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DETERMINATION OF JUMPS OF A FUNCTION OF $\mathcal{V}_P$ CLASS BY ITS INTEGRATED FOURIER-JACOBI SERIES

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Abstract. The problem of determination of jump discontinuities in piecewise smooth functions from their spectral data is relevant in signal processing. We obtain new identity which determines the jumps of a periodic function of $\mathcal{V}_p, 1 \leqslant p < 2$, class with a finite number of discontinuities, by means of the tails of its integrated Fourier-Jacobi series. Next, we establish (C, α), $\alpha > 1 - \dfrac{1}{p}$, summability of the sequence $(n^2 a_n(w; f) \displaystyle\int P_n(w; x)dx)$, where $a_n(w; f) \displaystyle\int P_n(w; x)dx$ is the n-th term of the integrated Fourier-Jacobi series of a function f.

1. Introduction and Preliminaries

The problem of locating the discontinuities of a function by means of its truncated Fourier series, arises naturally from an attempt to overcome the Gibbs phenomenon, the poor approximative properties of the Fourier partial sums of a discontinuous function (i.e. the finite sum approximation of the discontinuous function overshoots the function itself, at a discontinuity by about 18 percent).

If a function f is integrable on $[-\pi, \pi]$, then it has a Fourier series with respect to the trigonometric system $\{1, cos\, nx, sin\, nx\}_{n=1}^{\infty}$, and we denote the n-th partial sum of the Fourier series of f by $S_n(x, f)$, i.e.,

$$S_n(x, f) = \frac{a_0(f)}{2} + \sum_{k=1}^{n}(a_k(f) cos\, kx + b_k(f) sin\, kx),$$

where $a_k(f) = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(t) cos\, kt\, dt$ and $b_k(f) = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(t) sin\, kt\, dt$ are the $k-$th Fourier coefficients of the function f.

The identity determining the jumps of a function of bounded variation by means of its differentiated Fourier partial sums has been known for a long time. Let $f(x)$ be a

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function of bounded variation with period 2π, and $S_n(x,f)$ be the partial sum of order n of its Fourier series. By the classical theorem of Fejer [16] the identity:

$$\lim_{n\to\infty} \frac{S_n'(x,f)}{n} = \frac{1}{\pi}(f(x+0)-f(x-0)), \tag{1.1}$$

holds at any point x. To characterize continuous periodic functions of BV in terms of their Fourier coefficients, Wiener [15] has introduced a concept of higher variation.

A function f is said to be of bounded p-variation, $p \geq 1$, on the segment $[a,b]$ and to belong to the class $\mathcal{V}_p[a,b]$ if

$$V_a^b{}_p(f) = \sup_{\Pi_{a,b}} \Big\{ \sum_i |f(x_i)-f(x_{i-1})|^p \Big\}^{\frac{1}{p}} < \infty,$$

where $\Pi_{a,b} = \{a = x_0 < x_1 < \ldots < x_n = b\}$ is an arbitrary partition of the segment $[a,b]$. $V_a^b{}_p(f)$ is the p-variation of f on $[a,b]$.

B.I. Golubov [8] has shown that identity (1.1) is valid for classes $\mathcal{V}_p$.

Theorem (A). *Let $f(x) \in \mathcal{V}_p$, $(1 \leq p < \infty)$ and $r \in \mathbb{N}_0$. Then for any point x one has the equation*

$$\lim_{n\to\infty} \frac{S_n^{(2r+1)}(x,f)}{n^{2r+1}} = \frac{(-1)^r}{(2r+1)\pi}(f(x+0)-f(x-0)). \tag{1.2}$$

Problems of everywhere convergence of Fourier series for every change of variable have led D. Waterman [14] to another type of generalization.
Let $\Lambda = \{\lambda_n\}$ be a nondecreasing sequence of positive numbers such that $\sum \frac{1}{\lambda_n}$ diverges and $\{I_n\}$ be a sequence of non overlapping segments $I_n = [a_n, b_n] \subset [a,b]$. A function f is said to be of Λ-bounded variation on $I = [a,b]$ ($f \in \Lambda BV$) if $\sum \frac{|f(b_n)-f(a_n)|}{\lambda_n} < \infty$ for every choice of $\{I_n\}$. The supremum of these sums is called the Λ-variation of f on I. In the case $\Lambda = \{n\}$, one speaks of harmonic bounded variation (HBV).

