ADV MATH SCI JOURNAL

Advances in Mathematics: Scientific Journal **2** (2013), no.1, 25–33
ISSN 1857-8365, UDC: 514.754:515.171

ODD-DIMENSIONAL RIEMANNIAN SPACES WITH ALMOST CONTACT AND ALMOST PARACONTACT STRUCTURES

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ABSTRACT. Riemannian spaces admitting almost contact and almost paracontact structures are studied from the point of view of compositions in spaces with a symmetric affine connection. Linear connections with torsion preserving by covariant differentiation the almost (para-)contact structure or the metric tensor are considered.

1. Introduction

Riemannian spaces with almost contact and almost paracontact structures have been studied by various authors, e.g. [1, 3, 4, 5, 8, 9, 10]. The almost contact structure is an odd-dimensional extension of the complex structure, and the almost paracontact structure can be considered as an extension of the almost product structure.

By the help of n independent vector fields in [13, 11, 12, 2] an apparatus for studying of spaces endowed with a symmetric affine connection is constructed.

In this work we apply this apparatus to study odd-dimensional Riemannian spaces V_{2n+1} admitting almost contact and almost paracontact structures. We prove that if these structures are parallel to the Levi-Civita connection of the Riemannian metric the space V_{2n+1} is a topological product of three differentiable manifolds $X_n \times \overline{X}_n \times X_1$. We also determine the projecting affinors of the structures and by their help obtain some characteristics of the considered space.

In the last section, we study linear connections with respect to which the structures of the space are parallel. We define a connection with torsion which preserves the metric tensor by covariant differentiation and compute the components of its curvature tensor.

2. Preliminaries

Let V_{2n+1} be a Riemannian space with metric tensor $g_{\alpha\beta}(\overset{\tau}{u})$ and Levi-Civita connection ∇ with Cristoffel symbols $\Gamma^{\sigma}_{\alpha\beta}$. Then, it is known that $\nabla_\sigma g_{\alpha\beta} = 0$.

2010 *Mathematics Subject Classification.* 53B05, 53B99.

Key words and phrases. Composition, almost paracontact structure, almost contact structure, affinor of composition.

We introduce the following notations

$$\begin{aligned} &\alpha,\beta,\gamma,\delta,\nu,\sigma,\tau = 1,2,...,2n+1,\\ &a,b,c,d,e = 1,2,...,2n, \\ &j,k,l,p,q,s = 1,2,...,n; \quad \bar{j},\bar{k},\bar{l},\bar{p},\bar{q},\bar{s} = n+1,n+2,...,2n. \end{aligned} \tag{2.1}$$

Let $\underset{\alpha}{v}^{\beta}$ $(\alpha = 1,2,...,2n+1)$ be independent vector fields satisfying the following conditions:

$$\begin{gathered} g_{\alpha\beta}\underset{\sigma}{v}^{\alpha}\underset{\sigma}{v}^{\beta} = 1, \quad g_{\alpha\beta}\underset{k}{v}^{\alpha}\underset{\bar{k}}{v}^{\beta} = 0, \quad g_{\alpha\beta}\underset{a}{v}^{\alpha}\underset{2n+1}{v}^{\beta} = 0, \\ g_{\alpha\beta}\underset{k}{v}^{\alpha}\underset{s}{v}^{\beta} = \cos\underset{ks}{\omega}, \qquad g_{\alpha\beta}\underset{\bar{k}}{v}^{\alpha}\underset{\bar{s}}{v}^{\beta} = \cos\underset{\bar{k}\bar{s}}{\omega}. \end{gathered} \tag{2.2}$$

The net defined by the vector fields $\underset{\alpha}{v}^{\beta}$ will be denoted by $\left\{\underset{\alpha}{v}\right\}$. The reciprocal covectors $\overset{\alpha}{v}_{\beta}$ of the vectors $\underset{\alpha}{v}^{\beta}$ are defined by

$$\underset{\sigma}{v}^{\beta}\,\overset{\sigma}{v}_{\alpha} = \delta_{\alpha}^{\beta} \quad \Leftrightarrow \quad \underset{\alpha}{v}^{\sigma}\,\overset{\beta}{v}_{\sigma} = \delta_{\alpha}^{\beta}, \tag{2.3}$$

where δ_{α}^{β} is the identity affinor.

If we choose the net $\left\{\underset{\alpha}{v}\right\}$ to be the coordinate net, we have

$$\begin{aligned} &\underset{1}{v}^{\beta}\left(\tfrac{1}{\sqrt{g_{11}}},0,0,...,0\right), \underset{2}{v}^{\beta}\left(0,\tfrac{1}{\sqrt{g_{22}}},0,...,0\right),..., \underset{2n+1}{v}^{\beta}\left(0,0,...,0,\tfrac{1}{\sqrt{g_{2n+1\ 2n+1}}}\right); \\ &\overset{1}{v}_{\beta}\left(\sqrt{g_{11}},0,0,...,0\right), \overset{2}{v}_{\beta}\left(0,\sqrt{g_{22}},0,...,0\right),..., \overset{2n+1}{v}_{\beta}\left(0,0,...,0,\sqrt{g_{2n+1\ 2n+1}}\right). \end{aligned} \tag{2.4}$$

According to (2.2) and (2.4), in the parameters of the coordinate net $\left\{\underset{\alpha}{v}\right\}$ the matrix of the metric tensor has the form

$$\|g_{\alpha\beta}\| = \begin{Vmatrix} g_{sk} & 0 & 0 \\ 0 & g_{\bar{s}\bar{k}} & 0 \\ 0 & 0 & g_{2n+1\ 2n+1} \end{Vmatrix}. \tag{2.5}$$

