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MULTI-WILSON SYSTEMS

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Abstract. We make use of sections of Wilson bases to construct a frame on $L^2(\mathbb{R})$. We prove the frame inequality for the multi-Wilson system and provide one dual frame; however, this dual frame does not have a multi-Wilson structure.

1. Introduction

Atomic decompositions of functions, successfully done via Gabor frames [8, 2] and Wilson bases [3], arise great research interest among mathematicians and engineers in the past decades. Motivated by the construction of multi-window Gabor frames [4], we compose a frame, using sections of Wilson bases, generated from tight Gabor frames with different canonical generator matrices (CGMs) [9]. As a result, we can adapt the local structure of our frame (to the local needs of the signal) and obtain *a multi-Wilson frame* with time-varying quality. Such frames can be beneficial when used on functions with variable bandwidth [1]. We restrict our work within this paper to functions in $L^2(\mathbb{R})$, but expect similar results to hold in weighted modulation spaces [5, 6, 8], since since Wilson bases construction is possible [7].

2. Preliminaries

In this paper $\mathcal{F}$ denotes the Fourier transform, a well-defined operator on $L^2(\mathbb{R})$. Given a function $f \in L^2(\mathbb{R})$, the *chirp*, *dilation*, *translation* and *modulation* operators are given by $\mathcal{N}_c f(t) = e^{-\pi i c t^2} f(t)$, $\mathcal{D}_b f(t) = |b|^{1/2} f(bt)$, $T_x f(t) = f(t-x)$ and $M_\omega f(t) = e^{2\pi i \omega t} f(t)$ for $t, x, \omega, b, c \in \mathbb{R}$.

Our construction of multi-Wilson systems/frames requires the use of a *bounded admissible partition of unity* (BAPU), consisting of non-negative, compactly supported functions $\phi_j \in L^2(\mathbb{R})$, $j \in \mathcal{I}$. We choose the family

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$$\{\phi_j\}_{j\in\mathcal{I}} \text{ such that } 0 \le \phi_j \le 1 \text{ for all } j \in \mathcal{I} \tag{2.1}$$

such that only direct neighbors have nonzero overlap. In addition, we assume

$$\sum_{j\in\mathcal{I}} \phi_j \equiv 1; \quad \text{hence for all } f \in L^2(\mathbb{R}) \text{ it holds} \quad f = \sum_{j\in\mathcal{I}} \phi_j f. \tag{2.2}$$

Under all the listed conditions, we call the family (2.1) a BAPU on $\mathbb{R}$. Observe that for each function $f \in L^2(\mathbb{R})$ it holds

$$\|f\|_2^2 \le \sum_{j\in\mathcal{I}} \|\phi_j f\|_2^2 \ \text{ and } \ \|\phi_j f\|_2 \le \|f\|_2 \text{ for all } j \in \mathcal{I}. \tag{2.3}$$

A *lattice* $\Lambda \subset \mathbb{R}^2$ is a discrete subgroup such that $\Lambda = \mathbb{A}\mathbb{Z}^2$. The matrix $\mathbb{A}$ is called the *canonical generator matrix* (CGM) and it shows it is unique [9]. We restrict our work on matrices with volume $\mathrm{vol}(\Lambda) = |det(\mathbb{A})| = 1/2$.

For $\lambda = (x, \omega) \in \mathbb{R}^2$ and $g \in L^2(\mathbb{R})$, let a *time-frequency shift* g_λ of a function g be defined by $g_\lambda = g_{x,\omega} = M_\omega T_x\, g$. The time-frequency shift is used to define the *short-time Fourier transform* (STFT) of a function $f \in L^2(\mathbb{R})$ with respect to a *window* function $g \in L^2(\mathbb{R})$ as