The class HBV contains all Wiener classes. Avdispahic has shown in [3] that HBV is the limiting case for validity of the identity (1.1):

Theorem (B). *The equation (1.1) holds for any function $f \in HBV$ at any point x.*

The third interesting generalization of the Jordan variation was given by Z. A. Chanturiya [5]. The modulus of variation of a bounded 2π periodic function f is the function ν_f with domain the positive integers, given by

$$\nu_f(n) = \sup_{\Pi_n} \sum_{k=1}^{n} |f(b_k)-f(a_k)|,$$

where $\Pi_n = \{[a_k,b_k]; k = 1,\ldots,n\}$ is an arbitrary partition of $[0,2\pi]$ into n non overlapping segments.

By a theorem of Avdispahic [1], there exist the following inclusion relations between Wiener's, Waterman's and Chanturiya's classes:

Theorem (C).
$$\{n^\alpha\}BV \subset \mathcal{V}_{\frac{1}{1-\alpha}} \subset V[n^\alpha] \subset \{n^\beta\}BV,$$
for $0 < \alpha < \beta < 1$.

Clearly, Fejer's identity (1.1) is a statement about Cesaro summability of the sequence $\{kb_k \cos kx - ka_k \sin kx\}$, $a_k = a_k(f)$ and $b_k = b_k(f)$ being the k-th cosine and sine coefficient, respectively. Looking at Fejer's theorem in this way, several mathematicians have extended it to more general summability methods. We note two results [2] which represent the extension to (C, α) summability, $\alpha > 0$:

Theorem (D). *If* $f \in \mathcal{V}_p$, $p > 1$ *the sequence* $\{kb_k \cos kx - ka_k \sin kx\}$ *is* (C, α) *summable to* $\frac{1}{\pi}(f(x+0) - f(x-0))$ *for any* $\alpha > 1 - \frac{1}{p}$ *and every* x.

Corollary(E). *If* $f \in V[n^\beta]$ $(\{n^\beta\}BV)$ *for some* $0 < \beta < 1$, *then the sequence* $\{kb_k \cos kx - ka_k \sin kx\}$ *is summable to* $\frac{1}{\pi}(f(x+0) - f(x-0))$ *by any Cesaro method of order* $\alpha > \beta$.

Theorem (D) and Corollary (E), are in some sense the most natural generalization of Fejer's theorem. Indicating the relationship between the order of Cesaro summability of the sequence $\{kb_k(f) \cos kx - ka_k(f) \sin kx\}$ and the "order of variation" of a function f, they complete the earlier picture whose elements were:
1) (C, α) summability for $\alpha > 0$ and the class BV;
2) (C, α) summability for $\alpha > 1$ and whole class of regulated functions (i.e. functions possessing the one-sided limits at each point);
3) $(C, 1)$ summability for the class HBV.

Similar identities hold if we consider the integrated rather than the differentiated Fourier series [9]. By $R_n(x, f)$ we denote the n-th order tails of the Fourier series of the function f, i.e.,
$$R_n(x, f) = \sum_{k=n}^{\infty} (a_k(f) coskx + b_k(f) sinkx),$$
for $n \in \mathbb{N}$.

For any function f, integrable on $[-\pi, \pi]$, $f^{(-r)}$, $r \in \mathbb{N}_0$, is defined as follows
$$f^{(-r-1)} \equiv \int f^{(-r)},$$
where $f^{(0)} \equiv f$, and the constants of integration are successively determined by the condition
$$\int_{-\pi}^{\pi} f^{(-r)}(t)dt = 0.$$

Theorem (F). *Let* $r \in \mathbb{N}_0$ *and suppose the function* $f \in \mathcal{V}_p$, $1 \leqslant p < 2$, *has a finite number of discontinuities. Then:*

1. the identity
$$\lim_{n\to\infty} n^{2r+1} R_n^{(-2r-1)}(f; x) = \frac{(-1)^{r+1}}{(2r+1)\pi}(f(x+) - f(x-))$$

is valid for each fixed $x \in [-\pi, \pi]$*;*
2. there is no way to determine the jump at the point $x \in [-\pi, \pi]$ *of an arbitrary function* $f \in \mathcal{V}_p$*,* $p \geqslant 1$*, by means of the sequence* $(R_n^{(-2r-2)}(f;.)), n \in \mathbb{N}$.

Such results find their application in recovering edges in piecewise smooth functions with finitely many jump discontinuities [6].