From (2.4) and (2.5) it follows that $g_{\alpha\beta}\underset{2n+1}{v}^{\alpha} = \overset{2n+1}{v}_{\beta}$. Also, the following equalities are valid [13]:

$$\nabla_{\sigma}\,\underset{\alpha}{v}^{\beta} = \underset{\alpha}{\overset{\nu}{T}}_{\sigma}\,\underset{\nu}{v}^{\beta}, \qquad \nabla_{\sigma}\,\overset{\alpha}{v}_{\beta} = -\underset{\nu}{\overset{\alpha}{T}}_{\sigma}\,\overset{\nu}{v}_{\beta}, \tag{2.6}$$

where $\nabla_{\sigma}\,\underset{\alpha}{v}^{\beta} = \partial_{\sigma}\,\underset{\alpha}{v}^{\beta} + \Gamma_{\sigma\nu}^{\beta}\,\underset{\alpha}{v}^{\nu}$ and $\nabla_{\sigma}\overset{\alpha}{v}_{\beta} = \partial_{\sigma}\,\overset{\alpha}{v}_{\beta} - \Gamma_{\sigma\beta}^{\nu}\,\overset{\alpha}{v}_{\nu}$.

After contracting with $\overset{\tau}{v}_{\beta}$ both sides of the first equality in (2.6) and taking into account (2.3), we obtain

$$\underset{\alpha}{\overset{\tau}{T}}_{\sigma} = \partial_{\sigma}\,\underset{\alpha}{v}^{\beta}\,\overset{\tau}{v}_{\beta} + \Gamma_{\sigma\nu}^{\beta}\,\underset{\alpha}{v}^{\nu}\,\overset{\tau}{v}_{\beta}. \tag{2.7}$$

According to (2.4), in the parameters of the coordinate net $\left\{\underset{\alpha}{v}\right\}$ equalities (2.7) take the form

$$\begin{aligned} &\underset{\alpha}{\overset{\tau}{T}}_{\sigma} = \tfrac{\sqrt{g_{\tau\tau}}}{\sqrt{g_{\alpha\alpha}}}\Gamma_{\sigma\alpha}^{\tau} \qquad \text{for } \tau \neq \alpha, \\ &\underset{\alpha}{\overset{\alpha}{T}}_{\sigma} = \Gamma_{\sigma\alpha}^{\alpha} - \tfrac{1}{2}\tfrac{\partial_{\sigma}\,g_{\alpha\alpha}}{g_{\alpha\alpha}} \quad \text{(no summing over } \alpha). \end{aligned} \tag{2.8}$$

Now, let us consider the following affinor [11, 12, 2]:

$$a_\alpha^\beta = \underset{a}{v}{}^\beta \overset{a}{v}{}_\alpha - \underset{2n+1}{v}{}^\beta \overset{2n+1}{v}{}_\alpha. \tag{2.9}$$

From (2.3) and (2.9) we obtain $a_\alpha^\beta\ a_\beta^\sigma = \delta_\alpha^\sigma$. Hence, the affinor a_α^β defines a composition $X_{2n} \times X_1$ of the basic manifolds X_{2n} and X_1.

The positions (tangent planes) of the basic manifolds X_{2n} and X_1 are denoted by $P(X_{2n})$ and $P(X_1)$, respectively [7].

According to [11, 12], the affinors

$$\overset{1}{a}{}_\alpha^\beta = \tfrac{1}{2}(\delta_\alpha^\beta + a_\alpha^\beta) = \underset{a}{v}{}^\beta \overset{a}{v}{}_\alpha, \qquad \overset{2}{a}{}_\alpha^\beta = \tfrac{1}{2}(\delta_\alpha^\beta - a_\alpha^\beta) = \underset{2n+1}{v}{}^\beta \overset{2n+1}{v}{}_\alpha$$

are the projecting affinors of the composition $X_{2n} \times X_1$. If v^β is an arbitrary vector, we have $v^\beta = \overset{1}{a}{}_\alpha^\beta\ v^\alpha + \overset{2}{a}{}_\alpha^\beta\ v^\alpha = \overset{1}{V}{}^\beta + \overset{2}{V}{}^\beta$, where $\overset{1}{V}{}^\beta = \overset{1}{a}{}_\alpha^\beta\ v^\alpha \in P(X_{2n})$ and $\overset{2}{V}{}^\beta = \overset{2}{a}{}_\alpha^\beta\ v^\alpha \in P(X_1)$. Obviously, $\underset{a}{v}{}^\alpha \in P(X_{2n})$, and $\underset{2n+1}{v}{}^\alpha \in P(X_1)$.

Let $X_a \times X_b$ $(a+b=n)$ be an arbitrary composition in the Riemannian space V_n, and $P(X_a)$ and $P(X_b)$ be the positions of the differentiable manifolds X_a and X_b, respectively. According to [7], the composition $X_a \times X_b$ is of the type (c, c), i.e. (Cartesian, Cartesian), if the positions $P(X_a)$ and $P(X_b)$ are translated parallelly along any line in the space V_n.

