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# ON A PRODUCT SUMMABILITY OF AN ORTHOGONAL SERIES

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ABSTRACT. In this paper we have defined a new product summability, in order to make an advanced study in the special topic of summability. Namely, we give some sufficient conditions, in terms of the coefficients of an orthogonal series, under which such series is product summable almost everywhere.

## 1. Introduction

The absolute summability is a generalization of the concept of the absolute convergence just as the summability is an extension of the concept of the convergence. There are a lot of notions of absolute summability defined by several authors. Particularly, by those authors such notions are employed for studying the absolute summability of an orthogonal series. As a recent result can be mentioned those of Y. Okuyama (see section 2) who has proved two theorems which give sufficient conditions in terms of the coefficients of an orthogonal series under which such series would be absolute generalized Nörlund summable almost everywhere. Moreover, an interested reader could find some new results, see as examples [4]-[6], where are given some statements which include all of the results previously proved by Y. Okuyama and T. Tsuchikura [8]-[9], and also are given some new consequences. In order to make an advance study in this direction, here we study the question when an orthogonal series is product summable almost everywhere.

## 2. Notations and Known Results

For two sequences of real or complex numbers $\{p_n\}$ and $\{q_n\}$, let

$$\begin{aligned} P_n &= p_0 + p_1 + p_2 + \cdots + p_n = \sum_{v=0}^{n} p_v, \\ Q_n &= q_0 + q_1 + q_2 + \cdots + q_n = \sum_{v=0}^{n} q_v, \end{aligned}$$

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and let the convolution $(p*q)_n$ be defined by

$$R_n := (p*q)_n := \sum_{v=0}^{n} p_v q_{n-v}, \quad \text{and} \quad \text{denote} \quad R_n^j := \sum_{v=j}^{n} p_v q_{n-v}.$$

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with the sequence of its $n-$th partial sums $\{s_n\}$. We write

$$t_n^{p,q} = \frac{1}{R_n}\sum_{v=0}^{n} p_{n-v} q_v s_v.$$

If $R_n \neq 0$ for all $n$, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$.

The infinite series $\sum_{n=0}^{\infty} a_n$ is said to be absolutely summable $(N,p,q)$ if the series

$$\sum_{n=1}^{\infty} |t_n^{p,q} - t_{n-1}^{p,q}|$$

converges, and we write in brief

$$\sum_{n=0}^{\infty} a_n \in |N,p,q|.$$

The $|N,p,q|$ summability was introduced by Tanaka [3].

Let $\{\varphi_j(x)\}$ be an orthonormal system defined in the interval $(a,b)$. We assume that $f$ belongs to $L^2(a,b)$ and

$$f(x) \sim \sum_{j=0}^{\infty} c_j \varphi_j(x), \tag{2.1}$$

where $c_j = \int_a^b f(x)\varphi_j(x)dx$, $(j=0,1,2,\dots)$.

Regarding to the orthogonal series (2.1) Y. Okuyama has proved the following two theorems:

**Theorem 2.1** ([8]). *If the series*

$$\sum_{n=1}^{\infty}\left\{\sum_{j=1}^{n}\left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}}\right)^2 |c_j|^2\right\}^{\frac{1}{2}}$$

*converges, then the orthogonal series*

$$\sum_{j=0}^{\infty} c_j \varphi_j(x)$$

*is summable $|N,p,q|$ almost everywhere.*

**Theorem 2.2** ([8]). *Let $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series*

$$\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$$

*converges. Let* $\{p_n\}$ *and* $\{q_n\}$ *be non-negative. If the series*

$$\sum_{n=1}^{\infty} |c_n|^2 \Omega(n) w(n)$$

*converges, then the orthogonal series*

$$\sum_{j=0}^{\infty} c_j \varphi_j(x) \in |N, p, q|$$

*almost everywhere, where* $w(n)$ *is defined by*

$$w(j) := j^{-1} \sum_{n=j}^{\infty} n^2 \left( \frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 .$$

