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SOLVING SOME FRACTIONAL ORDER DIFFERENTIAL EQUATIONS BY MEANS OF INTEGRAL TRANSFORMS AND SPECIAL FUNCTIONS

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Abstract. In this paper we propose some methods for solving fractional order differential equations with variable coefficients. To this aim, we consider suitable generalizations of the classical integral transforms of Fourier, Mellin and Laplace, and study their basic properties. First we consider some ordinary differential equations of fractional order with variable coefficients. The solutions obtained by means of integral transforms are expressed in terms of special functions, as the Wright function and 1- and 2-parametric Mittag-Leffler functions. The method of integral transforms is used also to solve partial differential equations of the same kind. Namely, a generalized Laplace transform is applied to solve the so-called Giona equation and the fractional wave equation. Special attention is paid also to the generalized fractional heat equation involving a generalization of the Riemann-Liouville fractional derivative. A combined application of the Laplace transform and of the generalized Fourier transform leads to a solution of Cauchy problem for this equation in explicit integral form, where the kernel is represented by the 1-parametric Mittag-Leffler function.

1. Introduction

The theory of integrals and derivatives of arbitrary order, usually referred to as Fractional Calculus (see for example [4], [15]), is one of the most intensively developing areas of mathematical analysis as a result of its increasing range of applications in biology, physics, electrochemistry, economics, probability theory and statistics. The operators for fractional differentiation and integration have been used in various fields such as hydraulics of dams, potential fields, diffusion problems and waves in liquids and gases. The main advantage of the fractional calculus is that the fractional order models provide an instrument for the adequate description of the memory and hereditary properties of materials and processes. In 1980 Namias [7] introduced the Fractional Fourier transform (FRFT) as a way to solve certain classes of ordinary and partial differential equations

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appearing in quantum mechanics. Most probably, Namias was unaware of Wiener's paper [16] published in 1929, where the FRFT in the form of fractional powers of Fourier operator has been already studied. The FRFT [6] became more popular after 1995 because of its numerous applications in quantum mechanics, chemistry, optics, dynamical systems, stochastic processes and signal processing.

The main goal of this paper is to apply the techniques of some generalized integral transforms for solving some fractional differential equations with variable coefficients. Since there are no common methods for solving fractional differential equations, specially of this kind, here we propose some four approaches based on integral transforms useful for this purpose.

2. Preliminaries

For our purposes we use the notions of *Riemann-Liouville* $(R-L)$ *fractional derivative* and *Riemann-Liouville fractional integral* of a function $f(x)$ of order α, resp. of order $n-\alpha$, defined as follows:

$$ {}_0D_x^{\alpha} f(x) = \frac{d^n}{dx^n}\left[{}_0D_x^{-(n-\alpha)} f(x)\right] \ , \ {}_0D_x^{-(n-\alpha)} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\tau)^{n-\alpha-1} f(\tau) d\tau, $$

where n is positive integer, and $n-1 \leq \alpha < n$.

Let us mention that in a contrast to the classical calculus, if $\alpha \geq 0$, $x > 0$ and $\beta > -1$, then the fractional derivative of the power function x^{β} is given by ${}_0D_x^{\alpha} x^{\beta} = \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}$, that in the particular case $\beta = 0$ and $0 \leq \alpha < 1$ implies ${}_0D_x^{\alpha} 1 = \dfrac{x^{-\alpha}}{\Gamma(1-\alpha)}$. Thus, generally the R-L derivative of a constant is not zero (as in the classical calculus). It is of primary importance for our later considerations to outline the composition property valid for $\alpha > 0$, $x > 0$, $n \in N$, $n-1 \leq \alpha < n$, and $n-1 \leq \beta < n$,

$$ {}_0D_x^{-\alpha}[{}_0D_x^{\alpha} f(t)] = f(t) - \sum_{k=1}^{n} \left[{}_0D_x^{\alpha-k} f(t)\right]_{t=0} \frac{t^{\alpha-k}}{\Gamma(\alpha-k+1)} \, . $$

Following an idea developed in [1], we consider the *generalized Laplace-type integral transform* defined as $L_A^{\Phi}[f(x);p] = \int_0^{\infty} A'(x) e^{-A(x)\Phi(p)} f(x) dx$, where p is complex variable and $f(x)$ is piecewise continuous function such that $|f(x)| \leq M e^{A(x)\Phi(c)}$, for some constants M and c. The functions $A(x)$ and Φ are supposed to be increasing invertible functions with $A(0) = 0$. Obviously, from here the classical Laplace transform follows as a very special case $(A(x) = x, \Phi(p) = p)$.