3. Almost contact and almost paracontact structures on V_{2n+1}

Let us consider the following affinors

$$\underset{\lambda}{b}{}_\alpha^\beta = \lambda\left(\underset{k}{v}{}^\beta \overset{k}{v}{}_\alpha - \underset{\bar{k}}{v}{}^\beta \overset{\bar{k}}{v}{}_\alpha\right), \tag{3.1}$$

where $\lambda = 1, i$ (i is the imaginary unit, i.e. $i^2 = -1$). According to (2.3) and (3.1) we have $\underset{\lambda}{b}{}_\alpha^\beta \underset{2n+1}{v}{}^\alpha = 0$ and $\underset{\lambda}{b}{}_\alpha^\beta \overset{2n+1}{v}{}_\beta = 0$.

Let $\lambda = 1$. From (2.3) and (3.1) we obtain

$$\underset{1}{b}{}_\alpha^\beta\ \underset{1}{b}{}_\beta^\sigma = \delta_\alpha^\sigma - \underset{2n+1}{v}{}^\sigma \overset{2n+1}{v}{}_\alpha,$$

i.e. the affinor $\underset{1}{b}{}_\alpha^\beta$ defines an almost paracontact structure on V_{2n+1}.

In the parameters of the coordinate net, it is easy to prove that

$$g_{\sigma\nu}\ \underset{1}{b}{}_\alpha^\sigma\ \underset{1}{b}{}_\beta^\nu = g_{\alpha\beta} - \overset{2n+1}{v}{}_\alpha \overset{2n+1}{v}{}_\beta, \tag{3.2}$$

i.e. the almost paracontact structure $\underset{1}{b}{}_\alpha^\beta$ is compatible with the Riemannian metric $g_{\alpha\beta}$, and hence V_{2n+1} is an almost paracontact Riemannian manifold [1, 8].

In the case $\lambda = i$ the affinor (3.1) defines an almost contact structure in V_{2n+1} which is not compatible with the Riemannian metric $g_{\alpha\beta}$, i.e. (3.2) does not hold for $\underset{i}{b}{}_\alpha^\beta$.

Theorem 3.1. *The affinor $\underset{\lambda}{b}{}_\alpha^\beta$ is parallel to the Levi-Civita connection ∇, i.e. $\nabla_\sigma\ \underset{\lambda}{b}{}_\alpha^\beta = 0$, iff the coefficients of the derivative equations (2.6) satisfy*

$$\underset{k}{\overset{\bar{s}}{T}}{}_\sigma = \underset{\bar{k}}{\overset{s}{T}}{}_\sigma = 0, \qquad \underset{2n+1}{\overset{a}{T}}{}_\sigma = \underset{a}{\overset{2n+1}{T}}{}_\sigma = 0. \tag{3.3}$$

Proof. Let

$$\nabla_\sigma \underset{\lambda}{b}{}_\alpha^\beta = 0. \tag{3.4}$$

According to (2.6) and (3.1), equality (3.4) takes the form

$$\overset{\nu}{\underset{k}{T}}{}_\sigma \underset{\nu}{v}{}^\beta \overset{k}{v}_\alpha - \overset{k}{\underset{\nu}{T}}{}_\sigma \underset{k}{v}{}^\beta \overset{\nu}{v}_\alpha - \overset{\nu}{\underset{\bar{k}}{T}}{}_\sigma \underset{\nu}{v}{}^\beta \overset{\bar{k}}{v}_\alpha + \overset{\bar{k}}{\underset{\nu}{T}}{}_\sigma \underset{\bar{k}}{v}{}^\beta \overset{\nu}{v}_\alpha = 0. \tag{3.5}$$

After contracting (3.5) with $\underset{s}{\nu}{}^\alpha$, $\underset{\bar{s}}{v}{}^\alpha$ and $\underset{2n+1}{v}{}^\alpha$, we obtain the following equalities which are equivalent to (3.5):

$$\begin{gathered} 2\,\overset{\bar{k}}{\underset{s}{T}}{}_\sigma \underset{\bar{k}}{v}{}^\beta + \overset{2n+1}{\underset{s}{T}}{}_\sigma \underset{2n+1}{v}{}^\beta = 0, \qquad 2\,\overset{k}{\underset{\bar{s}}{T}}{}_\sigma \underset{k}{v}{}^\beta + \overset{2n+1}{\underset{\bar{s}}{T}}{}_\sigma \underset{2n+1}{v}{}^\beta = 0, \\ \overset{k}{\underset{2n+1}{T}}{}_\sigma \underset{k}{v}{}^\beta - \overset{\bar{k}}{\underset{2n+1}{T}}{}_\sigma \underset{\bar{k}}{v}{}^\beta = 0. \end{gathered} \tag{3.6}$$

From the independency of the vectors $\underset{\nu}{v}{}^\beta$ it follows that equalities (3.6) are equivalent to conditions (3.3) which proves the statement. □

Let us note that manifolds satisfying (3.4) are contact and paracontact analogues to Kähler manifolds.

Corollary 3.1. *If* $\nabla_\sigma \underset{\lambda}{b}{}_\alpha^\beta = 0$, *in the parameters of the net* $\left\{\underset{\alpha}{v}\right\}$, *the Christoffel symbols* $\Gamma^\nu_{\alpha\beta}$ *satisfy*

$$\Gamma^{\bar{k}}_{\sigma s} = 0, \quad \Gamma^{k}_{\sigma \bar{s}} = 0, \quad \Gamma^{a}_{\sigma 2n+1} = 0, \quad \Gamma^{2n+1}_{\sigma a} = 0. \tag{3.7}$$

Proof. According to (2.8), equalities (3.3) take the form (3.7). □

Corollary 3.2. *If* $\nabla_\sigma \underset{\lambda}{b}{}_\alpha^\beta = 0$, *the composition* $X_{2n} \times X_1$ *defined by the affinor (2.9), is of the type* (c, c).