$$V_g f(x,\omega) = \langle f, M_\omega T_x g \rangle .$$

Whenever $f, g \in L^2(\mathbb{R})$, it holds

$$\|V_g f\|_2 = \|f\|_2 \|g\|_2. \tag{2.4}$$

We denote by $\mathcal{G} = \mathcal{G}(g, \Lambda)$ the *Gabor system* of shifts $\{g_\lambda \mid \lambda = (x, \omega) \in \Lambda\}$. As usual, the redundancy of $\mathcal{G}(g, \Lambda)$ is given by $1/\mathrm{vol}(\Lambda)$. $\mathcal{G}$ is called a *Gabor frame* for $L^2(\mathbb{R})$, if it satisfies the *frame inequality* i.e. there exist $A > 0$ (*lower frame bound*) and $B > 0$ (*Bessel bound*) such that for all $f \in L^2(\mathbb{R})$ it holds

$$A\|f\|_2 \le \left(\sum_{\lambda\in\Lambda} |\langle f, g_\lambda \rangle\, g_\lambda|^2 \right)^{1/2} \le B\|f\|_2. \tag{2.5}$$

If $A = B$, then we have a tight frame. If in addition $A = 1$, then we have a basis at our hands.

For our purposes, we work with tight Gabor frames with redundancy 2 ($A = 2$), since these are suitable to compose Wilson bases [3] with elements composed out of Gabor windows as in (3.3). The frame operator S, defined by $Sf = \sum_{\lambda\in\Lambda} \langle f, g_\lambda\rangle g_\lambda$, is positive definite and bounded. A *dual* frame $\{\tilde{g_\lambda}\}_{\lambda\in\Lambda}$ for $\mathcal{G}(g, \Lambda)$, with Gabor structure on the same lattice Λ, always exists and it holds $f = \sum_\Lambda \langle f, \tilde{g_\lambda}\rangle g_\lambda = \sum_\Lambda \langle f, g_\lambda\rangle\, \tilde{g_\lambda}$ for every $f \in L^2(\mathbb{R})$. We refer to [2, 8] as a good source for frames and Gabor frames.

The construction of a multi-window Gabor frame [4], is possible on $L^2(\mathbb{R})$ out of tight Gabor frames. We cite here a result from [4] regarding compactly supported windows.

Theorem 2.1. [4] *Let $\mathcal{G}_j = \mathcal{G}(g^j, \Lambda_j)$, $j \in \mathcal{I}$, be a family of tight Gabor frames for $L^2(\mathbb{R})$, generated by compactly supported windows g_j and let $\{\varphi_j\}_{j\in\mathcal{I}}$ be a BAPU defined with* (2.1)*, which satisfy* (2.2) *and* (2.3)*. Set the index sets $\chi_j \subset \Lambda_j$, $j \in \mathcal{I}$, be such that $\chi_j = \Lambda_j \cap (\mathrm{supp}(\varphi_j) \times \mathbb{R})$. Then the union of subfamilies*

$$\cup_{j\in\mathcal{I}}\mathcal{G}(g^j, \chi_j) \tag{2.6}$$

is a frame for $L^2(\mathbb{R})$.

3. Wilson basis with a non-standard lattice

We first recall the result from [9] about the construction of a Wilson basis on a lattice Λ, related to a CGM $\mathbb{A}$. If $\mathcal{G}(g, \Lambda)$ is a Gabor system of redundancy 2 and

$$\mathcal{A} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \tag{3.1}$$

is the CGM for Λ, with $a, b, d \in \mathbb{R}$, then the associated Wilson system

$$\mathcal{W} = \mathcal{W}_{\Lambda,g} = \{\psi^{\Lambda}_{m,n}\}_{m\in\mathbb{Z},n\geq 0} \tag{3.2}$$

consists of the functions

$$\psi^{\Lambda}_{k,n} = c_{k,n}\left(g_{ka+nb,nd} + (-1)^{n+k} g_{ka-nb,-nd}\right) \tag{3.3}$$

with $c_{k,0} = \frac{1}{2}$, $c_{k,n} = \frac{1}{\sqrt{2}}e^{-i\pi bdn^2}$ when $n+k$ is even, and $c_{k,n} = \frac{i}{\sqrt{2}}e^{-i\pi bdn^2}$ otherwise. Notice that the lattice for the Wilson system $\mathcal{W}$ is symmetric with respect to the x-axis and we shall denote it by $\Lambda_{\mathcal{W}}$.