We say that a function w is a generalized Jacobi weight and write $w \in GJ$, if

$$w(t) = h(t)(1-t)^{\alpha}(1+t)^{\beta}|t-x_1|^{\delta_1}...|t-x_M|^{\delta_M},$$

$$h \in C[-1,1],\ h(t) > 0\,(|t| \leq 1),\ \ \omega(h;t;[-1,1])t^{-1} \in L[0,1],$$

$$-1 < x_1 < ... < x_M < 1, \quad \alpha, \beta, \delta_1, ..., \delta_M > -1.$$

By $\sigma(w) = (P_n(w;x))_{n=0}^{\infty}$ we denote the system of algebraic polynomials $P_n(w;x) = \gamma(w)x^n +$ lower degree terms with positive leading coefficients $\gamma_n(w)$, which are orthonormal on $[-1,1]$ with respect to the weight $w \in GJ$, i.e.,

$$\int_{-1}^{1} P_n(w;t)P_m(w;t)w(t)dt = \delta_{nm}.$$

Such polynomials are called the generalized Jacobi polynomials. If $fw \in L[-1,1]$, and $w \in GJ$, then the n th partial sum of the Fourier series of f with respect to the system $\sigma(w)$ is given by

$$S_n(w;f;x) = \sum_{k=0}^{n-1} a_k(w;f)P_k(w;x) = \int_{-1}^{1} f(t)K_n(w;x;t)w(t)dt\,,$$

where $a_k(w;f) = \displaystyle\int_{-1}^{1} f(t)P_k(w;t)w(t)dt$ is the k th Fourier coefficient of the function f, and

$$K_n(w;x;t) = \sum_{k=0}^{n-1} P_k(w;x)P_k(w;t)\,,$$

is the Dirichlet kernel of the system $\sigma(w)$.
For a given weight $w \in GJ$ it is assumed that $x_0 = -1$ and $x_{M+1} = 1$. In addition,

$$\Delta(\nu;\varepsilon) = [x_\nu + \varepsilon; x_{\nu+1} - \varepsilon]\,,$$

for a fixed $\varepsilon \in (0, \frac{x_{\nu+1}-x_\nu}{2})$, $\ \nu = 1,2,...,M.$

For functions of $\Lambda-$bounded variation G. Kvernadze [10] has proved the following theorem:

Theorem (G). *Let* $r \in \mathbb{N}_0$*,* $w \in GJ$*, and suppose* ΛBV *is the class of functions of* Λ*-bounded variation determined by the sequence* $\Lambda = (\lambda_k)_{k=1}^{\infty}$*. Then the identity*

$$\lim_{n\to\infty} \frac{(S_n(w;f;x))^{2r+1}}{n^{2r+1}} = \frac{(-1)^r(1-x^2)^{-r-\frac{1}{2}}}{(2r+1)\pi}(f(x+0)-f(x-0)) \tag{1.3}$$

is valid for every $f \in \Lambda BV$ *and each fixed* $x \in (-1,1)$*,* $x \neq x_1, ..., x_M$*, if* $\Lambda BV \subseteq HBV$*. If, in addition, the weight* $w \in GJ$ *satisfies the following conditions:*

$$\alpha \geq -\frac{1}{2},\ \ \beta \geq -\frac{1}{2},\ \delta_1 \geq 0, ..., \delta_M \geq 0,\ \ \omega(h;t)t^{-1}\ln t \in L[0,1], \tag{1.4}$$

then condition $\Lambda BV \subseteq HBV$ is necessary for the validity of identity (1.3) for every $f \in \Lambda BV$ and each fixed $x \in (-1,1)$, $x \neq x_1, ..., x_M$ as well.

In [11], [12] is shown that the jump of a function f belonging to the Wiener class $\mathcal{V}_p$, $p > 1$, can be determined through (C,α), $\alpha > 1 - \dfrac{1}{p}$, summability of the sequence of terms of it's differentiated Fourier-Jacobi series. Consesequently, the corresponding (C,α) summability result holds for the Waterman classes $\{n^\beta\}BV$ and the Chanturiya classes $V[n^\beta]$ if $\alpha > \beta$, $0 < \beta < 1$.

2. Main Results

Theorem 2.1. *Let $r \in \mathbb{N}_0$ and suppose the function $f \in \mathcal{V}_p$, $1 \leqslant p < 2$, has a finite number of discontinuities and $fw \in L[-1,1], w \in GJ$. Then the identity*

$$\lim_{n\to\infty} nR_n^{(-1)}(w; f; x) = -\frac{1}{\pi}(1-x^2)^{\frac{1}{2}}(f(x+) - f(x-))$$

is valid for each fixed $x \in [-1,1]$, where $R_n^{(-1)}(w; f; x)$ is the n-th order tails of the integrated Fourier-Jacobi series of the function f.

