ADV MATH SCI JOURNAL

Advances in Mathematics: Scientific Journal **1** (2012), no.2, 127–131
ISSN 1857-8365, UDC: 517.546.12

NOTES ON NEW CLASS FOR CERTAIN ANALYTIC FUNCTIONS

KAZUO KUROKI AND SHIGEYOSHI OWA

Presented at the 8^{th} International Symposium GEOMETRIC FUNCTION THEORY AND APPLICATIONS, 27-31 August 2012, Ohrid, Republic of Macedonia.

Abstract. By considering a certain univalent function in the open unit disk $\mathbb{U}$ which maps $\mathbb{U}$ onto the strip domain w with $\alpha < \operatorname{Re} w < \beta$, some properties for a new class of certain analytic functions are discussed.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. The subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in $\mathbb{U}$ is denoted by $\mathcal{S}$.

A function $f(z) \in \mathcal{A}$ is said to be starlike of order α in $\mathbb{U}$ if it satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U})$$

for some real number α with $0 \leq \alpha < 1$. This class is denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{S}^*(0) = \mathcal{S}^*$. The class $\mathcal{S}^*(\alpha)$ was introduced by Robertson [2]. It is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^* \subset \mathcal{S}$.

Furthermore, let $\mathcal{M}(\beta)$ be the class of functions $f(z) \in \mathcal{A}$ which satisfy

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \beta \qquad (z \in \mathbb{U})$$

for some real number β with $\beta > 1$. The class $\mathcal{M}(\beta)$ was investigated by Uralegaddi, Ganigi and Sarangi [4].

2010 *Mathematics Subject Classification.* 30C45, 30C50.
Key words and phrases. Analytic function, univalent function, starlike function, subordination, coefficient estimate.

Let $p(z)$ and $q(z)$ be analytic in $\mathbb{U}$. Then the function $p(z)$ is said to be subordinate to $q(z)$ in $\mathbb{U}$, written by

$$p(z) \prec q(z) \qquad (z \in \mathbb{U}), \tag{1.1}$$

if there exists a function $w(z)$ which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1 \quad (z \in \mathbb{U})$, and such that $p(z) = q\big(w(z)\big) \quad (z \in \mathbb{U})$. From the definition of the subordinations, it is easy to show that the subordination (1.1) implies that

$$p(0) = q(0) \quad \text{and} \quad p(\mathbb{U}) \subset q(\mathbb{U}). \tag{1.2}$$

In particular, if $q(z)$ is univalent in $\mathbb{U}$, then the subordination (1.1) is equivalent to the condition (1.2).

Remark 1.1. *Let $p(z)$ and $q(z)$ be analytic in $\mathbb{U}$. Then the subordination* (1.1) *implies that*

$$|p'(0)| \leq |q'(0)|, \tag{1.3}$$

and $|p'(0)| = |q'(0)|$ if and only if $p(z) = q(xz)$ for some real number x with $|x| = 1$ (cf. [1]*).*

Motivated by the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{M}(\beta)$, we define new class for certain analytic functions. Let $\mathcal{S}(\alpha, \beta)$ denote the class of functions $f(z) \in \mathcal{A}$ which satisfy the inequality

$$\alpha < \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \beta \qquad (z \in \mathbb{U}) \tag{1.4}$$

for some real number α ($\alpha < 1$) and some real number β ($\beta > 1$). Note that α is not necessary to be positive in the class $\mathcal{S}(\alpha, \beta)$.

Remark 1.2. *Let $f(z) \in \mathcal{S}(\alpha, \beta)$. If $\alpha \geq 0$, then $f(z)$ is starlike in $\mathbb{U}$, which implies that $f(z)$ is univalent in $\mathbb{U}$.*

In order to discuss our new class $\mathcal{S}(\alpha, \beta)$, we need to consider a certain univalent function in $\mathbb{U}$ which maps $\mathbb{U}$ onto the strip domain w with $\alpha < \operatorname{Re} w < \beta$.