For a function u of the class S of a rapidly decreasing test functions on the real axis $\mathbb{R}$, the Fourier transformation is defined as $\hat{u}(\omega) = F[u(t);\omega] = \displaystyle\int_{-\infty}^{\infty} u(t) e^{i\omega t} dt$, $\omega \in \mathbb{R}$, where the inverse Fourier transformation has the form $u(t) = F^{-1}[\hat{u}(\omega);t] = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \hat{u}(\omega) e^{-i\omega t} d\omega$

$t \in \mathbb{R}$. Denote further by $V(R)$ the set of functions $v(t) \in S$, satisfying the conditions $\hat{u}_\alpha(\omega) = F \left.\frac{d^n v}{dt^n}\right|_{t=0} = 0$, $n = 0, 1, 2, \ldots$. The Fourier pre-image of the space $V(R)$, i.e. $\Phi(\mathbf{R}) = \{\varphi \in S : \hat{\varphi} \in V(\mathbf{R})\}$, is called the Lizorkin space. As it is stated in [15], the Lizorkin space is invariant with respect to the fractional integration and differentiation operators. The notion of *Fractional Fourier transformation* (FRFT) was introduced also in Podlubny [15]. For a function $u \in \Phi(\mathbb{R})$ the FRFT $\hat{u}_\alpha$ of the order α $(0 < \alpha \leq 1)$ is defined as

$$\hat{u}_\alpha(\omega) = F_\alpha[u(t);\omega] = \int_{-\infty}^{\infty} e_\alpha(\omega,t)u(t)dt,\ \omega \in \mathbb{R},\ \text{where } e_\alpha(\omega,t) := \begin{cases} e^{-i|\omega|^{1/\alpha}t}, & \omega \leq 0 \\ e^{i|\omega|^{1/\alpha}t}, & \omega > 0 \end{cases}.$$

If $\alpha = 1$, the kernel e_α coincides with the kernel of the classical Fourier transformation: $e_1(\omega,t) := \begin{cases} e^{-i|\omega|t}, & \omega \leq 0 \\ e^{i|\omega|t}, & \omega > 0 \end{cases} \equiv e^{i\omega t}$, $\omega \in \mathbb{R}$. Evidently, the fractional Fourier transformation of the order 1 coincides with the Fourier transformation: $F_1 \equiv F$.

The relation between the fractional and conventional Fourier transformation is given by the following simple formula:

$$\hat{u}_\alpha(\omega) = F_\alpha[u(t);\omega] \equiv F_1[u(t);w] = \hat{u}(w), \quad \text{where } w = \begin{cases} -|\omega|^{1/\alpha}, & \omega \leq 0 \\ |\omega|^{1/\alpha}, & \omega > 0 \end{cases}.$$

Thus, if $F_\alpha[u(t);\omega] = F_1[u(t);w] = \phi(w)$, then $u(t) := {F_\alpha}^{-1}[\hat{u}_\alpha(\omega);t] = F_1^{-1}[\phi(w);t]$.

We study the fractional diffusion equation in terms of the Caputo fractional derivative [2]:

$$ {}_aD_*^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_a^x \frac{f^{(m)}(\tau)}{(x-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m \\ D_x^m f(x), & \alpha = m \end{cases},$$

where m is a positive integer. The method we follow uses the rule for the Laplace transform of Caputo derivative, see e.g. [15]:

$$L[{}_0D_*^\alpha f(x);s] = s^\alpha L[f(t);s] - \sum_{k=0}^{m-1} s^{\alpha-k-1} f_{(0)}^{(k)}, \quad m-1 < \alpha < m.$$

We use also a *generalized fractional derivative operator* of the form $D_\beta^\alpha u(x) = (1-\beta)D_+^\alpha u(x) - \beta D_-^\alpha u(x)$, $0 < \alpha \leq 1$, $\beta \in \mathbf{R}$ where D_+^α and D_-^α are the Riemann-Liouville fractional derivatives on the real axis, given as

$$D_+^\alpha u(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\infty}^{x}(x-t)^{\alpha-1}u(t)dt, \quad D_-^\alpha u(x) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x}^{+\infty}(t-x)^{\alpha-1}u(t)dt.$$

