ADV MATH SCI JOURNAL

Advances in Mathematics: Scientific Journal **2** (2013), no.2, 43–47
ISSN 1857-8365, UDC: 517.546/.547

SOME PROPERTIES OF FUNCTIONS CONCERNED WITH OZAKI AND NUNOKAWA RESULT

YUTAKA SHIMODA,YAYOI NAKAMURA AND SHIGEYOSHI OWA

ABSTRACT. For analytic functions $f(z)$ with $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$ and $f^{(n)}(0) \neq 0$ in the open unit disk $\mathbb{U}$, a new class $\mathcal{A}_n(\lambda)$ concerned with Ozaki and Nunokawa result in 1972 is introduced. The object of the present paper is to discuss some properties of functions $f(z)$ in the class $\mathcal{A}_n(\lambda)$.

1. INTRODUCTION

Let $\mathcal{A}_n$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=n}^{\infty} a_k z^k \quad (n = 2, 3, 4, \cdots) \tag{1.1}$$

with $a_n \neq 0$ which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

For $f(z) \in \mathcal{A}_n$, we define the subclass $\mathcal{A}_n(\lambda)$ of $\mathcal{A}_n$ consisting of $f(z)$ which satisfy

$$\left| \frac{z^2 f'(z)}{f(z)^2} - 1 \right| < \lambda \qquad (z \in \mathbb{U}) \tag{1.2}$$

for some real $\lambda > 0$. We note that $f(z) \in \mathcal{A}_2(1)$ is univalent in $\mathbb{U}$ by Ozaki and Nunokawa [5].

The class $\mathcal{A}_2(\lambda)$ was defined and discussed by Obradović and Ponnusamy [4]. The class $\mathcal{A}_3(\lambda)$ with $0 < \lambda < 1$ was considered by Singh [7].

Resently, Shimoda, Nakamura and Owa [6] discussed some radius problems for $f(z) \in \mathcal{A}_2(\lambda)$ with $a_3 - a_2^2 = 0$.

In order to discuss our problems for $f(z) \in \mathcal{A}_n(\lambda)$, we have to recall here the following lemmas.

Lemma 1.1. *Let the function $w(z)$ defined by*

$$w(z) = b_n z^n + b_{n+1} z^{n+1} + b_{n+2} z^{n+2} + \cdots \qquad (b_n \neq 0) \tag{1.3}$$

2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Analytic, Jack's lemma, Carathéodory function.

be analytic in $\mathbb{U}$. *If* $|w(z)|$ *attains its maximum value at a point* $z_0 \in \mathbb{U}$ *on the circle* $|z| = r$, *then there exists a real number* $k \geq n$ *such that*

$$z_0 w'(z_0) = k w(z_0) \tag{1.4}$$

and

$$\mathrm{Re}\left(\frac{z_0 w''(z_0)}{w'(z_0)}\right) \geq k - 1. \tag{1.5}$$

This lemma was given by Jack [2] (also, due to Miller and Mocanu [3]).

Lemma 1.2. *Let the function* $p(z)$ *given by*

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \tag{1.6}$$

be analytic in $\mathbb{U}$. *If* $p(z)$ *satisfies* $\mathrm{Re}p(z) > 0 \quad (z \in \mathbb{U})$, *then*

$$|c_n| \leq 2 \quad (n = 1, 2, 3, \cdots). \tag{1.7}$$

The equality in (1.7) is attained for $p(z)$ *given by*

$$p(z) = \frac{1 + z}{1 - z}. \tag{1.8}$$

This lemma is called as Carathéodory's lemma [1].

2. Some properties for the class $\mathcal{A}_n(\lambda)$

First result of our problems is contained in:

Theorem 2.1. *If* $f(z) \in \mathcal{A}_n(n-2)$ *with* $n \geq 3$, *then*

$$\left|\frac{z}{f(z)} - 1\right| < 1 \qquad (z \in \mathbb{U}) \tag{2.1}$$

and

$$\mathrm{Re}\left(\frac{f(z)}{z}\right) > \frac{1}{2} \qquad (z \in \mathbb{U}). \tag{2.2}$$

Proof. For $\mathcal{A}_n(n-2)$, we define the function $w(z)$ by

$$w(z) = \frac{z}{f(z)} - 1 = b_n z^{n-1} + b_{n+1} z^n + b_{n+2} z^{n+1} + \cdots .$$

Then $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$.

Further, we see that

$$\left|\frac{z^2 f'(z)}{f(z)^2} - 1\right| = |w(z) - z w'(z)| < n - 2 \quad (z \in \mathbb{U}). \tag{2.3}$$

We suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z|=r} |w(z)| = |w(z_0)| = 1. \tag{2.4}$$

Then, Lemma 1.1 gives us that

$$z_0 w'(z_0) = k w(z_0) \quad (k \geq n - 1). \tag{2.5}$$

It follows from (2.3) , (2.4), and (2.5) that

$$\left|\frac{z_0^2 f'(z_0)}{f(z_0)^2} - 1\right| = \left|w(z_0)\left(1 - \frac{z_0 w'(z_0)}{w(z_0)}\right)\right|$$

$$= |k-1| \geq n-2.$$

This contradicts that $f(z) \in \mathcal{A}_n(n-2)$. This means that there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. Therefore, we conclude that $|w(z)| < 1$ $(z \in \mathbb{U})$, and prove that equation (2.1) Furthermore, it is easy to see that (2.1) gives equation (2.2), proving the theorem. □

