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STRONGLY ALMOST SUMMABLE DIFFERENCE SEQUENCES AND STATISTICAL CONVERGENCE

M. AIYUB

Abstract. The idea of difference sequence space was introduced by Kizmaz [12] and was generalized by Et and Çolak [6]. In this paper, we introduce and examine some properties of three sequence spaces defined by using a modulus function and give various properties and inclusion relation on these spaces.

1. Introduction and Preliminaries

Let ω be the set of all sequences of real numbers and ℓ_∞, c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $\|x\| = \sup |x_k|$, where $k \in \mathbb{N} = \{1, 2, 3.....\}$, the positive integers.

A sequence $x \in \ell_\infty$ is said to be almost convergent [14] if all Banach limits of x coincide. Lorentz [14] defined:

$$\hat{c} = \Big\{ x : \lim_n \frac{1}{n} \sum_{k=1}^{n} x_{k+m} \text{ exists, uniformly in } m \Big\}.$$

Several authors including Lorentz [14], Duran [2] and King [11], have studied almost convergent sequences. Maddox [16, 17] has defined x to be strongly almost convergent to a number L if

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \big| x_{k+m} - L \big| = 0 \text{ uniformly in } m.$$

By $\big[\hat{c}\big]$ we denote the spaces of all strongly almost convergent sequences. It is easy to see that $c \subset \big[\hat{c}\big] \subset \hat{c} \subset \ell_\infty$.

The space of strongly almost convergent sequences was generalized by Nanda [20, 21].

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Let $p=(p_k)$ be a sequence of strictly positive numbers. Nanda [20] defined:

$$\left[\hat{c},p\right]=\left\{x=(x_k):\lim_n\frac{1}{n}\sum_{k=1}^{n}\left|x_{k+m}-L\right|^{p_k}=0,\ \text{uniformly in } m\right\},$$

$$\left[\hat{c},p\right]_0=\left\{x=(x_k):\lim_n\frac{1}{n}\sum_{k=1}^{n}\left|x_{k+m}\right|^{p_k}=0,\ \text{uniformly in } m\right\},$$

$$\left[\hat{c},p\right]_\infty=\left\{x=(x_k):\sup_{m,n}\frac{1}{n}\sum_{k=1}^{n}\left|x_{k+m}\right|^{p_k}<\infty\right\}.$$

Let $\lambda=(\lambda_k)$ be a nondecreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1}\leq\lambda_n+1,\ \lambda_1=1.$

The generalized de la Vallée-poussin mean is defined by

$$t_n(x)=\frac{1}{\lambda_n}\sum_{k\in I_n}x_k,$$

where $I_n=[n-\lambda_n+1,n]$ for $n=1,2,....$

A sequence $x=(x_k)$ is said to be (V,λ) summable to a number L (see [13]), if $t_n(x)\to L$ as $n\to\infty$. If $\lambda_n=n$, then (V,λ) summability and strongly (V,λ) summability are reduced to $(C,1)$ summability and $[C,1]$ summability, respectively.

The idea of difference sequence spaces was introduced by Kizmaz in [12]. In 1981, Kizmaz defined the sequence spaces:

$$X(\triangle)=\left\{x=(x_k):\triangle x\in X\right\}$$

for $X=\ell_\infty,c$ and c_0, where $\triangle x=\big(x_k-x_{k+1}\big)$.

Then Et and Çolak [6] generalized the above sequence spaces as below:

$$X(\triangle^r)=\left\{x=(x_k):\triangle^r x\in X\right\}$$

for $X=\ell_\infty,c$ and c_0, where $r\in\mathbb{N}$, $\triangle^0x=(x_k)$, $\triangle x=(x_k-x_{k+1})$, $\triangle^r x=(\triangle^{r-1}x_k-\triangle^{r-1}x_{k+1})$, and so that

$$\triangle^r x=\sum_{v=0}^{r}(-1)^v\begin{bmatrix}r\\v\end{bmatrix}x_{k+v}.$$

Recently, Et and Basarir [5] extended the above sequence spaces to the sequence spaces $X(\triangle^r)$ for $X=\ell_\infty(p)$, $c(p)$, $c_0(p)$, $\left[\hat{c},p\right]$, $\left[\hat{c},p\right]_0$ and $\left[\hat{c},p\right]_\infty$.