A key role in our considerations is given to a relation established in Luchko, Martinez and Trujillo [5], according to which for $0 < \alpha \leq 1$, any value of β and a function $u(x) = \Phi(\mathbf{R})$, $F_\alpha[D_\beta^\alpha u(x);\omega] = (-ic_\alpha\omega)F_\alpha[u(x);\omega]$, $\omega \in \mathbf{R}$ and $c_\alpha = \sin(\alpha\pi/2) + i\,\text{sign}\omega(1-2\beta)\cos(\alpha\pi/2)$. If $0 < \alpha \leq 1$ and the variable ð is a complex number, such that

$p = \begin{cases} \text{Re}\,p - i|\omega|^{1/\alpha}, & \omega \leq 0 \\ \text{Re}\,p + i|\omega|^{1/\alpha}, & \omega > 0 \end{cases}$, then the integral transform of the form $M_\alpha\{f(t);p\} = M(p\text{sign}\omega) = \int\limits_0^\infty t^{p\ \text{sign}\omega - 1} f(t)dt$, is called α-Mellin transform of the function $f(t)$.

Using this definition, we prove some basic properties of the α-Mellin transforms, see for example Nikolova [8, 9]. Namely, in [8] we give a generalization of the classical Mellin transformation, see also Debnath et al. [3].

3. Main results

3.1. Solving a fractional order differential equation by the α-Mellin transform. We consider the *Bessel differential equation of fractional order*, called also generalized Bessel differential equation:

$$t^{\beta+1}{}_0D_t^{\beta+1}y(t) + t^{\beta}{}_0D_t^{\beta}y(t) = f(t), \quad 0 < \beta < 1, t > 0,$$

and look for the solution of the boundary value problem for this equation with the conditions $y(0) = y'(0) = 0$, $y(\infty) = y'(\infty) = 0$. The solution is found by applying the α-Mellin and the inverse α-Mellin transforms consecutively, and it has the following form:

$$y(t) = \int\limits_0^\infty f(t\tau)g(\tau)d\tau,$$

where $g(t) = M_\alpha^{-1}\{G(p\text{sign}\omega);t\}$, $G(p\text{sign}\omega) = \dfrac{\Gamma(p\text{sign}\omega - \beta)}{(p\text{sign}\omega - \beta)\Gamma(p\text{sign}\omega)}$. The detailed results are given in the paper of Nikolova [9].

3.2. Solving a fractional order differential equation by generalized Laplace-type integral transform. We solve now the *two-terms fractional differential equation* of the form

$${}_0D_x^{\beta}[x^{2\mu}f(x)] - a^{\mu}{}_0D_x^{\alpha}f(x) = 0, \quad \text{where} \;\; n-1 \leq \beta < n,\, n \in N,\, a > 0,$$

and α is real number such that $n - 1 \leq \alpha < \beta < n$ and $\mu = \beta - \alpha$. The particular case $\beta = n$ of this equation was studied by Nonnenmacher. First, we start by solving this equation for the particular case as $a = \pi$ and $\mu = \frac{1}{2}$. Then the solution has the form $f(x) = x^{-3/2}e^{-\pi/x}$. In general, if $f(x)$ is piecewise continuous on $(0, +\infty)$ and integrable on any finite subinterval of $[0, +\infty)$, this two-terms fractional differential equation with variable coefficients is solvable, and its solution is given by $f(x) = \dfrac{a^{\mu}}{\Gamma(\mu)}x^{-\mu-1}e^{-a/x}$.

This solution is obtained by using the Laplace transform, the inverse Laplace transform (for details see e.g. [14]), the property of the Γ - function and the generalized Laplace-type integral transform defined as L_A^{Φ} [1].

Details on these results are can be found in the paper of Nikolova [12].

3.3. **Using a generalization of the Frobenius method.** We consider the fractional differential equation of the form

$$x^{2\alpha}{}_0D_x^{2\alpha}y(x) + (x^{2\alpha} + ax^{\alpha} + b)y(x) = 0, \text{ where } x > 0,\ 0 < \alpha \leq 1,\ a, b \in R,$$