Proof. Having in mind (3.4), equalities (3.7) hold.
Then, according to [7], from $\Gamma^a_{\sigma 2n+1} = \Gamma^{2n+1}_{\sigma a} = 0$ it follows that the composition $X_{2n} \times X_1$ is of the type (c, c). □

From (2.5) it follows that the composition $X_{2n} \times X_1$ is orthogonal. The coordinate net $\left\{\underset{\alpha}{v}\right\}$ gives rise to coordinates which are adapted to the composition $X_{2n} \times X_1$. In accordance to [6], the line element of the space V_{2n+1} is of the form

$$\mathrm{d}s^2 = g_{ab}(\overset{c}{u})\mathrm{d}\overset{a}{u}\mathrm{d}\overset{b}{u} + g_{2n+1\ 2n+1}(\overset{2n+1}{u})\mathrm{d}(\overset{2n+1}{u})^2, \tag{3.8}$$

where g_{ab} is the metric tensor of the manifold X_{2n}.

Theorem 3.2. *If condition (3.4) holds, the Riemannian space* X_{2n} *is a space of the composition* $X_n \times \overline{X}_n$ *with line element defined in the parameters of the net* $\left\{\underset{\alpha}{v}\right\}$ *by*

$$\mathrm{d}s^2 = g_{ks}(\overset{j}{u})\mathrm{d}\overset{k}{u}\mathrm{d}\overset{s}{u} + g_{\bar{k}\bar{s}}(\overset{\bar{j}}{u})\mathrm{d}\overset{\bar{k}}{u}\mathrm{d}\overset{\bar{s}}{u}\,. \tag{3.9}$$

Proof. The tensors $\underset{\lambda}{b}{}_a^d$, $\nabla_c \underset{\lambda}{b}{}_a^d$ and g_{ab} are the full projections of the tensors $\underset{\lambda}{b}{}_\alpha^\beta$, $\nabla_\sigma \underset{\lambda}{b}{}_\alpha^\beta$ and $g_{\alpha\beta}$, respectively, over the positions $P(X_{2n})$.

From (3.1) it follows that $\underset{\lambda}{b}{}_a^d \underset{\lambda}{b}{}_d^c = \pm\delta_a^c$. Hence, the affinor $\underset{\lambda}{b}{}_a^d$ defines a composition $X_n \times \overline{X}_n$ in the manifold X_{2n}. Because of the condition $\nabla_c \underset{\lambda}{b}{}_a^d = 0$, the composition $X_n \times \overline{X}_n$ is of the type (c, c) [7]. From (2.5) it follows that the composition $X_n \times \overline{X}_n$ is orthogonal. Then, according to [6], the line element of $X_n \times \overline{X}_n$ is of the form (3.9). $\square$

Let $P(X_n)$ and $P(\overline{X}_n)$ are the positions of the differentiable manifolds X_n and $\overline{X}_n$, respectively. The projecting affinors of the composition $X_n \times \overline{X}_n$ are:

$$\overset{1}{b}{}_\alpha^\beta = \lambda \underset{k}{v}{}^\beta \overset{k}{v}{}_\alpha, \qquad \overset{2}{b}{}_\alpha^\beta = \lambda \underset{\bar{k}}{v}{}^\beta \overset{\bar{k}}{v}{}_\alpha.$$

For an arbitrary vector $w^\alpha \in P(X_{2n})$ we have $w^\beta = \overset{1}{b}{}_\alpha^\beta w^\alpha + \overset{2}{b}{}_\alpha^\beta w^\alpha = \overset{1}{W}{}^\beta + \overset{2}{W}{}^\beta$, where $\overset{1}{W}{}^\beta = \overset{1}{b}{}_\alpha^\beta w^\alpha \in P(X_n)$, and $\overset{2}{W}{}^\beta = \overset{2}{b}{}_\alpha^\beta w^\alpha \in P(\overline{X}_n)$. Obviously, $\underset{k}{v}{}^\beta \in P(X_n)$, and $\underset{\bar{k}}{v}{}^\beta \in P(\overline{X}_n)$.

The following statements are immediate consequences of our results:

Proposition 3.1. *If condition (3.4) holds, the Riemannian space V_{2n+1} is a topological product of three basic differentiable manifolds X_n, $\overline{X}_n$ and X_1, i.e. V_{2n+1} is a space of the composition $X_n \times \overline{X}_n \times X_1$.* ■

Proposition 3.2. *If (3.4) holds, in the parameters of the coordinate net $\left\{\underset{\alpha}{v}\right\}$ the line element of the space V_{2n+1} is of the form*

$$\text{(3.10)} \qquad ds^2 = g_{ks}(\overset{j}{u})d\overset{k}{u}d\overset{s}{u} + g_{\bar{k}\bar{s}}(\overset{\bar{j}}{u})d\overset{\bar{k}}{u}d\overset{\bar{s}}{u} + g_{2n+1\ 2n+1}(\overset{2n+1}{u})d(\overset{2n+1}{u})^2. \ ■$$

Now we will prove the following theorem.