Theorem 1.1. *Let α and β be real numbers with $\alpha < 1$ and $\beta > 1$. Then the function $S_{\alpha,\beta}(z)$ defined by*

$$S_{\alpha,\beta}(z) = 1 + \frac{\beta - \alpha}{\pi} i \log\left(\frac{1 - e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}} z}{1 - e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}} z}\right) \qquad (z \in \mathbb{U}) \tag{1.5}$$

is analytic and univalent in $\mathbb{U}$ with $S_{\alpha,\beta}(0) = 1$. In addition, $S_{\alpha,\beta}(z)$ maps $\mathbb{U}$ onto the strip domain w with $\alpha < \operatorname{Re} w < \beta$.

Remark 1.3.

$$S_{\alpha,\beta}(z) = 1 + \frac{\beta - \alpha}{\pi} i \log\left(\frac{1 - e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}} z}{1 - e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}} z}\right) = 1 + \sum_{n=1}^{\infty} B_n z^n,$$

where

$$B_n = \frac{2(\beta - \alpha)}{n\pi} \sin \frac{n\pi(1-\alpha)}{\beta - \alpha} \qquad (n = 1, 2, \cdots). \tag{1.6}$$

Specially, we note that the coefficient B_n defined by (1.6) is the real number. Since

$$\lim_{\beta\to+\infty} B_n = \lim_{\beta\to+\infty}\left\{2(1-\alpha)\frac{\sin\frac{n\pi(1-\alpha)}{\beta-\alpha}}{\frac{n\pi(1-\alpha)}{\beta-\alpha}}\right\} = 2(1-\alpha),$$

a simple check gives us that

$$S_{\alpha,\beta}(z) = 1+\sum_{n=1}^{\infty} 2(1-\alpha)z^n = \frac{1+(1-2\alpha)z}{1-z} \qquad (\beta\to+\infty),$$

which implies that $S_{\alpha,\beta}(z) \quad (\beta\to+\infty)$ maps $\mathbb{U}$ onto the right half-plane w with $\operatorname{Re} w > \alpha$.

On the other hand, it is easy to see that

$$\lim_{\alpha\to-\infty} B_n = \lim_{\alpha\to-\infty}\left\{\frac{2(\beta-\alpha)}{n\pi}\sin\left(\frac{n\pi(1-\beta)}{\beta-\alpha}+n\pi\right)\right\}$$

$$= \lim_{\alpha\to-\infty}\left\{\frac{2(\beta-\alpha)}{n\pi}(-1)^n\sin\frac{n\pi(1-\beta)}{\beta-\alpha}\right\}$$

$$= \lim_{\alpha\to-\infty}\left\{2(\beta-1)(-1)^{n-1}\frac{\sin\frac{n\pi(1-\beta)}{\beta-\alpha}}{\frac{n\pi(1-\beta)}{\beta-\alpha}}\right\} = 2(\beta-1)(-1)^{n-1}.$$

Therefore, we find that

$$S_{\alpha,\beta}(z) = 1+\sum_{n=1}^{\infty} 2(\beta-1)(-1)^{n-1}z^n = \frac{1-(1-2\beta)z}{1+z} \qquad (\alpha\to-\infty),$$

which implies that $S_{\alpha,\beta}(z) \quad (\alpha\to-\infty)$ maps $\mathbb{U}$ onto the left half-plane w with $\operatorname{Re} w < \beta$.

We give some example for $f(z)\in\mathcal{S}(\alpha,\beta)$ as follows.

Example 1.1. *Let us consider the function $f(z)$ given by*

$$f(z) = z\exp\left\{\frac{\beta-\alpha}{\pi}i\int_0^z \frac{1}{t}\log\left(\frac{1-e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}}t}{1-e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}}t}\right)dt\right\} \tag{1.7}$$

$$= z+\frac{2(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}z^2+\cdots \qquad (z\in\mathbb{U})$$

with $\alpha<1$ and $\beta>1$. Then we have

$$\frac{zf'(z)}{f(z)} = 1+\frac{\beta-\alpha}{\pi}i\log\left(\frac{1-e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}}z}{1-e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}}z}\right) = S_{\alpha,\beta}(z) \qquad (z\in\mathbb{U}).$$

According to Theorem 1.1, it is clear that the function $f(z)$ given by (1.7) *satisfies the inequality* (1.4)*, which implies that $f(z)\in\mathcal{S}(\alpha,\beta)$.*

Applying the function $S_{\alpha,\beta}(z)$ defined by (1.5), we give a necessary and sufficient condition for $f(z)\in\mathcal{A}$ to beling to the class $\mathcal{S}(\alpha,\beta)$.