We recall that a modulus f is a function from $[0,\infty)\to[0,\infty)$ such that

(i) $f(x)=0$ if and only if $x=0$,
(ii) $f(x+y)\leq f(x)+f(y)$ for all $x\geq0,\ y\geq0$,
(iii) f is increasing,
(iv) f is continuous from right at 0.

It follows that f must be a continuous everywhere on $[0,\infty)$. The modulus function may be bounded or unbounded. Ruckle [23] and Maddox [15] used a modulus function f to construct some sequence spaces. Subsequently modulus function has been discussed in [3, 4, 19, 22, 26].

Further, let $X, Y \subset \omega$. Then we shall write ([27]):

$$M(X,Y) = \bigcap_{x\in X} x^{-1} * Y = \Big\{ a \in \omega : ax \in Y \ \text{ for all } x \in X \Big\}.$$

The set $X^{\alpha} = M(X, \ell_1)$ is called the Köthe-Toeplitz dual space or α- dual of X. Let X be a sequence space. Then X is called:

(*i*) Solid (or normal) if $(\alpha_k x_k) \in X$, whenever, $(x_k) \in X$ for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.

(*ii*) Symmetric if $(x_k) \in X$ implies $\big(x_{\pi(k)}\big) \in X$ whenever $\pi(k)$ is a permutation of $\mathbb{N}$.

(*iii*) Perfect if $X = X^{\alpha\alpha}$.

(*iv*) Sequence algebra if $x.y \in X$, whenever $x, y \in X$. It is well known that if X is perfect then X is normal [10].

The following inequality will be used throughout this paper:

$$\big|a_k + b_k\big|^{p_k} \leq C\big[\ |a_k|^{p_k} + |b_k|^{p_k}\big]\,, \tag{1.1}$$

where $a_k, b_k \in \mathbb{C}$, $0 \leq p_k \leq \sup_k p_k = H$, $C = \max\big(1,\ 2^{H-1}\big)$, (see [18]).

2. Main Results

In this section we prove some results involving the sequence space $\big[\hat{V}, \lambda, f, p\big]_0(\triangle^r, E)$, $\big[\hat{V}, \lambda, f, p\big]_1(\triangle^r, E)$ and $\big[\hat{V}, \lambda, f, p\big]_\infty(\triangle^r, E)$.

Definition 2.1. *Let E be Banach space. We define $\omega(E)$ to be the vector space of all E-valued sequences that is $\omega(E) = \Big\{x = (x_k) : x_k \in E\Big\}$. Let f be a modulus function and $p = (p_k)$ be any sequence of strictly positive real numbers.*

We define the following sequence sets:

$$\big[\hat{V}, \lambda, f, p\big]_1(\triangle^r, E) =$$

$$= \Big\{x \in \omega(E) : \lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} \Big[f\big(\big|\triangle^r x_{k+m} - L\big|\big)\Big]^{p_k} = 0,\ \text{uniformly in } m \text{ for some } L > 0\Big\},$$

$$\big[\hat{V}, \lambda, f, p\big]_0(\triangle^r, E) = \Big\{x \in \omega(E) : \lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} \Big[f\big(\big|\triangle^r x_{k+m}\big|\big)\Big]^{p_k} = 0, \text{uniformly in } m\Big\},$$

and

$$\big[\hat{V}, \lambda, f, p\big]_\infty(\triangle^r, E) = \Big\{x \in \omega(E) : \sup_{n,m} \frac{1}{\lambda_n} \sum_{k\in I_n} \Big[f\big(\big|\triangle^r x_{k+m}\big|\big)\Big]^{p_k} \leq \infty\Big\}.$$

If $x \in \big[\hat{V}, \lambda, f, p\big]_1(\triangle^r, E)$ then we shall write $x_k \to L\big[\hat{V}, \lambda, f, p\big]_1(\triangle^r, E)$ and L will be called λ_E- strongly almost difference limit of x with respect to the modulus f.

Through this paper, Z will denote any one of the notions $0, 1$ or ∞.