Theorem 3.1. [9] *Let Λ be a lattice in $\mathbb{R}^2$, with $\mathrm{vol}(\Lambda) = 1/2$ and CGM is as in* (3.1)*. Define U by $U = \mathcal{D}_{1/d} \circ \mathcal{F} \circ \mathcal{N}_{-b/d} \circ \mathcal{F}^{-1}$ and let $g \in L^2(\mathbb{R})$ be such that $\mathcal{F}(Ug)$ is real-valued and $\{g_{ka+nb,nd}\}_{k,n\in\mathbb{Z}}$ is a tight frame for $L^2(\mathbb{R})$ with frame bound 2. Then the system* (3.2) *is an orthonormal basis for $L^2(\mathbb{R})$.*

The definition of a Wilson basis we use here, reduces to the usual one ([8], Def. 8.5.1), with taking $a = 0.5$, $b = 0$, $d = 1$ in (3.1).

The frame operator is a simple multiplication operator

$$f = \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{N}}\langle f, \psi_{k,n}\rangle\psi_{k,n} \tag{3.4}$$

that converges unconditionally. In other words, finite linear combinations of $\mathcal{W}_{\Lambda,g}$ are dense in $L^2(\mathbb{R})$.

4. The frame inequality on $L^2(\mathbb{R})$

We start with a collection of Wilson bases $\mathcal{W}_j$ on $L^2(\mathbb{R})$ of form (3.2), each generated with the technology from Theorem 3.1, by a tight Gabor frame $\mathcal{G}_j = \mathcal{G}(g_j, \Lambda_j)$, for $j \in \mathcal{I}$. Notice that the set of indexes $\mathcal{I}$ may be finite, or at most countable. Each $\mathcal{W}_j$ is also a tight frame, that is

$$\|f\|_2^2 = \sum_{\Lambda_j} | \left\langle f, \psi^j_{k,n} \right\rangle |^2, \tag{4.1}$$

where Λ_j is the lattice related to $\mathcal{W}_j$, $\mathcal{G}_j$ and $A_j = \begin{bmatrix} a_j & b_j \\ 0 & d_j \end{bmatrix}$.

We use (2.1) to cut out χ_j, a lattice section of Λ_j on which each ϕ_j from (2.1) is non-zero and denote with $\mathcal{W}_{\chi_j}$ the respective subset of $\mathcal{W}_j$. We apply (2.1), (2.3) and (4.1) in the following calculation of the **lower bound**:

$$\begin{aligned} \|f\|_2^2 &\leq \sum_{j\in\mathcal{I}} \|\phi_j f\|_2^2 = \sum_{j\in\mathcal{I}} \|\phi_j \sum_{\Lambda_j} \left\langle f, \psi^j_{k,n} \right\rangle \psi^j_{k,n}\|_2^2 \\ &= \sum_{j\in\mathcal{I}} \|\phi_j \sum_{\chi_j} \left\langle f, \psi^j_{k,n} \right\rangle \psi^j_{k,n}\|_2^2 \leq \sum_{j\in\mathcal{I}} \| \sum_{\chi_j} \left\langle f, \psi^j_{k,n} \right\rangle \psi^j_{k,n}\|_2^2 \\ &= \sum_{j\in\mathcal{I}} \|h_j\|_2^2 = \sum_{j\in\mathcal{I}} \sum_{\Lambda_j} | \left\langle h_j, \psi^j_{k,n} \right\rangle |^2 = \sum_{j\in\mathcal{I}} \sum_{\chi_j} | \left\langle f, \psi^j_{k,n} \right\rangle |^2. \end{aligned} \tag{4.2}$$

Iquality (4.2) holds due to the orthogonality property of each $\mathcal{W}_j$. An easy calculation shows that each function

$$h_j = \sum_{\chi_j} \left\langle f, \psi^j_{k,n} \right\rangle \psi^j_{k,n}$$

has Wilson coefficients $\left\langle h_j, \psi^j_{k,n} \right\rangle = \left\langle f, \psi^j_{k,n} \right\rangle$ on the lattice section χ_j, and $\left\langle h_j, \psi^j_{k,n} \right\rangle = 0$ out of χ_j within each fixed Wilson basis $\mathcal{W}_j$.