If we take $p_v = 1$ for all $v$ then, the sequence-to-sequence transformation $t_n^{p,q}$ reduces to transformation $R_n^q := \frac{1}{Q_n} \sum_{v=0}^{n} q_v s_v$, while for $q_v = 1$ we obtain the transformation $R_n^p := \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v$. G. Das [2] defined the transformation

$$U_n := \frac{1}{P_n} \sum_{v=0}^{n} \frac{p_{n-v}}{Q_v} \sum_{j=0}^{v} q_{v-j} s_j \, ,$$

and gave the following definition:

**Definition 2.1.** *The infinite series* $\sum_{n=0}^{\infty} a_n$ *is said to be summable* $|(N, p)(N, q)|$, *if the sequence* $\{U_n\}$ *is of bonded variation, i.e. the series*

$$\sum_{n=1}^{\infty} |U_n - U_{n-1}|$$

*converges.*

Later on, W. T. Sulaiman [1] considered the transformation

$$V_n := \frac{1}{Q_n} \sum_{v=0}^{n} \frac{q_v}{P_v} \sum_{j=0}^{v} p_j s_j$$

of the sequence $\{s_n\}$, and presented the definition:

**Definition 2.2.** *The infinite series* $\sum_{n=0}^{\infty} a_n$ *is said to be summable* $|(R, q_n)(R, p_n)|_k$, $k \geq 1$, *if*

$$\sum_{n=1}^{\infty} n^{k-1} |V_n - V_{n-1}|^k$$

*converges, and we write in brief*

$$\sum_{n=0}^{\infty} a_n \in |(R, q_n)(R, p_n)|_k .$$

Let us denote by $D_n$ the transformation

$$D_n := \frac{1}{R_n}\sum_{v=0}^{n}\frac{p_{n-v}q_v}{R_v}\sum_{j=0}^{v}p_jq_{v-j}s_j$$

of the sequence $\{s_n\}$.

Now we shall introduce the following definition:

**Definition 2.3.** *The infinite series* $\sum_{n=0}^{\infty}a_n$ *is said to be summable* $|(N,p_n,q_n)(N,q_n,p_n)|_k$, $k\geq 1$, *if*

$$\sum_{n=1}^{\infty}n^{k-1}|D_n-D_{n-1}|^k$$

*converges, and we write shortly* $\sum_{n=0}^{\infty}a_n\in|(N,p_n,q_n)(N,q_n,p_n)|_k$.

The main purpose of the present paper is to study the $|(N,p_n,q_n)(N,q_n,p_n)|_k$ summability of the orthogonal series (2.1) for $1\leq k\leq 2$.

Throughout $K$ denotes a positive constant that it may depends only on $k$, and be different in different relations.

The following lemma due to Beppo Levi (see, for example [7]) is often used in the theory of functions. It will need us to prove main results.

**Lemma 2.1.** *If* $f_n(t)\in L(E)$ *are non-negative functions and*

$$\sum_{n=1}^{\infty}\int_E f_n(t)dt<\infty, \tag{2.2}$$

*then the series*

$$\sum_{n=1}^{\infty}f_n(t)$$

*converges almost everywhere on* $E$ *to a function* $f(t)\in L(E)$. *Moreover, the series (2.2) is also convergent to* $f$ *in the norm of* $L(E)$.

## 3. Main Results

We prove the following theorem.

**Theorem 3.1.** *If for* $1\leq k\leq 2$ *the series*

$$\sum_{n=1}^{\infty}\left[n^{2-\frac{2}{k}}\sum_{i=1}^{n}\left(\frac{R_n^i\widetilde{R}_n^i}{R_n}-\frac{R_{n-1}^i\widetilde{R}_{n-1}^i}{R_{n-1}}\right)^2|c_i|^2\right]^{\frac{k}{2}}$$

*converges, then the orthogonal series*

$$\sum_{n=0}^{\infty}c_n\varphi_n(x)$$

*is summable* $|(N,p_n,q_n)(N,q_n,p_n)|_k$ *almost everywhere.*

*Proof.* First we consider the case $k \in (1,2)$. We use the notations

$$\widetilde{R}_n^i := \sum_{v=i}^{n} \frac{q_{n-v}p_v}{R_v}; \quad \widetilde{R}_{n-1}^n = 0.$$