Proof. By $S_n^{(-\frac{1}{2},-\frac{1}{2})}(f; x)$ we denote the n-th partial sum of the Fourier-Tchebycheff series of function f [4]. We use the uniform equiconvergence of Fourier-Tchebycheff series and Fourier series with respect to the system of generalized Jacobi polynomials for an arbitrary function $f \in HBV$ and a fixed $\varepsilon \in (0, \frac{x_{\nu+1}-x_\nu}{2})$, $\nu = 0, 1, 2, ..., M$

$$\|S_n(w; f; x) - S_n^{(-\frac{1}{2},-\frac{1}{2})}(f; x)\|_{C[\Delta(\nu;\frac{\varepsilon}{2})]} = o(1), \tag{2.1}$$

proved by Kvernadze [10, p.185].

From the equiconvergence formula (2.1) and from the identities:

$$S_n(w; f; x) = f(x) - R_n(w; f; x),$$

$$S_n^{(-\frac{1}{2},-\frac{1}{2})}(f; x) = f(x) - R_n^{(-\frac{1}{2},-\frac{1}{2})}(f; x),$$

we obtain

$$\|R_n(w; f; x) - R_n^{(-\frac{1}{2},-\frac{1}{2})}(f; x)\|_{C[\Delta(\nu;\frac{\varepsilon}{2})]} = o(1). \tag{2.2}$$

From an obvious identity [13]

$$S_n^{(-\frac{1}{2},-\frac{1}{2})}(f; x) = S_n(g, \theta),$$

where $x = \cos\theta$, $g(\theta) = f(cos\theta)$ one has

$$R_n^{(-\frac{1}{2},-\frac{1}{2})}(f; x) = R_n(g, \theta).$$

Integrating the last identity with respect to x we obtain

$$[R_n^{(-\frac{1}{2},-\frac{1}{2})}(f; x)]^{(-1)} = -(1-x^2)^{\frac{1}{2}} R_n^{(-1)}(g; \theta) + \int R_n^{(-1)}(g; \theta) cos\theta d\theta\,. \tag{2.3}$$

Multiplying by n the identity (2.3), we get

(2.4)
$$\lim_{n\to\infty} n[R_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x)]^{(-1)} = -(1-x^2)^{\frac{1}{2}} \lim_{n\to\infty} nR_n^{(-1)}(g;\theta) + \lim_{n\to\infty} n\int R_n^{(-1)}(g;\theta)cos\theta d\theta\,.$$

Since
$$|n\int R_n^{(-1)}(g;\theta)cos\theta d\theta| \leqslant \int n|R_n^{(-1)}(g;\theta)|d\theta\,,$$
it is enough to estimate the term $n|R_n^{(-1)}(g;\theta)|$.

If $G(\theta) = \dfrac{\pi-\theta}{2}$, $\theta \in (0, 2\pi)$, is the 2π periodic sawtooth function, then the function g can be represented as follows [9]:

$$g_c(\theta) \equiv g(\theta) - \frac{1}{\pi}\sum_{m=0}^{M-1}[g]_m G(\theta-\theta_m), \tag{2.5}$$

where θ_m and $[g]_m$, $m = 0, 1, ...M-1$, are the locations of discontinuities and the associated jumps of the function g, and g_c is the $2\pi-$ periodic continuous function, which is piecewise smooth on $[-\pi, \pi]$.

Obviously,

$$g_c \in C \cap V_p. \tag{2.6}$$

Continuity of g_c follows from (2.5). Besides, since $G \in V \subset V_p$ and V_p is a linear vector space, $g_c \in V_p$ as well.

It is known that if $g_c \in V_p$, $1 \leqslant p < 2$, then the function g is continous if and only if its Fourier coefficients satisfy the following condition [7]:

$$\sum_{k=n}^{\infty}\left(a_k(f)^2 + b_k(f)^2\right) = o\left(\frac{1}{n}\right)\,. \tag{2.7}$$

Thus, according to (2.6), (2.7) and Cauchy-Schwartz inequality we have:

$$n|R_n^{(-1)}(g_c;\theta)| \leqslant n\sum_{k=n}^{\infty}\frac{|a_k(g_c)|+|b_k(g_c)|}{k}$$
$$\leqslant \sqrt{2}\,n\left(\sum_{k=n}^{\infty}(a_k(g_c)^2+b_k(g_c)^2\right)^{\frac{1}{2}}\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}$$
$$= n\;o\left(n^{-\frac{1}{2}}\right)\;O\left(n^{-\frac{1}{2}}\right) = o(1)\,, \tag{2.8}$$

uniformly with respect to $\theta \in [-\pi, \pi]$.

