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TRANSFORMATIONS ON SPACES WITH SPECIAL COMPOSITIONS

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ABSTRACT. Transformations of symmetric affine connections into connections with torsion on spaces endowed with Chebyshev and geodesic compositions are considered. Some of the components of the curvature tensor for the non-symmetric connections are computed with respect to the composition coordinates. Two special cases for the symmetric affine connection are discussed: a Weyl connection and an equiaffine connection.

1. INTRODUCTION

Multidimensional spaces of compositions equipped with a symmetric affine connection have been studied in [1, 3, 4, 11]. Weyl spaces of compositions are considered in [5, 6, 10], and equiaffine spaces are investigated in [9].

In this paper we consider spaces endowed with a symmetric affine connection and two special types of compositions: Chebyshev and geodesic ones. We study transformations of the symmetric connection into connections with torsion so that the composition satisfies an analogous characteristic condition with respect to the second connection. We compute some of the components of the curvature tensors of the introduced connections in the adapted to the composition coordinates. We also consider two special cases for the symmetric connection: a Weyl connection and an equiaffine connection.

2. PRELIMINARIES

Let A_N be a space with a symmetric affine connection denoted by $\Gamma^{\nu}_{\alpha\beta}$. In A_N, we consider a composition (i.e. a topological product) $X_n \times X_m$ of two basic manifolds X_n and X_m $(n+m=N)$. The positions (tangent planes) of the composition are denoted by $P(X_n)$ and $P(X_m)$, see [3, 4].

In addition to the usual coordinates u^{α} $(\alpha = 1, 2, ..., N)$, in A_N there are defined coordinates $(u^i, u^{\bar{i}})$ $(i = 1, 2, ..., n; \bar{i} = n+1, n+2, ..., n+m)$ which are adapted to the composition $X_n \times X_m$ (see [3]).

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The existence of a composition in A_N is equivalent to the definition of an affinor field a_α^β, satisfying the condition ([3, 8]):

$$a_\alpha^\sigma \; a_\sigma^\beta = \delta_\alpha^\beta \,, \tag{2.1}$$

and the integrability condition

$$a_\beta^\sigma \; \nabla_{[\alpha} \; a_{\sigma]}^\nu - a_\alpha^\sigma \; \nabla_{[\beta} \; a_{\sigma]}^\nu = 0 \,,$$

where ∇ is the covariant derivative with respect to $\Gamma_{\alpha\beta}^\nu$, and δ_α^β is the identity affinor.

The affinor a_α^β is called affinor of the composition ([3]).

According to [7], the projecting affinors have the following form

$$\overset{n}{a}{}_\alpha^\beta = \tfrac{1}{2}(\delta_\alpha^\beta + a_\alpha^\beta), \qquad \overset{m}{a}{}_\alpha^\beta = \tfrac{1}{2}(\delta_\alpha^\beta - a_\alpha^\beta).$$

For an arbitrary vector v^α we have the decomposition $v^\alpha = \overset{n}{a}{}_\beta^\alpha \; v^\beta + \overset{m}{a}{}_\beta^\alpha \; v^\beta$, where $\overset{n}{a}{}_\beta^\alpha \; v^\beta \in P(X_n)$, and $\overset{m}{a}{}_\beta^\alpha \; v^\beta \in P(X_m)$.

In the adapted to the composition coordinates, the matrices of the affinors a_α^β, $\overset{n}{a}{}_\alpha^\beta$ and $\overset{m}{a}{}_\alpha^\beta$ have the form (see [4, 5]):

$$(a_\alpha^\beta) = \begin{pmatrix} \delta_j^i & 0 \\ 0 & -\delta_{\bar{j}}^{\bar{i}} \end{pmatrix}, \qquad (\overset{n}{a}{}_\alpha^\beta) = \begin{pmatrix} \delta_j^i & 0 \\ 0 & 0 \end{pmatrix}, \qquad (\overset{m}{a}{}_\alpha^\beta) = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{\bar{j}}^{\bar{i}} \end{pmatrix}. \tag{2.2}$$

Further in this paper, we consider two special types of compositions.

A composition $X_n \times X_m$ is said to be Chebyshev, i.e. of the type (ch, ch), if the positions $P(X_n)$ and $P(X_m)$ are parallel transport along any line in the manifold X_m and X_n, (see [4]). The characteristic condition of a (ch, ch)-composition is given by:

$$\nabla_{[\alpha} \; a_{\beta]}^\nu = 0 \,. \tag{2.3}$$

In the adapted coordinates condition (2.3) is equivalent to the condition (see [4]):

$$\Gamma_{\bar{i}j}^{\bar{k}} = \Gamma_{i\bar{j}}^{k} = 0 \,. \tag{2.4}$$

A composition $X_n \times X_m$ is said to be geodesic, i.e. of the type (g, g), if the positions $P(X_n)$ and $P(X_m)$ are parallel transport along any line in the manifold X_n and X_m, see [4]. The characteristic condition of a (g, g)-composition is given by:

$$a_\alpha^\sigma \nabla_\beta a_\sigma^\nu + a_\beta^\sigma \nabla_\sigma a_\alpha^\nu = 0 \,. \tag{2.5}$$