Theorem 3.3. *Condition (3.4) is equivalent to the following:*

$$\text{(3.11)} \qquad \overset{1}{b}{}_\nu^\sigma \nabla_\alpha \overset{1}{b}{}_\sigma^\beta = 0, \qquad \overset{2}{b}{}_\nu^\sigma \nabla_\alpha \overset{2}{b}{}_\sigma^\beta = 0, \qquad \overset{2}{a}{}_\nu^\sigma \nabla_\alpha \overset{2}{a}{}_\sigma^\beta = 0,$$

where $\overset{1}{b}{}_\nu^\sigma$, $\overset{2}{b}{}_\nu^\sigma$ and $\overset{2}{a}{}_\nu^\sigma$ are the projecting affinors of the composition $X_n \times \overline{X}_n \times X_1$.

Proof. Because of $\overset{1}{b}{}_\nu^\sigma = \lambda \underset{k}{v}{}^\sigma \overset{k}{v}{}_\nu$, $\overset{2}{b}{}_\nu^\sigma = \lambda \underset{\bar{k}}{v}{}^\sigma \overset{\bar{k}}{v}{}_\nu$ and $\overset{2}{a}{}_\nu^\sigma = \underset{2n+1}{v}{}^\sigma \overset{2n+1}{v}{}_\nu$, we obtain

$$\text{(3.12)} \qquad \begin{aligned} &\overset{1}{b}{}_\nu^\sigma \nabla_\alpha \overset{1}{b}{}_\sigma^\beta = \pm \underset{k}{v}{}^\sigma \overset{k}{v}{}_\nu \nabla_\alpha \left(\underset{s}{v}{}^\beta \overset{s}{v}{}_\sigma\right), \\ &\overset{2}{b}{}_\nu^\sigma \nabla_\alpha \overset{2}{b}{}_\sigma^\beta = \pm \underset{\bar{k}}{v}{}^\sigma \overset{\bar{k}}{v}{}_\nu \nabla_\alpha \left(\underset{\bar{s}}{v}{}^\beta \overset{\bar{s}}{v}{}_\sigma\right), \\ &\overset{2}{a}{}_\nu^\sigma \nabla_\alpha \overset{2}{a}{}_\sigma^\beta = \underset{2n+1}{v}{}^\sigma \overset{2n+1}{v}{}_\nu \nabla_\alpha \left(\underset{2n+1}{v}{}^\beta \overset{2n+1}{v}{}_\sigma\right). \end{aligned}$$

According to (2.6) and (3.12), we get

$$\overset{1}{b}{}^{\sigma}_{\nu} \nabla_{\alpha} \overset{1}{b}{}^{\beta}_{\sigma} = \pm \left(\underset{k}{\overset{\bar{s}}{T}}{}_{\alpha} \underset{\bar{s}}{v}{}^{\beta} + \underset{k}{\overset{2n+1}{T}}{}_{\alpha} \underset{2n+1}{v}{}^{\beta} \right) \overset{k}{v}{}_{\nu},$$
$$\overset{2}{b}{}^{\sigma}_{\nu} \nabla_{\alpha} \overset{2}{b}{}^{\beta}_{\sigma} = \pm \left(\underset{\bar{k}}{\overset{s}{T}}{}_{\alpha} \underset{s}{v}{}^{\beta} + \underset{\bar{k}}{\overset{2n+1}{T}}{}_{\alpha} \underset{2n+1}{v}{}^{\beta} \right) \overset{\bar{k}}{v}{}_{\nu}, \tag{3.13}$$
$$\overset{2}{a}{}^{\sigma}_{\nu} \nabla_{\alpha} \overset{2}{a}{}^{\beta}_{\sigma} = \underset{2n+1}{\overset{a}{T}}{}_{\alpha} \underset{a}{v}{}^{\beta} \overset{2n+1}{v}{}_{\nu}.$$

From (3.13) it follows that conditions (3.11) hold iff conditions (3.3) hold, too. And, according to Theorem 3.1, (3.3) are equivalent to condition (3.4). Then, (3.4) and (3.11) are also equivalent which completes the proof. □

In accordance to (3.7), for the components of the curvature tensor $R_{\alpha\beta\sigma}{}^{\nu} = \partial_\alpha \Gamma^{\nu}_{\beta\sigma} - \partial_\beta \Gamma^{\nu}_{\alpha\sigma} + \Gamma^{\nu}_{\alpha\delta}\Gamma^{\delta}_{\beta\sigma} - \Gamma^{\nu}_{\beta\delta}\Gamma^{\delta}_{\alpha\sigma}$ we obtain

$$R_{\alpha ks}{}^{\bar{j}} = R_{ks\alpha}{}^{\bar{j}} = R_{\alpha\bar{k}\bar{s}}{}^{j} = R_{\bar{k}\bar{s}\alpha}{}^{j} = R_{\alpha ab}{}^{2n+1} = R_{ab\alpha}{}^{2n+1} = 0. \tag{3.14}$$

4. Transformations of linear connections

4.1. **Linear connections with torsion.** Let us consider the linear connection

$$ {}^{1}\Gamma^{\nu}_{\alpha\beta} = \Gamma^{\nu}_{\alpha\beta} + S^{\nu}_{\alpha\beta}, \tag{4.1}$$

where $S^{\nu}_{\alpha\beta}$ is the deformation tensor. The covariant derivative and the curvature tensor with respect to ${}^{1}\Gamma$ are denoted by ${}^{1}\nabla$ and ${}^{1}R$.