Let

$$s_j(x) = \sum_{i=0}^{j} c_i\varphi_i(x)$$

be the partial sums of order $j$ of the series (2.1). Then, for the transform $D_n(x)$ of the partial sums $s_j(x)$, we have

$$\begin{aligned} D_n(x) &= \frac{1}{R_n}\sum_{v=0}^{n}\frac{p_{n-v}q_v}{R_v}\sum_{j=0}^{v}p_j q_{v-j}\sum_{i=0}^{j}c_i\varphi_i(x) \\ &= \frac{1}{R_n}\sum_{v=0}^{n}\frac{p_{n-v}q_v}{R_v}\sum_{i=0}^{v}c_i\varphi_i(x)\sum_{j=i}^{v}p_j q_{v-j} \\ &= \frac{1}{R_n}\sum_{v=0}^{n}\frac{p_{n-v}q_v}{R_v}\sum_{i=0}^{v}R_v^i c_i\varphi_i(x) \\ &= \frac{1}{R_n}\sum_{i=0}^{n}R_n^i c_i\varphi_i(x)\sum_{v=i}^{n}\frac{p_{n-v}q_v}{R_v} \\ &= \sum_{i=0}^{n}\frac{R_n^i\widetilde{R}_n^i}{R_n}c_i\varphi_i(x). \end{aligned}$$

Whence,

$$\begin{aligned} \triangle D_n(x) &= D_n(x) - D_{n-1}(x) \\ &= \sum_{i=0}^{n}\frac{R_n^i\widetilde{R}_n^i}{R_n}c_i\varphi_i(x) - \sum_{i=0}^{n-1}\frac{R_{n-1}^i\widetilde{R}_{n-1}^i}{R_{n-1}}c_i\varphi_i(x) \\ &= \sum_{i=1}^{n}\left(\frac{R_n^i\widetilde{R}_n^i}{R_n} - \frac{R_{n-1}^i\widetilde{R}_{n-1}^i}{R_{n-1}}\right)c_i\varphi_i(x). \end{aligned}$$

Using the Hölder's inequality and orthogonality to the latter equality, we obtain

$$\begin{aligned} \int_a^b |\triangle D_n(x)|^k dx &\le (b-a)^{1-\frac{k}{2}}\left(\int_a^b |D_n(x) - D_{n-1}(x)|^2 dx\right)^{\frac{k}{2}} \\ &= (b-a)^{1-\frac{k}{2}}\left[\sum_{i=1}^{n}\left(\frac{R_n^i\widetilde{R}_n^i}{R_n} - \frac{R_{n-1}^i\widetilde{R}_{n-1}^i}{R_{n-1}}\right)^2 |c_i|^2\right]^{\frac{k}{2}}. \end{aligned}$$

Subsequently, the series

$$(3.1)\quad \sum_{n=1}^{\infty} n^{k-1}\int_a^b |\triangle D_n(x)|^k dx \le K\sum_{n=1}^{\infty} n^{k-1}\left[\sum_{i=1}^{n}\left(\frac{R_n^i\widetilde{R}_n^i}{R_n} - \frac{R_{n-1}^i\widetilde{R}_{n-1}^i}{R_{n-1}}\right)^2 |c_i|^2\right]^{\frac{k}{2}}$$

converges, since the last one does. Now according to the Lemma 2.1 the series (2.1) is summable $|(N,p_n,q_n)(N,q_n,p_n)|_k$ almost everywhere. For $k = 2$ we apply only the orthogonality, as far as for $k = 1$ we apply the well-known Schwarz's inequality. This completes the proof of the theorem. □

Now we shall prove the counterpart of Theorem 3.1 (it can be seen also as the counterpart of a theorem of P. L. Ul'yanov [10]). It is a general theorem which involves in it a new positive sequence with some additional conditions. For this reason first we put:

$$\mathfrak{R}^{(k)}(i) := \frac{1}{i^{\frac{2}{k}-1}} \sum_{n=i}^{\infty} n^{\frac{2}{k}} \left( \frac{R_n^i \widetilde{R}_n^i}{R_n} - \frac{R_{n-1}^i \widetilde{R}_{n-1}^i}{R_{n-1}} \right)^2, \tag{3.2}$$

and then the following theorem holds true.