Lemma 1.1. *Let $f(z) \in \mathcal{A}$. Then $f(z) \in \mathcal{S}(\alpha, \beta)$ if and only if*

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{\beta - \alpha}{\pi} i \log \left(\frac{1 - e^{i \frac{\pi(1-\alpha)}{\beta - \alpha}} z}{1 - e^{-i \frac{\pi(1-\alpha)}{\beta - \alpha}} z} \right) \qquad (z \in \mathbb{U}), \tag{1.8}$$

where $\alpha < 1$ and $\beta > 1$.

By using the subordination (1.8), we discuss some properties for $f(z) \in \mathcal{S}(\alpha, \beta)$.

2. Some results

Rogosinski [3] proved the coefficient estimates for subordinate functions.

Lemma 2.1. *Let $q(z) = \sum_{n=1}^{\infty} B_n z^n$ be analytic and univalent in $\mathbb{U}$, and suppose that $q(z)$ maps $\mathbb{U}$ onto a convex domain. If $p(z) = \sum_{n=1}^{\infty} A_n z^n$ is analytic in $\mathbb{U}$ and satisfies the following subordination*

$$p(z) \prec q(z) \qquad (z \in \mathbb{U}),$$

then

$$|A_n| \leq |B_1| \qquad (n = 1, 2, \cdots).$$

Applying Lemma 2.1, we deduce some coefficient estimates for $f(z) \in \mathcal{S}(\alpha, \beta)$ bellow.

Theorem 2.1. *If the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}(\alpha, \beta)$, then*

$$|a_n| \leq \prod_{k=2}^{n} \frac{k - 2 + \frac{2(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}}{k-1} \qquad (n = 2, 3, \cdots).$$

We next give sharp bounds on the second and third coefficients for $f(z) \in \mathcal{S}(\alpha, \beta)$. To obtain some sharp coefficient estimates, we need the following lemma due to Rogosinski [3].

Lemma 2.2. *Let $p(z) = \sum_{n=1}^{\infty} A_n z^n$ and $q(z) = \sum_{n=1}^{\infty} B_n z^n$ be analytic in $\mathbb{U}$. If $p(z) \prec q(z)$ $(z \in \mathbb{U})$, then*

$$\sum_{k=1}^{m} |A_k|^2 \leq \sum_{k=1}^{m} |B_k|^2 \qquad (m = 1, 2, \cdots).$$

By Remark 1.1 and Lemma 2.2, we obtain

Theorem 2.2. *If the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}(\alpha, \beta)$, then*

$$|a_2| \leq \frac{2(\beta - \alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta - \alpha}$$

and

$$|a_3| \leq \frac{\beta - \alpha}{\pi} \sin \frac{\pi(1-\alpha)}{\beta - \alpha} \left(\cos \frac{\pi(1-\alpha)}{\beta - \alpha} + \frac{2(\beta - \alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta - \alpha} \right).$$

Moreover, the equality holds in either inequality if and only if

$$f(z) = z \exp\left\{ \int_0^z \frac{S_{\alpha,\beta}(e^{i\theta}t) - 1}{t}\, dt \right\}$$

for some real number θ $(0 \leq \theta < 2\pi)$*, where* $S_{\alpha,\beta}(z)$ *is defined by* (1.5).

References

[1] A.W.Goodman: *Univalent Functions, Vol.I and II*, Mariner, Tampa, Florida, 1983.
[2] M.S.Robertson: *On the theory of univalent functions,* Ann. Math. **37** (1936), 374–408.
[3] W.Rogosinski: *On the coefficients of subordinate functions,* Proc. London Math. Soc. **48** (1943), 48–82.
[4] B.A.Uralegaddi, M.D.Ganigi, S.M.Sarangi: *Univalent functions with positive coefficients,* Tamkang J. Math. **25** (1994), 225–230.

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan
E-mail address: freedom@sakai.zaq.ne.jp

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan
E-mail address: shige21@ican.zaq.ne.jp