Theorem 4.1. *The affinors (3.1) are parallel to* ∇ *and* ${}^{1}\nabla$ *iff in parameters of the net* $\left\{\underset{\alpha}{v}\right\}$ *the tensor* $S^{\nu}_{\alpha\beta}$ *satisfies*

$$S^{s}_{\alpha\bar{k}} = S^{s}_{\alpha 2n+1} = S^{\bar{s}}_{\alpha k} = S^{\bar{s}}_{\alpha 2n+1} = S^{2n+1}_{\alpha a} = 0. \tag{4.2}$$

Proof. Let conditions (3.4) hold and let

$$ {}^{1}\nabla_{\sigma} \underset{\lambda}{b}{}^{\beta}_{\alpha} = 0. \tag{4.3}$$

According to (4.1), we have ${}^{1}\nabla_{\sigma} \underset{\lambda}{b}{}^{\beta}_{\alpha} = \nabla_{\sigma} \underset{\lambda}{b}{}^{\beta}_{\alpha} + S^{\beta}_{\sigma\nu} \underset{\lambda}{b}{}^{\nu}_{\alpha} - S^{\nu}_{\sigma\alpha} \underset{\lambda}{b}{}^{\beta}_{\nu}$, from where it follows that equalities (3.4) and (4.3) hold iff

$$P^{\beta}_{\sigma\alpha} = S^{\beta}_{\sigma\nu} \underset{\lambda}{b}{}^{\nu}_{\alpha} - S^{\nu}_{\sigma\alpha} \underset{\lambda}{b}{}^{\beta}_{\nu} = 0. \tag{4.4}$$

We choose $\left\{\underset{\alpha}{v}\right\}$ for the coordinate net. In its parameters of the net, the matrix of the affinor $\underset{\lambda}{b}{}^{\beta}_{\alpha}$ has the form

$$\left\| \underset{\lambda}{b}{}^{\beta}_{\alpha} \right\| = \left\| \begin{matrix} \lambda\delta^{k}_{s} & 0 & 0 \\ 0 & -\lambda\delta^{\bar{k}}_{\bar{s}} & \vdots \\ 0 & \dots & 0 \end{matrix} \right\|. \tag{4.5}$$

From (4.4) and (4.5) we compute the following non-zero components of P:

$$
(4.6)\qquad
\begin{array}{lll}
P^{j}_{s\bar{k}} = -2\lambda S^{j}_{s\bar{k}}, & P^{j}_{\bar{s}\bar{k}} = -2\lambda S^{j}_{\bar{s}\bar{k}}, & P^{j}_{s2n+1} = -\lambda S^{j}_{s2n+1},\\
P^{j}_{2n+1,2n+1} = -\lambda S^{j}_{2n+1,2n+1}, & P^{j}_{\bar{s}2n+1} = -\lambda S^{j}_{\bar{s}2n+1}, & P^{j}_{2n+1\bar{s}} = -2\lambda S^{j}_{2n+1\bar{s}},\\
P^{\bar{j}}_{\bar{s}k} = 2\lambda S^{\bar{j}}_{\bar{s}k}, & P^{\bar{j}}_{sk} = 2\lambda S^{\bar{j}}_{sk}, & P^{\bar{j}}_{\bar{s}2n+1} = \lambda S^{\bar{j}}_{\bar{s}2n+1}\\
P^{\bar{j}}_{2n+1,2n+1} = \lambda S^{\bar{j}}_{2n+1,2n+1}, & P^{\bar{j}}_{s2n+1} = \lambda S^{\bar{j}}_{s2n+1}, & P^{\bar{j}}_{2n+1s} = 2\lambda S^{\bar{j}}_{2n+1s},\\
P^{2n+1}_{sk} = \lambda S^{2n+1}_{sk}, & P^{2n+1}_{\bar{s}k} = \lambda S^{2n+1}_{\bar{s}k}, & P^{2n+1}_{s\bar{k}} = -\lambda S^{2n+1}_{k\bar{s}},\\
P^{2n+1}_{\bar{s}\bar{k}} = -\lambda S^{2n+1}_{\bar{s}\bar{k}}, & P^{2n+1}_{2n+1s} = \lambda S^{2n+1}_{2n+1s}, & P^{2n+1}_{2n+1\bar{s}} = -\lambda S^{2n+1}_{2n+1\bar{s}},
\end{array}
$$

Then, according to (4.6), equalities (4.4) hold iff (4.2) hold, too. □

From (4.1) and (4.2) we get the non-zero components of ${}^1\Gamma$ expressed by the components of Γ and S:

$$
(4.7)\qquad
\begin{array}{lll}
{}^1\Gamma^{j}_{sk} = \Gamma^{j}_{sk} + S^{j}_{sk}, & {}^1\Gamma^{j}_{\bar{k}s} = S^{j}_{\bar{k}s}, & {}^1\Gamma^{j}_{2n+1s} = S^{j}_{2n+1s}\\
{}^1\Gamma^{\bar{j}}_{\bar{s}\bar{k}} = \Gamma^{\bar{j}}_{\bar{s}\bar{k}} + S^{\bar{j}}_{\bar{s}\bar{k}}, & {}^1\Gamma^{\bar{j}}_{k\bar{s}} = S^{\bar{j}}_{k\bar{s}}, & {}^1\Gamma^{\bar{j}}_{2n+1\bar{s}} = S^{\bar{j}}_{2n+1\bar{s}},\\
{}^1\Gamma^{2n+1}_{s2n+1} = S^{2n+1}_{s2n+1}, & {}^1\Gamma^{2n+1}_{\bar{s}2n+1} = S^{2n+1}_{\bar{s}2n+1}, & {}^1\Gamma^{2n+1}_{2n+1,2n+1} = S^{2n+1}_{2n+1,2n+1}.
\end{array}
$$

