{D}$ is given by $f(z)=h(z)+\overline{g(z)}$ where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$. We call $h(z)$ and $g(z)$ the analytic part and the co-analytic part of $f(z)$, respectively. The Jacobian of $f(z)$ denoted by $\mathcal{J}_f$ can be computed by $\mathcal{J}_f=|h'(z)|^2-|g'(z)|^2$. A necessary and sufficient condition for $f(z)$ to be locally univalent and sense-preserving in $\mathbb{D}$ is that $\mathcal{J}_f>0$, that is, that $|g'(z)|<|h'(z)|$ in $\mathbb{D}$ (see, [2] or [9]). Let $\mathcal{H}$ denote the class of harmonic functions $f(z)$ in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ with $f(0)=h(0)=0$ and $h'(0)=1$. Thus, all functions $f(z)\in\mathcal{H}$ can be written by

$$f(z)=h(z)+\overline{g(z)}=z+\sum_{n=2}^{\infty}a_nz^n+\overline{\sum_{n=1}^{\infty}b_nz^n}$$

where $a_1=1$ and $b_0=0$, for convenience.

2010 *Mathematics Subject Classification.* 30C45, 58E20.
Key words and phrases. Coefficient conditions, harmonic functions, univalent functions, close-to-convex functions.

We next denote by $\mathcal{S}_{\mathcal{H}}$ the class of functions $f(z) \in \mathcal{H}$ which are univalent and sense-preserving in $\mathbb{U}$. Since the sense-preserving property of $f(z)$, we see that $|b_1| = |g'(0)| < |h'(0)| = 1$. If $g(z) \equiv 0$, then $\mathcal{S}_{\mathcal{H}}$ reduces to the class $\mathcal{S}$ consisting of normalized analytic univalent functions. Furthermore, for every function $f(z) \in \mathcal{S}_{\mathcal{H}}$, the function

$$F(z) = \frac{f(z) - \overline{b_1 f(z)}}{1 - |b_1|^2} = z + \sum_{n=2}^{\infty} \frac{a_n - \overline{b_1} b_n}{1 - |b_1|^2} z^n + \overline{\sum_{n=2}^{\infty} \frac{b_n - b_1 a_n}{1 - |b_1|^2} z^n}$$

is also a member of $\mathcal{S}_{\mathcal{H}}$. Therefore, we consider the subclass $\mathcal{S}_{\mathcal{H}}^0$ of $\mathcal{S}_{\mathcal{H}}$ defined as

$$\mathcal{S}_{\mathcal{H}}^0 = \{f(z) \in \mathcal{S}_{\mathcal{H}} : b_1 = g'(0) = 0\}.$$

Conversely, if $F(z) \in \mathcal{S}_{\mathcal{H}}^0$, then $f(z) = F(z) + \overline{b_1 F(z)} \in \mathcal{S}_{\mathcal{H}}$ for any b_1 $(|b_1| < 1)$.

We say that $\mathbb{D}$ is a close-to-convex domain if the complement of $\mathbb{D}$ can be written as a union of non-intersecting half-lines (except that the origin of one half-line may lie on one of the other half-lines). Let $\mathcal{C}$, $\mathcal{C}_{\mathcal{H}}$ and $\mathcal{C}_{\mathcal{H}}^0$ be the respective subclasses of $\mathcal{S}$, $\mathcal{S}_{\mathcal{H}}$ and $\mathcal{S}_{\mathcal{H}}^0$ consisting of all functions $f(z)$ which map $\mathbb{U}$ onto a certain close-to-convex domain.

A simple and interesting example is below.

Example 1.1. *The function*

$$f(z) = \frac{1 - (1-z)^2}{2(1-z)^2} + \overline{\frac{z^2}{2(1-z)^2}} = z + \sum_{n=2}^{\infty} \frac{n+1}{2} z^n + \sum_{n=2}^{\infty} \frac{n-1}{2} \overline{z}^n$$

belongs to the class $\mathcal{C}_{\mathcal{H}}^0$.

The aim of this paper is to find new sufficient conditions for functions $f(z) \in \mathcal{H}$ to be in the class $\mathcal{C}_{\mathcal{H}}$. To obtain our results, we have to recall here the following lemmas due to Clunie and Sheil-small [2].

Lemma 1.1. *If $h(z)$ and $g(z)$ are analytic in $\mathbb{U}$ with $|h'(0)| > |g'(0)|$ and $h(z) + \varepsilon g(z)$ is close-to-convex for each ε $(|\varepsilon| = 1)$, then $f(z) = h(z) + \overline{g(z)}$ is harmonic close-to-convex.*

Lemma 1.2. *If $f(z) = h(z) + \overline{g(z)}$ is locally univalent in $\mathbb{U}$ and $h(z) + \varepsilon g(z)$ is convex for some ε $(|\varepsilon| \leq 1)$, then $f(z)$ is univalent close-to-convex.*

We also need the following result due to Hayami, Owa and Srivastava [5].

