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A NEW CLASS OF LOG-HARMONIC FUNCTIONS

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ABSTRACT. In this paper, we consider a new class of log-harmonic mappings of the form $f = zh(z)\overline{g(z)}$ defined on the open disc $\mathbb{D} = \{z| \quad |z| < 1\}$ which are univalent and satisfying the condition $\left|\frac{h(z)}{g(z)} - M\right| < M$ for every $z \in \mathbb{D}$.

1. INTRODUCTION

Let $H(\mathbb{D})$ be the linear space of analytic functions defined on the open unit disc $\mathbb{D}$. A log-harmonic mapping is a solution of the non-linear elliptic partial differential equation

$$\frac{\overline{f_{\overline{z}}}}{\overline{f}} = w(z).\frac{f_z}{f} \tag{1.1}$$

where $w(z)$ is the second dilatation function of f and $w(z) \in H(\mathbb{D})$ such that $|w(z)| < 1$ for all $z \in \mathbb{D}$. It has been shown that if f is a non-vanishing log-harmonic mapping, then f can be expressed as

$$f = h(z).\overline{g(z)} \tag{1.2}$$

where $h(z)$ and $g(z)$ are analytic functions in $\mathbb{D}$. On the other hand, if f vanishes at $z = 0$ but is not identically zero, then f admits the representation,

$$f = z\,|z|^{2\beta}\,h(z)\overline{g(z)} \tag{1.3}$$

where $Re\beta > -\frac{1}{2}$ and $h(z)$ and $g(z)$ are analytic functions in $\mathbb{D}$, $g(0) = h(0) = 1$. Univalent log-harmonic mapping have been studied extensively (for details see [1],[2],[3],[4],[5],[6]). The class of all log-harmonic mappings is denoted by S_{LH}.

Let $f = z\,|z|^{2\beta}\,h(z)\overline{g(z)}$ be a univalent harmonic mapping. We say that f is a starlike log-harmonic mapping if

$$\frac{\partial}{\partial\theta}\arg f(r.e^{i\theta}) = Re[\frac{zf_z - \overline{z}f_{\overline{z}}}{f}] > 0 \tag{1.4}$$

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for all $z \in \mathbb{D}$. Denote by ST_{LH} the set of all starlike log-harmonic mappings.

Finally, let Ω be the family of functions $\phi(z)$ regular in $\mathbb{D}$ and satisfying $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Next, let $F(z) = z + \alpha_2 z^2 + \alpha_3 z^3 + \cdots$ and $G(z) = z + \beta_2 z^2 + \beta_3 z^3 + \cdots$ be analytic functions in $\mathbb{D}$, if there exist a function $\phi(z) \in \Omega$ such that $F(z) = G(\phi(z))$ for every $z \in \mathbb{D}$, then we say that $F(z)$ is subordinate to $G(z)$, and we write $F(z) \prec G(z)$ [7].

In this paper we will investigate the class of log-harmonic mappings defined by

$$S_{LH(M)} = \left\{ f \in S_{LH} \mid \left| \frac{h(z)}{g(z)} - M \right| < M, M \geq 1 \text{ is a fixed number, } h(0) = g(0) = 1 \right\}.$$

2. Main Results

Theorem 2.1. *Let $f = zh(z)\overline{g(z)}$ be an element of $S_{LH(M)}$, then*

$$\frac{h(z)}{g(z)} \prec \frac{1+z}{1-(1-\frac{1}{M})z} \tag{2.1}$$

$$\frac{1-r}{1+(1-\frac{1}{M})r} \leq \left| \frac{h(z)}{g(z)} \right| \leq \frac{1+r}{1-(1-\frac{1}{M})r} \tag{2.2}$$

Proof.

$$\left| \frac{h(z)}{g(z)} - M \right| < M \Rightarrow \left| \frac{1}{M} \frac{h(z)}{g(z)} - 1 \right| < 1,$$

then the function $\psi(z) = (\frac{1}{M}\frac{h(z)}{g(z)} - 1)$ has modulus at 1 in the unit disc $\mathbb{D}$, so that

$$\phi(z) = \frac{\psi(z) - \psi(0)}{1 - \psi(0)\psi(z)} = \frac{(\frac{1}{M}\frac{h(z)}{g(z)} - 1) - (\frac{1}{M} - 1)}{1 - (\frac{1}{M} - 1)(\frac{1}{M}\frac{h(z)}{g(z)} - 1)}. \tag{2.3}$$

Then $\phi(0) = 0$ and $|\phi(z)| < 1$, therefore by Schwarz's Lemma

$$|\phi(z)| \leq |z|. \tag{2.4}$$

From (2.3) and (2.4) we obtain,

$$\frac{h(z)}{g(z)} = \frac{1+\phi(z)}{1-(1-\frac{1}{M})\phi(z)}$$

and using the subordination principle we get (2.1). On the other hand $\omega = \dfrac{1+z}{1-(1-\frac{1}{M})z}$ maps $|z| = r$ onto the circle with the center

$$C(r) = (\frac{1+(1-\frac{1}{M})r^2}{1-(1-\frac{1}{M})r^2}, 0)$$

and the radius

$$\rho(r) = \frac{2(1-\frac{1}{M})r}{1-(1-\frac{1}{M})r^2}$$

and again using the subordination principle, then we have

$$\left| \frac{h(z)}{g(z)} - \frac{1+(1-\frac{1}{M})r^2}{1-(1-\frac{1}{M})r^2} \right| \leq \frac{2(1-\frac{1}{M})r}{1-(1-\frac{1}{M})r^2}. \tag{2.5}$$

The last inequality gives (2.2). ☐

Theorem 2.2. *The radius of starlikeness of the class* $S_{LH(M)}$ *is* $r = \frac{1}{1+\sqrt{2-\frac{1}{M}}}$.