Having in mind (4.7), we compute the following components of the curvature tensor ${}^1R_{\alpha\beta\sigma}{}^{\nu}$:

$$
{}^1R_{\alpha sk}{}^{\bar{j}} = {}^1R_{\alpha\bar{s}\bar{k}}{}^{j} = {}^1R_{\alpha ab}{}^{2n+1} = 0,
$$

$$
{}^1R_{ks\alpha}{}^{\bar{j}} = 2\left(\partial_{[k}S^{\bar{j}}_{s]\alpha} + S^{\bar{j}}_{[k|\bar{l}|}S^{\bar{l}}_{s]\alpha}\right), \quad {}^1R_{\bar{k}\bar{s}\alpha}{}^{j} = 2\left(\partial_{[\bar{k}}S^{j}_{\bar{s}]\alpha} + S^{j}_{[\bar{k}|l|}S^{l}_{\bar{s}]\alpha}\right),
$$

$$
{}^1R^{2n+1}_{ab\alpha} = 2\left(\partial_{[a}S^{2n+1}_{b]\alpha} + S^{2n+1}_{[a|2n+1|}S^{2n+1}_{b]\alpha}\right),
$$

$$
{}^1R_{skl}{}^{j} = R_{skl}{}^{j} + 2\left(\partial_{[s}S^{j}_{k]l} + \Gamma^{j}_{[s|p|}S^{p}_{k]l} + S^{j}_{[s|p|}\Gamma^{p}_{k]l} + S^{j}_{[s|p|}S^{p}_{k]l}\right),
$$

$$
{}^1R_{\bar{s}\bar{k}\bar{l}}{}^{\bar{j}} = R_{\bar{s}\bar{k}\bar{l}}{}^{\bar{j}} + 2\left(\partial_{[\bar{s}}S^{\bar{j}}_{\bar{k}]\bar{l}} + \Gamma^{\bar{j}}_{[\bar{s}|\bar{p}|}S^{\bar{p}}_{\bar{k}]\bar{l}} + S^{\bar{j}}_{[\bar{s}|\bar{p}|}\Gamma^{\bar{p}}_{\bar{k}]\bar{l}} + S^{\bar{j}}_{[\bar{s}|\bar{p}|}S^{\bar{p}}_{\bar{k}]\bar{l}}\right).
$$

4.2. **A metric connection.** Let V_{2n+1} be a space with $\nabla_\sigma \underset{\lambda}{b}{}^{\beta}_{\alpha} = 0$, and let us consider the connection

$$
(4.8)\qquad {}^2\Gamma^{\nu}_{\alpha\beta} = \Gamma^{\nu}_{\alpha\beta} + \bar{S}^{\nu}_{\alpha\beta},
$$

where

$$
(4.9)\qquad \bar{S}^{\nu}_{\alpha\beta} = \sum_{\tau=1}^{2n+1} \overset{\tau}{v}_{\alpha}\, g_{\beta\delta} \sum_{k=1}^{n}\left(\underset{k}{v}^{\delta}\ \underset{k+n}{v}{}^{\nu} - \underset{k}{v}^{\nu}\ \underset{k+n}{v}{}^{\delta}\right).
$$

The covariant derivative and the curvature tensor with respect to the connection ${}^2\Gamma$ are denoted by ${}^2\nabla$ and ${}^2R$.

Theorem 4.2. *The metric tensor of the space V_{2n+1} is parallel to the connection ${}^2\Gamma$, i.e.*

$$
(4.10)\qquad {}^2\nabla_\sigma g_{\alpha\beta} = 0\,.
$$

Proof. From (4.8) and (4.10) we get

$$ {}^2\nabla_\sigma g_{\alpha\beta} = \nabla_\sigma g_{\alpha\beta} - \bar{S}^{\nu}_{\sigma\alpha}\; g_{\nu\beta} - \bar{S}^{\nu}_{\sigma\beta}\; g_{\nu\alpha}. \tag{4.11} $$

Let us consider the tensor

$$ T_{\sigma\alpha\beta} = \bar{S}^{\nu}_{\sigma\alpha} g_{\nu\beta}. \tag{4.12} $$

According to (4.9) and (4.12), we have

$$ T_{\sigma\alpha\beta} = \sum_{\tau=1}^{2n+1} \overset{\tau}{v}_\sigma\; g_{\alpha\delta} \sum_{k=1}^{n} \left(\underset{k}{v}^{\nu} \;\underset{k+n}{v}^{\;\delta} - \underset{k}{v}^{\delta} \;\underset{k+n}{v}^{\;\nu} \right) g_{\nu\beta}. \tag{4.13} $$