In order to prove the **Bessel bound** for the general multi-Wilson system, we need to observe that, for each j, k, n fixed in (3.3), we have $2c^2_{k,n} \leq 1$ and it holds

$$| \left\langle f, \psi^j_{k,n} \right\rangle |^2 \leq \left| \left\langle f, g^j_{ka+nb,nd} \right\rangle \right|^2 + \left| \left\langle f, g^j_{ka-nb,-nd} \right\rangle \right|^2. \tag{4.3}$$

Applying the inequality (4.3) in (4.2), we obtain a frame inequality

$$\begin{aligned} \|f\|_2^2 &\leq \sum_{j\in\mathcal{I}} \sum_{\chi_j} | \left\langle f, \psi^j_{k,n} \right\rangle |^2 \leq \sum_{j\in\mathcal{I}} \sum_{\chi_j} \left| \left\langle f, g^j_{ka+nb,nd} \right\rangle \right|^2 \\ &+ \sum_{j\in\mathcal{I}} \sum_{\chi_j} \left| \left\langle f, g^j_{ka-nb,-nd} \right\rangle \right|^2 \leq 2B^2 \|f\|_2^2 . \end{aligned} \tag{4.4}$$

In the last inequality, B denotes the Bessel bound for the respective multi-window Gabor system, which exist due to Theorem 2.1. We have proved the following theorem:

Theorem 4.1. *Let $\mathcal{W}_j$, $j \in \mathcal{I}$ be Wilson bases for $L^2(\mathbb{R})$, each generated via a tight Gabor frame $\mathcal{G}_j = \mathcal{G}(g_j, \Lambda_j)$ with redundancy 2. Let $\{\phi_j\}_{j\in\mathcal{I}}$ form a BAPU of compactly supported atoms ϕ_j. Let B denote the Bessel bound for the respective multi-window Gabor system, related to $\{\phi_j\}_{j\in\mathcal{I}}$ and the sub-lattices*

$$\chi_j = (\mathrm{supp}(\phi_j) \times \mathbb{R}) \cap \Lambda_j \quad \textit{for each } \; j \in \mathcal{I}.$$

Then the multi-Wilson system

$$\bigcup_{j\in\mathcal{I}} \mathcal{W}_{\chi_j} = \bigcup_{j\in\mathcal{I}} \{\psi^j_{m,n}\}_{(m,n)\in\chi_j} \tag{4.5}$$

is a frame for $L^2(\mathbb{R})$ and it holds

$$\|f\|_2 \leq \left(\sum_{j\in\mathcal{I}} \sum_{\chi_j} | \left\langle f, \psi^j_{k,n} \right\rangle |^2 \right)^{1/2} \leq B\sqrt{2}\, \|f\|_2 . \tag{4.6}$$

Locally, at each χ_j this frame obviously acts as a **basis**. Also, note that from the simple equality

$$f = \sum_{j\in\mathcal{I}} \sum_{(k,n)\in\chi_j} \varphi_j \left\langle f, \psi^j_{k,n} \right\rangle \psi^j_{k,n} = \sum_{\cup\chi_j} \left\langle f, \psi^j_{k,n} \right\rangle \varphi_j \psi^j_{k,n}$$

we derive one dual frame

$$\left\{ \varphi_j \psi^j_{k,n} : j \in \mathcal{I}, \psi^j_{k,n} \in \mathcal{W}_{\chi_j} \right\}, \tag{4.7}$$

which has a distorted multi-Wilson structure.

Remark 4.1. *More generally, due to* (4.3)*, if we have a collection of tight Gabor frames, such that their multi-window Gabor system (composed in any other way) is a frame, then the multi-Wilson system is also a frame.*

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