**Theorem 3.2.** *Let $1 \leq k \leq 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges.*

*Let $\{p_n\}$ and $\{q_n\}$ be non-negative. If the series*

$$\sum_{n=1}^{\infty} |c_n|^2 \Omega^{\frac{2}{k}-1}(n) \mathfrak{R}^{(k)}(n)$$

*converges, then the orthogonal series*

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \in |(N, p_n, q_n)(N, q_n, p_n)|_k$$

*almost everywhere, where $\mathfrak{R}^{(k)}(n)$ is defined by (3.2).*

*Proof.* Applying Hölder's inequality to the inequality (3.1) we get that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{k-1} \int_a^b |\triangle D_n(x)|^k dx \\
&\leq K \sum_{n=1}^{\infty} n^{k-1} \left[ \sum_{i=1}^{n} \left( \frac{R_n^i \widetilde{R}_n^i}{R_n} - \frac{R_{n-1}^i \widetilde{R}_{n-1}^i}{R_{n-1}} \right)^2 |c_i|^2 \right]^{\frac{k}{2}} \\
&= K \sum_{n=1}^{\infty} \frac{1}{(n\Omega(n))^{\frac{2-k}{2}}} \left[ (n\Omega(n))^{\frac{2}{k}-1} n^{2-\frac{2}{k}} \sum_{i=1}^{n} \left( \frac{R_n^i \widetilde{R}_n^i}{R_n} - \frac{R_{n-1}^i \widetilde{R}_{n-1}^i}{R_{n-1}} \right)^2 |c_i|^2 \right]^{\frac{k}{2}} \\
&\leq K \left( \sum_{n=1}^{\infty} \frac{1}{n\Omega(n)} \right)^{\frac{2-k}{2}} \left[ \sum_{n=1}^{\infty} (n\Omega(n))^{\frac{2}{k}-1} n^{2-\frac{2}{k}} \sum_{i=1}^{n} \left( \frac{R_n^i \widetilde{R}_n^i}{R_n} - \frac{R_{n-1}^i \widetilde{R}_{n-1}^i}{R_{n-1}} \right)^2 |c_i|^2 \right]^{\frac{k}{2}} \\
&\leq K \left\{ \sum_{i=1}^{\infty} |c_i|^2 \sum_{n=i}^{\infty} (n\Omega(n))^{\frac{2}{k}-1} n^{2-\frac{2}{k}} \left( \frac{R_n^i \widetilde{R}_n^i}{R_n} - \frac{R_{n-1}^i \widetilde{R}_{n-1}^i}{R_{n-1}} \right)^2 \right\}^{\frac{k}{2}} \\
&\leq K \left\{ \sum_{i=1}^{\infty} |c_i|^2 \left( \frac{\Omega(i)}{i} \right)^{\frac{2}{k}-1} \sum_{n=i}^{\infty} n^{\frac{2}{k}} \left( \frac{R_n^i \widetilde{R}_n^i}{R_n} - \frac{R_{n-1}^i \widetilde{R}_{n-1}^i}{R_{n-1}} \right)^2 \right\}^{\frac{k}{2}} \\
&\leq K \left\{ \sum_{i=1}^{\infty} |c_i|^2 \Omega^{\frac{2}{k}-1}(i) \mathfrak{R}^{(k)}(i) \right\}^{\frac{k}{2}},
\end{aligned}$$

which is finite by assumption. Doing the same reasoning as in the proof of Theorem 3.1 we easy arrive to finish the proof. □

It is obvious that the transformation $D_n$ can never be the same as $\sigma_n^{p,q}$, therefore Theorems 3.1-3.2 bring new results. Moreover, transformation $D_n$ can not reduces to the transformations $U_n$ or $V_n$, i.e. it would be of particular interest to answer questions: Under what conditions an orthogonal series of the form (2.1) is $(N,p)(N,q)$ or $(R,q_n)(R,p_n)_k$ summable? Regarding to these questions, in the following, we shall give four theorems without their proofs.