In this case $f(x) = x$ and $p_k = 1$ for all $k \in \mathbb{N}$, we shall write $\big[\hat{V}, \lambda\big]_z(\triangle^r, E)$ and $\big[\hat{V}, \lambda, f\big]_z(\triangle^r, E)$ instead of $\big[\hat{V}, \lambda, f, p\big]_z(\triangle^r, E)$. If $x \in \big[\hat{V}, \lambda\big]_1(\triangle^r, E)$ then we say that x is $\triangle^r_{\lambda,E}$ strongly almost convergent to L.

The proofs of the following two theorems are obtained by using the known standard techniques, therefore we give them without proofs (see for detail [3, 22]).

Theorem 2.1. *Let $p = (p_k)$ be a bounded. Then the spaces $\left[\hat{V}, \lambda, f, p\right]_z(\triangle^r, E)$ are linear spaces over the set of complex numbers $\mathbb{C}$.*

Theorem 2.2. *Let $p = (p_k)$ be a bounded and f be modulus function, then*

$$\left[\hat{V}, \lambda, f, p\right]_0(\triangle^r, E) \subset \left[\hat{V}, \lambda, f, p\right]_1(\triangle^r, E \subset \left[\hat{V}, \lambda, f, p\right]_\infty(\triangle^r, E).$$

Theorem 2.3. *If $r \geq 1$, then the inclusion $\left[\hat{V}, \lambda, f, p\right]_z(\triangle^{r-1}, E) \subset \left[\hat{V}, \lambda, f, p\right]_z(\triangle^r, E)$ is strict. In general $\left[\hat{V}, \lambda, f, p\right]_z(\triangle^i, E) \subset \left[\hat{V}, \lambda, f, p\right]_z(\triangle^r, E)$ for all $i = 1, 2, 3...r-1$ and the inclusion is strict.*

Proof. We give the proof for $Z = \infty$ only. It can be proved in similar way for $Z = 0, 1$. Let $x \in \left[\hat{V}, \lambda, f, p\right]_\infty(\triangle^{r-1}, E)$. Then we have:

$$\sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} f\left(\left|\triangle^{r-1} x_{k+m}\right|\right) < \infty\,.$$

By definition of f, we have:

$$\frac{1}{\lambda_n} \sum_{k \in I_n} f\left(\left|\triangle^{r} x_{k+m}\right|\right) \leq \frac{1}{\lambda_n} \sum_{k \in I_n} f\left(\left|\triangle^{r-1} x_{k+m}\right|\right) + \frac{1}{\lambda_n} \sum_{k \in I_n} f\left(\left|\triangle^{r-1} x_{k+m+1}\right|\right) < \infty\,.$$

Thus,

$$\left[\hat{V}, \lambda, f, p\right]_\infty(\triangle^{r-1}, E) \subset \left[\hat{V}, \lambda, f, p\right]_\infty(\triangle^r, E)\,.$$

Proceeding in this way, we have:

$$[\hat{V}, \lambda, f, p]_\infty(\triangle^i E) \subset [\hat{V}, \lambda, f, p]_\infty(\triangle^r, E)\,,$$

for all $i = 1, 2, 3....r-1$. Let $\lambda_n = n$ for all $n \in \mathbb{N}$. Then the sequence $x = (k^r)$, for example, belongs to $\left[\hat{V}, \lambda, f, p\right]_\infty(\triangle^r, E)$, but doesn't belong to $\left[\hat{V}, \lambda, f, p\right]_\infty(\triangle^{r-1}, E)$ for $f(x) = x$ $\big($if $x = (k^r)$, then $\triangle^r x_k = (-1)^r r!$ and $\triangle^{r-1} x_k = (-1)^{r+1} r!(k + \frac{(r-1)}{2})$ for all $k \in \mathbb{N}\big)$. ∎

Similarly, as in the previous theorems for the cases $[\hat{V}, \lambda, f, p]_0(\triangle^r, E)$ and $[\hat{V}, \lambda, f, p]_1(\triangle^r E)$ we have:

Proposition 2.1. *Let f be a sequence of modulus functions. Then:*

$$\left[\hat{V}, \lambda, f, p\right]_1(\triangle^{r-1}, E) \subset \left[\hat{V}, \lambda, f, p\right]_0(\triangle^r, E).$$

Theorem 2.4. *Let f_1 and f_2 be modulus functions. Then we have:*

(*i*) $\left[\hat{V}, \lambda, f_1, p\right]_z(\triangle^r, E) \subset \left[\hat{V}, \lambda, f_1 o f_2, p\right]_z(\triangle^r, E)$,
(*ii*) $\left[\hat{V}, \lambda, f_1, p\right]_z(\triangle^r, E) \cap \left[\hat{V}, \lambda, f_2, p\right]_z(\triangle^r, E) \subset \left[\hat{V}, \lambda, f_1 + f_2, p\right]_z(\triangle^r, E)$.