Lemma 1.3. *If a function $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$ is analytic in $\mathbb{U}$ and satisfies*

$$\sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (-1)^{k-j} j(j+1) \binom{\alpha}{k-j} A_j \right\} \binom{\beta}{n-k} \right| + \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (-1)^{k-j} j(j-1) \binom{\alpha}{k-j} A_j \right\} \binom{\beta}{n-k} \right| \right] \leq 2$$

for some real numbers α and β, then $F(z)$ is convex in $\mathbb{U}$.

2. Main results

Using Lemma 1.1 we receive

Theorem 2.1. *If $f(z) \in \mathcal{H}$ satisfies the following condition*

$$\sum_{n=2}^{\infty} \left| n a_n - e^{i\varphi}(n-1)a_{n-1} \right| + \sum_{n=1}^{\infty} \left| n b_n - e^{i\varphi}(n-1)b_{n-1} \right| \leq 1$$

for some real number φ $(0 \leq \varphi < 2\pi)$, then $f(z) \in \mathcal{C}_{\mathcal{H}}$.

Example 2.1. *The function*

$$f(z) = -\log(1-z) + \overline{\left(-mz - \log(1-z)\right)} = z + \sum_{n=2}^{\infty} \frac{1}{n} z^n + (1-m)\overline{z} + \sum_{n=2}^{\infty} \frac{1}{n} \overline{z}^n \quad (0 < m \leq 1)$$

satisfies the condition of Theorem 2.1 with $\varphi = 0$ and belongs to the class $\mathcal{C}_{\mathcal{H}}$.

By making use of Lemma 1.2 with $\varepsilon = 0$ and applying Lemma 1.3, we readily obtain the following theorem.

Theorem 2.2. *If $f(z) \in \mathcal{H}$ is locally univalent in $\mathbb{U}$ and satisfies*

$$\sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (-1)^{k-j} j(j+1) \binom{\alpha}{k-j} a_j \right\} \binom{\beta}{n-k} \right| \right.$$
$$\left. + \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (-1)^{k-j} j(j-1) \binom{\alpha}{k-j} a_j \right\} \binom{\beta}{n-k} \right| \right] \leq 2$$

for some real numbers α and β, then $f(z) \in \mathcal{C}_{\mathcal{H}}$.

Putting $\alpha = \beta = 0$ in the above theorem, we arrive at the following result due to Jahangiri and Silverman [8].

Corollary 2.1. *If $f(z) \in \mathcal{H}$ is locally univalent in $\mathbb{U}$ with*

$$\sum_{n=2}^{\infty} n^2 |a_n| \leq 1,$$

then $f(z) \in \mathcal{C}_{\mathcal{H}}$.

Furthermore, taking $\alpha = 1$ and $\beta = 0$ in the theorem, we have

Corollary 2.2. *If $f(z) \in \mathcal{H}$ is locally univalent in $\mathbb{U}$ and satisfies*

$$\sum_{n=2}^{\infty} \{n\,|(n+1)a_n-(n-1)a_{n-1}|+(n-1)\,|na_n-(n-2)a_{n-1}|\} \leq 2,$$

then $f(z) \in \mathcal{C}_{\mathcal{H}}$.

Example 2.2. *The function*

$$f(z)=-\int_0^z \frac{\log(1-t)}{t}dt+\overline{\Big(z+(1-z)\log(1-z)\Big)}=z+\sum_{n=2}^{\infty}\frac{1}{n^2}z^n+\sum_{n=2}^{\infty}\frac{1}{n(n-1)}\overline{z}^n$$

satisfies the conditions of Corollary 2.2 and belongs to the class $\mathcal{C}_{\mathcal{H}}$.

3. Appendix

A sequence $\{c_n\}_{n=0}^{\infty}$ of non-negative real numbers is called a convex null sequence if $c_n \to 0$ as $n \to \infty$ and

$$c_n - c_{n+1} \geq c_{n+1} - c_{n+2} \geq 0$$

for all n $(n=0,1,2,\cdots)$.

The next lemma was obtained by Fejér [4].

Lemma 3.1. *Let $\{c_n\}_{n=0}^{\infty}$ be a convex null sequence. Then, the function*

$$p(z)=\frac{c_0}{2}+\sum_{n=1}^{\infty} c_n z^n$$

is analytic and satisfies $\mathrm{Re}(p(z))>0$ in $\mathbb{U}$.