Denote

$$\widetilde{P}_n^{i,q} = \sum_{v=i}^{n} \frac{p_{n-v}}{Q_v}, \quad \widetilde{Q}_n^{i,p} = \sum_{v=i}^{n} \frac{q_v}{P_v}, \quad \text{and} \quad \widetilde{P}_{n-1}^{n,q} = \widetilde{Q}_{n-1}^{n,p} = 0.$$

**Theorem 3.3.** *If the series*

$$\sum_{n=1}^{\infty} \left[ \sum_{i=1}^{n} \left( \frac{Q_n^i \widetilde{P}_n^{i,q}}{P_n} - \frac{Q_{n-1}^i \widetilde{P}_{n-1}^{i,q}}{P_{n-1}} \right)^2 |c_i|^2 \right]^{\frac{1}{2}}$$

*converges, then the orthogonal series*

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

*is summable $|(N,p)(N,q)|$ almost everywhere.*

**Theorem 3.4.** *Let $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges.*

*Let $\{p_n\}$ and $\{q_n\}$ be non-negative. If the series*

$$\sum_{n=1}^{\infty} |c_n|^2 \Omega(n) \mathfrak{R}^{p,q}(n)$$

*converges, then the orthogonal series*

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \in |(N,p)(N,q)|$$

*almost everywhere, where $\mathfrak{R}^{p,q}(n)$ is defined by*

$$\mathfrak{R}^{p,q}(i) := \frac{1}{i} \sum_{n=i}^{\infty} n^2 \left( \frac{Q_n^i \widetilde{P}_n^{i,q}}{P_n} - \frac{Q_{n-1}^i \widetilde{P}_{n-1}^{i,q}}{P_{n-1}} \right)^2 .$$

**Theorem 3.5.** *If for $1 \leq k \leq 2$ the series*

$$\sum_{n=1}^{\infty} \left[ n^{2-\frac{2}{k}} \sum_{i=1}^{n} \left( \frac{P_n^i \widetilde{Q}_n^{i,p}}{Q_n} - \frac{P_{n-1}^i \widetilde{Q}_{n-1}^{i,p}}{Q_{n-1}} \right)^2 |c_i|^2 \right]^{\frac{k}{2}}$$

*converges, then the orthogonal series*

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

*is summable* $|(R,q_n)(R,p_n)|_k$ *almost everywhere.*

**Theorem 3.6.** *Let* $1 \le k \le 2$ *and* $\{\Omega(n)\}$ *be a positive sequence such that* $\{\Omega(n)/n\}$ *is a non-increasing sequence and the series* $\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}$ *converges.*

*Let* $\{p_n\}$ *and* $\{q_n\}$ *be non-negative. If the series*

$$\sum_{n=1}^{\infty}|c_n|^2\Omega^{\frac{2}{k}-1}(n)\widehat{\mathfrak{R}}^{(p,q;k)}(n)$$

*converges, then the orthogonal series*

$$\sum_{n=0}^{\infty}c_n\varphi_n(x)\in|(R,q_n)(R,p_n)|_k$$

*almost everywhere, where* $\widehat{\mathfrak{R}}^{(p,q;k)}(n)$ *is defined by*

$$\widehat{\mathfrak{R}}^{(p,q;k)}(i):=\frac{1}{i^{\frac{2}{k}-1}}\sum_{n=i}^{\infty}n^{\frac{2}{k}}\left(\frac{P_n^i\widetilde{Q}_n^{i,p}}{Q_n}-\frac{P_{n-1}^i\widetilde{Q}_{n-1}^{i,p}}{Q_{n-1}}\right)^2.$$

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