The following results are consequence of Theorem 2.4.

Proposition 2.2. *Let f be a modulus functions. Then:*

$$\left[\hat{V}, \lambda, p\right]_z(\triangle^r, E) \subset \left[\hat{V}, \lambda, f, p\right]_z(\triangle^r, E).$$

Theorem 2.5. *The sequence spaces* $\left[\hat{V},\lambda,f,p\right]_0(\triangle^r,E)$, $\left[\hat{V},\lambda,f,p\right]_1(\triangle^r,E)$ *and* $\left[\hat{V},\lambda,f,p\right]_\infty(\triangle^r,E)$ *are not solid for* $r\geq 1$.

Proof. Let $p_k=1$ for all k, $f(x)=x$ and $\lambda_n=n$ for all $n\in\mathbb{N}$. Then $(x_k)=(k^r)\in[\hat{V},\lambda,f,p]_\infty(\triangle^r,E)$ but $(\alpha_k x_k)\notin[\hat{V},\lambda,f,p]_\infty(\triangle^r,E)$, when $\alpha_k=(-1)^k$ for all $k\in\mathbb{N}$. Hence $[\hat{V},\lambda,f,p]_\infty(\triangle^r,E)$ is not solid. The other cases can be proved by considering similar examples. $\square$

From the above theorem we may give the following corollary.

Corollary 2.1. *The sequence spaces* $\left[\hat{V},\lambda,f,p\right]_0(\triangle^r,E)$, $\left[\hat{V},\lambda,f,p\right]_1(\triangle^r,E)$ *and* $\left[\hat{V},\lambda,f,p\right]_\infty(\triangle^r,E)$ *are not perfect for* $r\geq 1$.

Theorem 2.6. *The sequence spaces* $\left[\hat{V},\lambda,f,p\right]_1(\triangle^r,E)$ *and* $\left[\hat{V},\lambda,f,p\right]_\infty(\triangle^r,E)$ *are not symmetric for* $r\geq 1$.

Proof. Let $(p_k)=1$ for all k, $f(x)=x$ and $\lambda_n=n$ for all $n\in\mathbb{N}$. Then $x_k=(k^r)\in\left[\hat{V},\lambda,f,p\right]_\infty(\triangle^r,E)$. Let (y_k) be an arrangement of (x_k), which is defined by

$$(y_k)=\left\{x_1,x_2,x_4,x_3,x_9,x_5,x_{16},x_6,x_{25},x_7,x_{36},x_8,x_{49},x_{10}......\right\}.$$

Then $(y_k)\notin\left[\hat{V},\lambda,f,p\right]_\infty(\triangle^r,E)$. $\square$

Remark 2.1. *The space* $[\hat{V},\lambda,f,p]_0(\triangle^r,E)$ *is not symmetric for* $r\geq 2$.

Theorem 2.7. *The sequence spaces* $\left[\hat{V},\lambda,f,p\right]_z(\triangle^r,E)$*. are not sequence of algebras.*

Proof. Let $p_k=1$ for all $k\in\mathbb{N}$., $f(x)=x$ and $\lambda_n=n$ for all $n\in\mathbb{N}$. Then $x=(k^{r-2})$, $y=(k^{r-2})\in\left[\hat{V},\lambda,f,p\right]_z(\triangle^r,E)$, but $x,y\in\left[\hat{V},\lambda,f,p\right]_z(\triangle^r,E)$. $\square$

3. Statistical Convergent

The notion of statistical convergence was introduced by Fast [7] and studied by various authors [1, 9, 24, 25]. In this section we define $\triangle^r_{\lambda,E}$ almost statistically convergent sequences and give some inclusion relations between $\hat{s}(\triangle^r_\lambda,E)$ and $\left[\hat{V},\lambda,f,p\right]_1(\triangle^r,E)$.