Proof. Using (2.5) we have,

$$z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} = \frac{(2-\frac{1}{M})z\phi'(z)}{1-\frac{1}{M}\phi(z)-(1-\frac{1}{M})(\phi(z))^2}. \tag{2.6}$$

Therefore

$$\left|z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}\right| \le \frac{(2-\frac{1}{M})r}{1-r^2}\frac{1-|\phi(z)|^2}{1-\frac{1}{M}|\phi(z)|-(1-\frac{1}{M})|\phi(z)|^2}, \tag{2.7}$$

where we used the estimate

$$|\phi'(z)| \le \frac{(1-|\phi(z)|^2)}{1-r^2}.$$

This estimate can be found in [1]. It can be easily shown that,

$$\frac{1-|\phi(z)|^2}{1-\frac{1}{M}|\phi(z)|-(1-\frac{1}{M})|\phi(z)|^2} \le \frac{1-r^2}{1-\frac{1}{M}r^2-(1-\frac{1}{M})r^2}, \tag{2.8}$$

and relations (2.7) and (2.8) give

$$\left|z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}\right| \le \frac{(2-\frac{1}{M})r}{1-\frac{1}{M}r-(1-\frac{1}{M})r^2}. \tag{2.9}$$

On the other hand we have

$$\Phi(z) = z\frac{h(z)}{g(z)}, f = zh(z)\overline{g(z)} \Rightarrow$$

$$Re\frac{zf_z - \overline{z}f_{\overline{z}}}{f} = Rez\frac{\Phi'(z)}{\Phi(z)} = Re[1 + z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}],$$

and we deduce that f will be univalent and starlike if

$$\left|z\frac{\Phi'(z)}{\Phi(z)} - 1\right| < \left|z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}\right| < 1 \tag{2.10}$$

Using (2.9) we deduce that (2.10) will be satisfied if

$$\frac{(2-\frac{1}{M})r}{1-\frac{1}{M}r-(1-\frac{1}{M})r^2} < 1,$$

and this implies that

$$r = |z| < \frac{1}{1+\sqrt{2-\frac{1}{M}}}.$$

☐

Theorem 2.3. *Let* $f = zh(z)\overline{g(z)}$ *be an element of* $S_{LH(M)}$, *then*

$$\sum_{k=1}^{n}|a_k - b_k|^2 \le (2-\frac{1}{M})^2 + \sum_{k=1}^{n-1}\left|b_k + (1-\frac{1}{M})a_k\right|^2. \tag{2.11}$$

Proof. The proof of this theorem is based on the Clunie method [8]. We start with the equality,

$$\frac{h(z)}{g(z)} = \frac{1+\phi(z)}{1-(1-\frac{1}{M})\phi(z)} \Leftrightarrow h(z)-g(z) = [(1-\frac{1}{M})h(z)+g(z)]\phi(z)\,. \tag{2.12}$$

Then let $h(z) = 1+\sum_{n=1}^{\infty} a_n z^n$, $g(z) = 1+\sum_{n=1}^{\infty} b_n z^n$, $\phi(z) = \sum_{n=1}^{\infty} c_n z^n$ and note that $|\phi(z)| < 1$, for all $z \in \mathbb{D}$. Therefore equation (2.12) now takes the form,

(2.13)

$$\sum_{k=1}^{n}(a_k-b_k)z^k + \sum_{k=n+1}^{\infty}(a_k-b_k)z^k = [(a+1)+\sum_{k=1}^{n-1}(b_k+aa_k)z^k+\sum_{k=n}^{\infty}(b_k+aa_k)z^k](\sum_{k=1}^{\infty}c_k z^k)\,,$$

where $a = 1-\frac{1}{M}$ or

(2.14)

$$\sum_{k=1}^{n}(a_k-b_k)z^n + \sum_{k=n+1}^{\infty}(a_k-b_k)z^k - (\sum_{k=n}^{\infty}(b_k+aa_k)z^k)(\sum_{n=1}^{\infty}c_n z^n) = [(1+a)+\sum_{k=1}^{n-1}(b_k+aa_k)z^k]\phi(z)\,.$$

Equality (2.14) can be written in the following form:

$$\sum_{k=1}^{n}(a_k-b_k)z^k + \sum_{k=n+1}^{\infty} d_k z^k = [(1+a)+\sum_{k=1}^{n-1}(b_k+aa_k)z^k]\phi(z)\,. \tag{2.15}$$

Since (2.15) has the form $F(z) = \phi(z).G(z)$, where $|\phi(z)| < 1$, it follows that

$$\frac{1}{2\pi}\int_0^{2\pi}\left|F(re^{i\theta})\right|^2 d\theta \le \frac{1}{2\pi}\int_0^{2\pi}\left|G(re^{i\theta})\right|^2 d\theta\,, \tag{2.16}$$

for each r, $(0 < r < 1)$. Expressing (2.16) in terms of the coefficients in (2.15) we obtain the inequality

$$\sum_{k=1}^{n}|a_k-b_k|^2 r^{2k} + \sum_{k=n+1}^{\infty}|a'_k|^2 r^{2k} \le |1+a|^2 + \sum_{k=1}^{n-1}|aa_k+b_k|^2 r^{2k} \tag{2.17}$$

By letting $r \to 1$ in (2.17) we conclude that

$$\sum_{k=1}^{n}|a_k-b_k|^2 \le (2-\frac{1}{M})^2 + \sum_{k=1}^{n-1}\left|(1-\frac{1}{M})a_k+b_k\right|^2\,,$$

proving the theorem. □

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