Applying the above lemma, we deduce

Theorem 3.1. *For some b $(|b|<1)$ and some convex null sequence $\{c_n\}_{n=0}^{\infty}$ with $c_0=2$, the function*

$$f(z)=h(z)+\overline{g(z)}=z+\sum_{n=2}^{\infty}\frac{c_{n-1}}{n}z^n+\overline{b\left(z+\sum_{n=2}^{\infty}\frac{c_{n-1}}{n}z^n\right)}$$

belongs to the class $\mathcal{C}_{\mathcal{H}}$.

In the same manner, we also have

Theorem 3.2. *For some b $(|b|<1)$ and some convex null sequence $\{c_n\}_{n=0}^{\infty}$ with $c_0=2$, the function*

$$f(z)=h(z)+\overline{g(z)}=z+\sum_{n=2}^{\infty}\frac{1}{n}\left(1+\sum_{j=1}^{n-1}c_j\right)z^n+\overline{b\left(z+\sum_{n=2}^{\infty}\frac{1}{n}\left(1+\sum_{j=1}^{n-1}c_j\right)z^n\right)}$$

belongs to the class $\mathcal{C}_{\mathcal{H}}$.

Remark 3.1. *The sequence*

$$\{c_n\}_{n=0}^{\infty} = \left\{2, 1, \frac{1}{2}, \cdots, 2^{1-n}, \cdots\right\}$$

is a convex null sequence because

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} 2^{1-n} = 0, \qquad c_n - c_{n+1} = 2^{-n} \geq 0$$

and

$$(c_n - c_{n+1}) - (c_{n+1} - c_{n+2}) = 2^{-(n+1)} \geq 0 \qquad (n = 0, 1, 2, \cdots).$$

According to Remark 3.1, letting $b = 1/4$ in Theorem 3.2 with the sequence $\{c_n\}_{n=0}^{\infty} = \{2^{1-n}\}_{n=0}^{\infty}$, we have

Example 3.1. *The function*

$$\begin{aligned} f(z) &= -3\log(1-z) + 4\log\left(1-\frac{z}{2}\right) + \overline{\left(-\frac{3}{4}\log(1-z) + \log\left(1-\frac{z}{2}\right)\right)} \\ &= z + \sum_{n=2}^{\infty} \frac{1}{n}\left(1 + \sum_{j=1}^{n-1} 2^{1-j}\right) z^n + \overline{\frac{1}{4}\left(z + \sum_{n=2}^{\infty} \frac{1}{n}\left(1 + \sum_{j=1}^{n-1} 2^{1-j}\right) z^n\right)} \end{aligned}$$

is in the class $\mathcal{C}_{\mathcal{H}}$.

The details of this article can be found in the paper [7].

References

[1] D. Bshouty, A. Lyzzaik: *Close-to-convexity criteria for planar harmonic mappings*, Complex Anal. Oper. Theory **5**(2011), 767–774.

[2] J. Clunie, T. Sheil-Small: *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A I Math. **9** (1984), 3–25.

[3] P. L. Duren: *Harmonic Mappings in the Plane*, Cambridge University Press, Cambridge, 2004.

[4] L. Fejér: *Über die positivität von summen, die nach trigonometrischen oder Legendreschen, funktionen fortschreiten. I*, Acta Szeged **2** (1925), 75-86.

[5] T. Hayami, S. Owa, H. M. Srivastava: *Coefficient inequalities for certain classes of analytic and univalent functions*, J. Inequal. Pure Appl. Math. **8** (2007), Article 95, 10 pp.

[6] T. Hayami, S. Owa: *Hypocycloid of $n+1$ cusps harmonic function*, Bull. Math. Anal. Appl. **3** (2011), 239–246.

[7] T. Hayami: *Coefficient conditions for harmonic close-to-convex functions*, Abstr. Appl. Anal. (2012), Article ID 413965, 12 pp.

[8] J. M. Jahangiri, H. Silverman: *Harmonic close-to-convex mappings*, J. Appl. Math. Stoch. Anal. **15**(2002), 23–28.

[9] H. Lewy: *On the non-vanishing of the Jacobian in certain one-to-one mappings*, Bull. Amer. Math. Soc. **42** (1936), 689–692.

[10] P. T. Mocanu: *Three-cornered hat harmonic functions*, Complex Var. Elliptic Equ. **54** (2009), 1079–1084.

[11] P. T. Mocanu: *Injectively conditions in the complex plane*, Complex Anal. Oper. Theory **5** (2011), 759–766.

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502, Japan
E-mail address: ha_ya_to112@hotmail.com