Next our result is:

Theorem 2.2. *If $f(z) \in \mathcal{A}_n\left(\frac{n-3}{4}\right)$ with $n \geq 4$, then*

$$\left|\frac{f(z)}{z} - 1\right| < 1 \qquad (z \in \mathbb{U}). \tag{2.6}$$

Proof. Let us define the function $w(z)$ by

$$w(z) = \frac{f(z)}{z} - 1\,,$$

for $f(z) \in \mathcal{A}_n\left(\frac{n-3}{4}\right)$. It follows that:

$$\left|\frac{z^2 f'(z)}{f(z)^2} - 1\right| = \left|\frac{1}{w(z)+1}\left(\frac{zw'(z)}{w(z)+1} - w(z)\right)\right| < \frac{n-3}{4} \quad (z \in \mathbb{U}). \tag{2.7}$$

Since, $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$, we suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z|=r} |w(z)| = |w(z_0)| = 1.$$

Then, we can write $w(z_0) = e^{i\theta}$ and

$$z_0 w'(z_0) = k w(z_0) \quad (k \geq n-1).$$

Therefore, we have that

$$\left|\frac{z_0^2 f'(z_0)}{f(z_0)^2} - 1\right| = \left|\frac{w(z_0)}{w(z_0)+1}\left(\frac{k}{w(z_0)+1} - 1\right)\right|$$

$$\geq \frac{|k-1-w(z_0)|}{|w(z_0)+1|^2} \geq \frac{n-3}{4},$$

which contradicts our condition (2.7). This shows that there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$ for $z_0 \in \mathbb{U}$. Thus we have that $|w(z)| < 1$ for all $z \in \mathbb{U}$, proving (2.6). □

3. Coefficient inequalities for the class $\mathcal{A}_2(\lambda)$

In this section we consider some coefficient inequalities for $f(z)$ in the class $\mathcal{A}_2(\lambda)$.

Theorem 3.1. *If $f(z) \in \mathcal{A}_2(\lambda)$ with $0 < \lambda < 1$, then*

$$|a_3 - a_2^2| \leq 2\lambda, \tag{3.1}$$

$$|a_4 - 2a_2a_3 + a_2^3| \leq \lambda, \tag{3.2}$$

and

$$|a_5 - 2a_2a_4 + 3a_2^2a_3 - a_3^2 - a_2^4| \leq \frac{2}{3}\lambda. \tag{3.3}$$

Proof. Let us define the function $p(z)$ by

$$p(z) = \frac{\dfrac{z^2 f'(z)}{f(z)^2} - (1-\lambda)}{\lambda} \quad (z \in \mathbb{U}).$$

This gives us that $p(z)$ is analytic in $\mathbb{U}$, $\text{Re}p(z) > 0$ $(z \in \mathbb{U})$, and

$$p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots .$$

Noting that

$$\frac{z^2 f'(z)}{f(z)^2} - 1 + \lambda = \lambda + (a_3 - a_2^2)z^2 + 2(a_4 - 2a_2a_3 + a_2^3)z^3$$
$$+3(a_5 - 2a_2a_4 + 3a_2^2a_3 - a_3^2 - a_2^4)z^4 + \cdots ,$$

we have that

$$a_3 - a_2^2 = \lambda c_2,$$

$$2(a_4 - 2a_2a_3 + a_2^3) = \lambda c_3,$$

and

$$3(a_5 - 2a_2a_4 + 3a_2^2a_3 - a_3^2 - a_2^4) = \lambda c_4.$$

Therefore, applying Lemma 1.2, we prove (3.1), (3.2) and (3.3). □

If $a_2 = 0$ in Theorem 2.2, then we have:

Corollary 3.1. *If $f(z) \in \mathcal{A}_3(\lambda)$ with $0 < \lambda < 1$, then*

$$|a_3| \leq 2\lambda,$$

$$|a_4| \leq \lambda,$$

and

$$|a_5| \leq \frac{8}{3}\lambda.$$

References

[1] C. Carathéodory: *"Uber den Variabilitätsbereich der Fourier'schen Konstanten von positiven harmonischen Funktionen*, Rend. Circ. Mat. Palermo **32**(1911), 193 – 217.
[2] I. S. Jack: *Functions starlike and convex of order α*, J. London Math. Soc., **3**(1971), 469 – 474.
[3] S. S. Miller, P. T. Mocanu: *Second-order differential inequalities in the complex plane*, J. Math. Anal. Appl., **65**(1978), 289 – 305.
[4] M. Obradović, S. Ponnusamy: *Radius properties for subclasses of univalent functions*, Analysis, **25**(2005), 183 – 188.
[5] S. Ozaki, M. Nunokawa: *The Schwarzian derivative and univalernt functions*, Proc. Amer. Math. Soc., **33** (1972), 392 – 394.
[6] Y. Shimoda, Y. Nakamura, S. Owa: *Radius problems for certain strongly close-to-convex functions*, PanAmer. Math. Journal, **23**(2013), 53 – 58.
[7] V. Singh: *On a class of univalent functions*, Internet. J. Math. Math. Sci., **23**(2000), 855 – 857.

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502, Japan

E-mail address: yutakashimoda119@gmail.com

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502, Japan

E-mail address: yayoi@math.kindai.ac.jp

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502, Japan

E-mail address: shige21@ican.zaq.ne.jp