Definition 3.1. *A sequence* $x=(x_k)$ *is said to be* $\triangle^r_{\lambda,E}$*-almost statistically convergent to the number* L *if for every* $\epsilon>0$,

$$\lim_n \frac{1}{\lambda_n}\left|\left\{k\in I_n:\left|\triangle^r x_{k+m}-L\right|\geq\epsilon\right\}\right|=0 \text{ uniformly in } m.$$

In this case we write $\hat{s}(\triangle^r_\lambda,E)-\lim x=L$, or $x_k\to L\hat{s}(\triangle^r_\lambda,E)$.

When $\lambda_n=n$ and $L=0$ we shall write $\hat{s}(\triangle^r,E)$ instead of $\hat{s}(\triangle^r_\lambda,E)$.

The proof of the following theorem is easily obtained by using the same technique as in Theorem 2 in Savaş [25], therefore we give it without proof.

Theorem 3.1. *Let* $\lambda=(\lambda_n)$ *be the same as in section 1, then:*

(*i*) *If* $x_k\to L\left[\hat{V},\lambda\right]_1(\triangle^r,E)\Rightarrow x_k$ *then* $L\hat{s}\left(\triangle^r_\lambda,E\right)$;

(*ii*) *If* $x\in\ell_\infty\left(\triangle^r,E\right)$ *and* $x_k\to L\hat{s}\left(\triangle^r_\lambda,E\right)$, *then* $x_k\to L\left[\hat{V},\lambda\right]_1(\triangle^r,E)$;

(*iii*) $\hat{s}\left(\triangle^r_\lambda,E\right)\cap\ell_\infty(\triangle^r,E)=\left[\hat{V},\lambda\right]_1(\triangle^r,E)\cap\ell_\infty(\triangle^r,E)$.

Theorem 3.2. *$\hat{s}\big(\triangle^r, E\big) \subset \hat{s}\big(\triangle^r_\lambda, E\big)$ if and only if* $\liminf_n \frac{\lambda_n}{n} > 0$.

Proof. The sufficiency part of this proof can be obtained using the same technique as the sufficiency part of proof of Theorem 3 in Savaş [25].

For necessity suppose that $\liminf_n \frac{\lambda_n}{n} = 0$. As in ([8], p. 510) we can choose a subsequence $(n(j))$ such that $\frac{\lambda_{n(j)}}{n(j)} < \frac{1}{j}$. We define $x = (x_i)$ such that:

$$\triangle^r x_i = \begin{cases} 1 \text{ if } i \in I_n(j),\ j = 1, 2, 3.....; \\ 0 \text{ otherwise.} \end{cases}$$

Then $x \in \big[\hat{c}\big](\triangle^r, E)$ and by [4, Theorem 3.1 (i)], $x \in \hat{s}\big(\triangle^r, E\big)$. But $x \notin \big[\hat{V}, \lambda\big]_1(\triangle^r, E)$ and Theorem 3.1 (ii) implies that $x \notin \hat{s}\big(\triangle^r_\lambda, E\big)$. This completes the proof. □

Theorem 3.3. *Let f be a modulus function and $\sup_k p_k = H$. Then:*

$$\big[\hat{V}, \lambda, f, p\big]_1(\triangle^r, E) \subset \hat{s}(\triangle^r_\lambda, E).$$

Proof. Let $x \in \big[\hat{V}, \lambda, f, p\big]_1(\triangle^r, E)$ and $\epsilon > 0$ be given. Let Σ_1 denote the sum over $k \leq n$ such that $\big|\triangle^r x_{k+m} - L\big| \geq \epsilon$ and Σ_2 denote the sum of over $k \leq n$ such that $\big|\triangle^r x_{k+m} - L\big| < \epsilon$. Then:

$$\begin{aligned}
\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[f\big(\big|\triangle^r_{k+m} - L\big|\big)\Big]^{p_k} &= \frac{1}{\lambda_n}\sum_1\Big[f\big(\big|\triangle^r_{k+m} - L\big|\big)\big]^{p_k} + \frac{1}{\lambda_n}\sum_2\Big[f\big(\big|\triangle^r_{k+m} - L\big|\big)\Big]^{p_k} \\
&\geq \frac{1}{\lambda_n}\sum_1\Big[f\big(\big|\triangle^r_{k+m} - L\big|\big)\big]^{p_k} \geq \frac{1}{\lambda_n}\sum_1\big[f(\epsilon)\big]^{p_k} \\
&\geq \frac{1}{\lambda_n}\sum_1 \min\Big(\big[f(\epsilon)\big]^{\inf p_k}, \big[f(\epsilon)\big]^H\Big) \\
&\geq \big|\big\{k \in I_n : \big|\triangle^r x_{k+m} - L\big| \geq \epsilon\big\}\big| \min\Big(\big[f(\epsilon)\big]^{\inf p_k}, \big[f(\epsilon)\big]^H\Big).
\end{aligned}$$

Hence $x \in \hat{s}(\triangle^r_\lambda, E)$. □

Theorem 3.4. *Let f be a bounded and $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. Then: $\hat{s}(\triangle^r_\lambda, E) \subset \big[\hat{V}, \lambda, f, p\big]_1(\triangle^r, E)$.*

Proof. Suppose that f is bounded. Let $\epsilon > 0$ and Σ_1 and Σ_2 be denoted in the previous theorem. Since f is bounded there exists an integer K such that $f(x) < K$ for all $x \geq 0$. Then:

$$\begin{aligned}
\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[f\big(\big|\triangle^r_{k+m} - L\big|\big)\Big]^{p_k} &= \frac{1}{\lambda_n}\sum_1\Big[f\big(\big|\triangle^r_{k+m} - L\big|\big)\Big]^{p_k} + \frac{1}{\lambda_n}\sum_2\Big[f\big(\big|\triangle^r_{k+m} - L\big|\big)\Big]^{p_k} \\
&\leq \frac{1}{\lambda_n}\sum_1 \max\big(K^h, K^H\big) + \frac{1}{\lambda_n}\sum_2\big[f(\epsilon)\big]^{p_k} \\
&\leq \max\big(K^h, K^H\big)\frac{1}{\lambda_n}\big|\big\{k \in I_n : \big|\triangle^r_{k+m} - L\big| \geq \epsilon\big\}\big| + \\
&+ \max\big(f(\epsilon)^h, f(\epsilon)^H\big)
\end{aligned}$$

Hence $x \in \left[\hat{V}, \lambda, f, p\right]_1(\triangle^r, E)$. $\square$

Theorem 3.5. *Let* $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. *Then*

$$\hat{s}(\triangle^r_\lambda, E) = \left[\hat{V}, \lambda, f, p\right]_1(\triangle^r, E),$$

if and only if f *is bounded.*

Proof. Let f is bounded bounded. By Theorem 3.4 and Theorem 3.5 we have $\hat{s}\left(\triangle^r_\lambda, E\right) = \left[\hat{V}, \lambda, f, p\right]_1(\triangle^r, E)$. Conversely, suppose that f is unbounded. Then there exists a positive sequence (t_k) with $f(t_k) = k^2$, for $k = 1, 2, 3.....$ If we choose

$$\triangle^r x_i = \begin{cases} t_k \ i = k^2, \ i = 1, 2, 3...... \\ 0 \text{ otherwise.} \end{cases},$$

then we have:

$$\frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|\triangle^r_{k+m}\right| \geq \epsilon\right\}\right| \leq \frac{\sqrt{\lambda_{n-1}}}{\lambda_n} \text{ for all } n, m$$

and so that $x \in \hat{s}(\triangle^r_\lambda, E)$ but $x \notin \left[\hat{V}, \lambda, f, p\right]_1(\triangle^r, E)$ for $E = \mathbb{C}$. This contradict to $\hat{s}\left(\triangle^r_\lambda, E\right) = \left[\hat{V}, \lambda, f, p\right]_1(\triangle^r_\lambda, E)$ $\square$

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Department of Mathematics
University of Bahrain
P.O. Box-32038
Kingdom of Bahrain
E-mail address: maiyub2002@yahoo.com