and ${}_0D_x^{2\alpha}$ is the Riemann-Liouville fractional derivative of order 2α. Then we take ρ such that $\rho - \alpha > -1$ and $\dfrac{\Gamma(\alpha+\rho+1)}{\Gamma(-\alpha+\rho+1)} + b = 0$. If this equation has a solution given by convergent power series, then the solution has the form $y(x) = \sum\limits_{k=1}^{\infty} a_k(\alpha,\rho)x^{k\alpha+\rho}$, where the coefficients $\alpha_k(\alpha,\rho)$ satisfy the recurrence relations $a_2(\alpha,\rho)\left[\dfrac{\Gamma(2\alpha+\rho+1)}{\Gamma(\rho+1)} + b\right] + a.a_1(\alpha,\rho) = 0$, and for $k \geq 3$, $a_k(\alpha,\rho)\left\{\dfrac{\Gamma(k\alpha+\rho+1)}{\Gamma[(k-2)\alpha+\rho+1]} + b\right\} + a.a_{k-1}(\alpha,\rho) + a_{k-2}(\alpha,\rho) = 0$. By means of the Frobenious method, the fractional Coulomb equation is solved and the solution is obtained in power series form. Namely, we consider the Coulomb wave equation $\dfrac{d^2y}{dx^2} + \left[1 - \dfrac{2\eta}{x} - \dfrac{m(m+1)}{x^2}\right]y = 0$, that evidently is a particular case of the above fractional equation when $\alpha = 1$, $a = -2\eta$, $\rho = m$, $m = 1, 2$, and $b = -m(m+1)$. The solution of the fractional equation reduces in this particular case to the regular solution of the Coulomb wave equation of the form

$$y(x) = F_m(\eta, x) = C_m(\eta)x^{m+1}\sum_{k=m+1}^{\infty} A_{k,m}x^{k-m-1},$$

where

$$A_{m+1,m} = 1;\ A_{m+2,m} = \frac{\eta}{m+1};\ (k+m)(k-m-1)A_{k,m} = 2\eta\ A_{k-1,m} - A_{k-2,m} \quad (k > m+2)$$

and

$$C_m(\eta) = \frac{2^m e^{-\pi\eta/2}\,|\Gamma(m+1+i\eta)|}{\Gamma(2m+2)}.$$

The detailed results are described in the paper of Nikolova et al. [13].

3.4. **Solving a generalized fractional diffusion equation by fractional Fourier transform (FRFT).** We consider the Cauchy-type problem for the *fractional diffusion equation*

$$D_*^{\alpha}u(x,t) = \mu\frac{\partial^2 u}{\partial x^2},\quad x \in \mathbb{R},\ t > 0,$$

subject to the initial condition $u(x,t)|_{t=0} = f(x)$, where D_*^{α} is the Caputo time-fractional derivative of order α, $f(x) \in \Phi(\mathbf{R})$ and μ is a diffusivity constant. The solution for this fractional equation is given by the integral

$$u(x,t) = \int_{-\infty}^{\infty} G(x-\xi,t)f(\xi)d\xi,$$

where $G(x,t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{-iwx}E_{\alpha}(-\mu|w|^{\frac{2}{\alpha}}t^{\alpha})dw$. The solution is obtained by applying the Laplace transform to the Caputo derivative (see [15]) and FRFT to the equation and the initial condition. If $\alpha = 1$, the solution of the Cauchy-type problem for above diffusion equation is given by the well-known integral formula

$$u(x,t) = \frac{1}{\sqrt{4\pi\mu t}} \int_{-\infty}^{\infty} e^{-(x-\varsigma)^2/4\mu t} f(\xi) d\xi .$$

Consider the Cauchy-type problem for the generalization of the fractional diffusion equation $D_*^\gamma u(x,t) = \mu D_\beta^{\alpha+1} u(x,t)$, subject to the initial condition $u(x,t)|_{t=0} = f(x)$, where D_*^γ is the Caputo time-fractional derivative of order γ, whereas $D_\beta^{\alpha+1}$ is the space- fractional derivative. If $0 < \gamma \leq 1, 0 < \alpha \leq 1$ and for each value of $\beta \in R$, the Cauchy-type problem for above fractional diffusion equation is solvable and its solution $u(x,t)$ is given by the integral $u(x,t) = \int_{-\infty}^{\infty} G(x-\xi,t) f(\xi) d\xi$, where

$$G(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iwx} E_\gamma(-i\mu c_{\alpha+1} w t^\gamma) dw .$$

The solution is obtained by application of the Laplace transform to the Caputo derivative and FRFT $F_{\alpha+1}$ to the above equation and to the initial condition.

We consider also *a generalization of the time-space diffusion equation* involving the generalized fractional derivative

$$D_\beta^\alpha u(x) = (1-\beta) D_+^\alpha u(x) - \beta D_-^\alpha u(x), \quad 0 < \alpha \leq 1, \ \beta \in \mathbf{R} ,$$

where D_+^α and D_-^α are the R-L fractional derivatives on the real axis. For $\mu = 1$ and $\beta = 1/2$ the solution of the space-time diffusion equation is studied by Luchko, Martinez, Trujillo [5].

The details on our results are given in the papers [10, 11].

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