In the parameters of the coordinate net $\left\{ \underset{\alpha}{v} \right\}$ we obtain

$$ T_{\sigma\alpha\beta} = \sum_{\tau=1}^{2n+1} \overset{\tau}{v}_\sigma \sum_{k=1}^{n} \frac{1}{\sqrt{g_{kk}}\sqrt{g_{k+n\;k+n}}} \left(g_{\alpha\;k+n}\; g_{\beta k} - g_{\beta\;k+n}\; g_{\alpha k} \right), $$

from where it follows that

$$ T_{\sigma(\alpha\beta)} = 0. \tag{4.14} $$

Then, (4.11), (4.12) and (4.14) imply (4.10). □

By (2.4) and (4.9) we obtain the components of the deformation tensor $\bar{S}$ of ${}^2\nabla$ and then by (3.7) and (4.8) we get the non-zero Christoffel symbols of ${}^2\nabla$ in the parameters of the coordinate net as follows:

$$ \begin{aligned} {}^2\Gamma^{j}_{k\;n+s} &= -\frac{\sqrt{g_{kk}}}{\sqrt{g_{jj}}\sqrt{g_{n+j\;n+j}}}\; g_{n+s\;n+j}, \\ {}^2\Gamma^{j}_{n+k\;n+s} &= -\frac{\sqrt{g_{n+k\;n+k}}}{\sqrt{g_{jj}}\sqrt{g_{n+j\;n+j}}}\; g_{n+s\;n+j}, \\ {}^2\Gamma^{n+j}_{sk} &= \frac{\sqrt{g_{ss}}}{\sqrt{g_{jj}}\sqrt{g_{n+j\;n+j}}}\; g_{jk}, \\ {}^2\Gamma^{n+j}_{n+s\;k} &= \frac{\sqrt{g_{n+s\;n+s}}}{\sqrt{g_{jj}}\sqrt{g_{n+j\;n+j}}}\; g_{jk}, \\ {}^2\Gamma^{j}_{2n+1\;n+s} &= -\frac{\sqrt{g_{2n+1\;2n+1}}}{\sqrt{g_{jj}}\sqrt{g_{n+j\;n+j}}}\; g_{n+s\;n+j}, \\ {}^2\Gamma^{n+j}_{2n+1\;k} &= \frac{\sqrt{g_{2n+1\;2n+1}}}{\sqrt{g_{jj}}\sqrt{g_{n+j\;n+j}}}\; g_{jk}. \end{aligned} \tag{4.15} $$

By (4.15) we compute the components of the curvature tensor ${}^2R$, for example

$$ \begin{aligned} &{}^2R_{skp}{}^{j} = R_{skp}{}^{j}, \qquad {}^2R_{\bar{s}\bar{k}\bar{p}}{}^{\bar{j}} = R_{\bar{s}\bar{k}\bar{p}}{}^{\bar{j}}, \qquad {}^2R^{2n+1}_{abc} = 0, \\ &{}^2R_{pk\;n+s}{}^{j} = \tfrac{g_{n+s\;n+s}}{\sqrt{g_{n+j\;n+j}}} \left(\partial_k \tfrac{\sqrt{g_{pp}}}{\sqrt{g_{jj}}} - \partial_p \tfrac{\sqrt{g_{kk}}}{\sqrt{g_{jj}}} \right) + \sqrt{g_{pp}} \textstyle\sum_{l=1}^{n} \Gamma^{j}_{kl} \tfrac{g_{n+s\;n+l}}{\sqrt{g_{ll}}\sqrt{g_{n+l\;n+l}}} \\ &\qquad - \sqrt{g_{kk}} \textstyle\sum_{l=1}^{n} \Gamma^{j}_{pl} \tfrac{g_{n+s\;n+l}}{\sqrt{g_{ll}}\sqrt{g_{n+l\;n+l}}}, \\ &{}^2R^{n+j}_{2n+1\;ks} = \tfrac{\sqrt{g_{2n+1\;2n+1}}}{\sqrt{g_{n+j\;n+j}}} \left(\tfrac{1}{\sqrt{g_{jj}}} g_{lj} \Gamma^{l}_{ks} - \partial_k \tfrac{g_{sj}}{\sqrt{g_{jj}}} \right). \end{aligned} \tag{4.16} $$

As an example we consider a 5-dimensional Riemannian space V_5. The matrix (2.5) has the form

$$ ||g_{\alpha\beta}|| = \begin{Vmatrix} g_{sk} & 0 & 0 \\ 0 & g_{\bar{s}\bar{k}} & 0 \\ 0 & 0 & g_{55} \end{Vmatrix}, \tag{4.17} $$

where $j, k, s = 1, 2,\ \bar{j}, \bar{k}, \bar{s} = 3, 4$.

In the parameters of the net $\left\{\underset{\alpha}{v}\right\}$ the line element if given by

$$\mathrm{d}s^2 = g_{sk}(\overset{j}{u})\mathrm{d}\overset{k}{u}\mathrm{d}\overset{s}{u} + g_{\bar{k}\bar{s}}(\overset{\bar{j}}{u})\mathrm{d}\overset{\bar{k}}{u}\mathrm{d}\overset{\bar{s}}{u} + g_{55}(\overset{5}{u})\mathrm{d}\overset{5}{u}^2. \tag{4.18}$$

From the last one of the equalities (4.16) we get

$$ {}^2R_{512}{}^3 = \frac{\sqrt{g_{55}}}{\sqrt{g_{33}}}\left(\frac{1}{\sqrt{g_{11}}} g_{l1}\Gamma^l_{12} - \partial_1 \frac{g_{12}}{\sqrt{g_{11}}}\right). \tag{4.19}$$

Since $g_{l1}\Gamma^l_{12} = \frac{1}{2}\partial_2 g_{11}$, (4.19) implies

$$ {}^2R_{512}{}^3 = \frac{\sqrt{g_{55}}}{\sqrt{g_{33}}}\left(\frac{1}{2\sqrt{g_{11}}} \partial_2 g_{11} - \partial_1 \frac{g_{12}}{\sqrt{g_{11}}}\right). \tag{4.20}$$

Acknowledgment

This work is partially supported by the Scientific Research Fund of the University of Plovdiv, Bulgaria, Project NI13-FMI-002.

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