As, by means of a change of variables the problem can always be reduced to the case $\theta = 0$, according to [9, p.33] we have

$$nR_n^{(-1)}(G(\theta_m), 0) = o(1)\;. \tag{2.9}$$

By use of (2.8) and (2.9) it follows

$$\lim_{n\to\infty} n|R_n^{(-1)}(g;\theta)| = 0\,. \tag{2.10}$$

Using (2.3) and (2.10) we get

$$\lim_{n\to\infty} n[R_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x)]^{(-1)} = -(1-x^2)^{\frac{1}{2}} \lim_{n\to\infty} nR_n^{(-1)}(g;\theta)\,. \tag{2.11}$$

Further, using Theorem (F) for $r=0$ we have

$$\lim_{n\to\infty} n[R_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x)]^{(-1)} = -(1-x^2)^{\frac{1}{2}} \frac{-1}{\pi}(g(\theta+)-g(\theta-))\,. \tag{2.12}$$

Hence, taking into account that $f(x\pm) = g(\theta\mp)$, $\theta\in[0,\pi]$ in the identity (2.12), we get:

$$\lim_{n\to\infty} n[R_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x)]^{(-1)} = -(1-x^2)^{\frac{1}{2}} \frac{1}{\pi}(f(x+)-f(x-))\,. \tag{2.13}$$

Finally, result follows from the equiconvergence formula (2.2). ☐

Theorem 2.2. *Let f be a function of bounded p-variation, i.e. $f\in\mathcal{V}_p$, $1\leqslant p<2$, which has a finite number of discontinuities such that $fw\in L[-1,1], w\in GJ$. Then the sequence $\{n^2 a_n(w;f)\int P_n(w;x)dx\}$ is (C,α), $\alpha > 1-\dfrac{1}{p}$ summable to $\dfrac{(1-x^2)^{\frac{1}{2}}}{\pi}(f(x+0)-f(x-0))$ for every $x\in[-1,1]$, where $\{a_n(w;f)\int P_n(w;x)dx\}$ is the n-th term of the integrated Fourier-Jacobi series of f.*

Proof. By $a_n^{(-\frac{1}{2},-\frac{1}{2})}(f)P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)$ we denote the n -th term of the Fourier-Tchebycheff series of f [4]. From the equiconvergence formula (2.1) and identities

$$S_n(w;f;x) = S_{n-1}(w;f;x) + a_n(w;f)P_n(w;x),$$
$$S_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x) = S_{n-1}^{(-\frac{1}{2},-\frac{1}{2})}(f;x) + a_n^{(-\frac{1}{2},-\frac{1}{2})}(f)P_n^{(-\frac{1}{2},-\frac{1}{2})}(x),$$

by the triangle inequality we get:

$$\|a_n(w;f)P_n(w;x) - a_n^{(-\frac{1}{2},-\frac{1}{2})}(f)P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)\|_{C[\Delta(\nu;\frac{\varepsilon}{2})]} = o(1)\,. \tag{2.14}$$

According to the identity (2.11) we have:

$$\lim_{n\to\infty}(n+1)[R_{n+1}^{(-\frac{1}{2},-\frac{1}{2})}(f;x)]^{(-1)} = -(1-x^2)^{\frac{1}{2}} \lim_{n\to\infty}(n+1)R_{n+1}^{(-1)}(g;\theta)\,.$$

Subtracting the last identity from the identity (2.11), we get

$$\lim_{n\to\infty} na_n^{(-\frac{1}{2},-\frac{1}{2})}(f)\int P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)dx = -(1-x^2)^{\frac{1}{2}}\lim_{n\to\infty}(a_n\sin nx - b_n\cos nx) + \tag{2.15}$$
$$\lim_{n\to\infty}\Big([R_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x)]^{(-1)} + (1-x^2)^{\frac{1}{2}}R_n^{(-1)}(g;\theta)\Big)\,.$$

The second sumand on the right side of the equation (2.16) tends to zero according to (2.11). Now, multiplaing by n the identity

$$\lim_{n\to\infty} na_n^{(-\frac{1}{2},-\frac{1}{2})}(f)\int P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)dx = -(1-x^2)^{\frac{1}{2}}\lim_{n\to\infty}(a_n\sin nx - b_n\cos nx)\ , \tag{2.16}$$

we have:

$$\lim_{n\to\infty} n^2a_n^{(-\frac{1}{2},-\frac{1}{2})}(f)\int P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)dx = (1-x^2)^{\frac{1}{2}}\lim_{n\to\infty}(nb_n\cos nx - na_n\sin nx)\,. \tag{2.17}$$

Finaly, the result follows from the Theorem (D) and the equiconvergence formula (2.14). □

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