In the adapted coordinates condition (2.5) is equivalent to:

$$\Gamma_{ij}^{\bar{k}} = \Gamma_{\bar{i}\bar{j}}^{k} = 0 \,. \tag{2.6}$$

The curvature tensor $R_{\alpha\beta\sigma.}^{\;\;\;\;\;\nu}$ of the space A_N , see [2], is defined as usually by:

$$R_{\alpha\beta\sigma.}^{\;\;\;\;\;\nu} = \partial_\alpha \Gamma_{\beta\sigma}^\nu - \partial_\beta \Gamma_{\alpha\sigma}^\nu + \Gamma_{\alpha\rho}^\nu \Gamma_{\beta\sigma}^\rho - \Gamma_{\beta\rho}^\nu \Gamma_{\alpha\sigma}^\rho \,. \tag{2.7}$$

3. Transformations of connections

Let us consider the connections ${}^1\Gamma^\nu_{\alpha\beta}$ defined by

$$ {}^1\Gamma^\nu_{\alpha\beta} = \Gamma^\nu_{\alpha\beta} + S^\nu_{\alpha\beta}, \tag{3.1} $$

where $S^\nu_{\alpha\beta}$ is the deformation tensor.

We denote by ${}^1\nabla$ and ${}^1R_{\alpha\beta\sigma.}{}^{\nu}$ the covariant derivative and the curvature tensor with respect to ${}^1\Gamma^\nu_{\alpha\beta}$.

3.1. **Transformations of connections on spaces of Chebyshev compositions.**

Let A_N be a space endowed with a composition $X_n \times X_m$ of the type (ch, ch). Then, A_N will be called a space of Chebyshev composition.

We shall prove the following theorem.

Theorem 3.1. *Let the composition $X_n \times X_m$ be Chebyshev copmposition, i.e. condition (2.3) holds. Then, ${}^1\nabla_{[\alpha}\ a^\nu_{\beta]} = 0$ if and only if in the adapted to the composition coordinates the deformation tensor $S^\nu_{\alpha\beta}$ satisfies the conditions:*

$$ S^k_{i\bar{j}} = S^{\bar{k}}_{\bar{i}j} = S^{\bar{k}}_{[ij]} = S^k_{[\bar{i}\bar{j}]} = 0\,. \tag{3.2} $$

Proof. We have

$$ {}^1\nabla_{[\alpha}\ a^\nu_{\beta]} = \nabla_{[\alpha}\ a^\nu_{\beta]} + P^\nu_{\alpha\beta}\,, \tag{3.3} $$

where

$$ P^\nu_{\alpha\beta} = S^\nu_{\alpha\sigma}a^\sigma_\beta - S^\nu_{\beta\sigma}a^\sigma_\alpha - S^\sigma_{\alpha\beta}a^\nu_\sigma + S^\sigma_{\beta\alpha}a^\nu_\sigma\,. \tag{3.4} $$

According to (2.2) and (3.4), in the adapted coordinates, for the components of the tensor $P^\nu_{\alpha\beta}$ we get:

$$ \begin{aligned} &P^k_{\bar{i}j} = 2S^k_{j\bar{i}}, \quad P^k_{i\bar{j}} = -2S^k_{i\bar{j}}, \quad P^{\bar{k}}_{i\bar{j}} = -2S^{\bar{k}}_{\bar{j}i}, \quad P^{\bar{k}}_{\bar{i}j} = 2S^{\bar{k}}_{\bar{i}j}, \\ &P^{\bar{k}}_{ij} = 4S^{\bar{k}}_{[ij]}, \quad P^k_{\bar{i}\bar{j}} = -4S^k_{[\bar{i}\bar{j}]}, \quad P^k_{ij} = P^{\bar{k}}_{\bar{i}\bar{j}} = 0\,. \end{aligned} \tag{3.5} $$

By (3.3) it follows that if $\nabla_{[\alpha}a^\nu_{\beta]} = 0$, then ${}^1\nabla_{[\alpha}a^\nu_{\beta]} = 0$ if and only if $P^\nu_{\alpha\beta} = 0$. Then, by (3.5) we obtain that the last condition is equivalent to (3.2) which proves the theorem. □

In the case of $\nabla_{[\alpha}a^\nu_{\beta]} = {}^1\nabla_{[\alpha}a^\nu_{\beta]} = 0$, by (2.4), (3.1) and (3.2) we obtain the components of the connection ${}^1\Gamma^\nu_{\alpha\beta}$ as follows:

$$ \begin{aligned} &{}^1\Gamma^k_{ij} = \Gamma^k_{ij} + S^k_{ij}, \quad {}^1\Gamma^{\bar{k}}_{\bar{i}\bar{j}} = \Gamma^{\bar{k}}_{\bar{i}\bar{j}} + S^{\bar{k}}_{\bar{i}\bar{j}}, \\ &{}^1\Gamma^{\bar{k}}_{ij} = \Gamma^{\bar{k}}_{ij} + S^{\bar{k}}_{ij} \quad ({}^1\Gamma^{\bar{k}}_{ij} ={}^1\Gamma^{\bar{k}}_{ji}), \quad {}^1\Gamma^k_{\bar{i}\bar{j}} = \Gamma^k_{\bar{i}\bar{j}} + S^k_{\bar{i}\bar{j}} \quad ({}^1\Gamma^k_{\bar{i}\bar{j}} ={}^1\Gamma^k_{\bar{j}\bar{i}}), \\ &{}^1\Gamma^k_{\bar{i}j} = S^k_{\bar{i}j}, \quad {}^1\Gamma^{\bar{k}}_{i\bar{j}} = S^{\bar{k}}_{i\bar{j}}, \quad {}^1\Gamma^k_{i\bar{j}} ={}^1\Gamma^{\bar{k}}_{\bar{i}j} = 0\,. \end{aligned} \tag{3.6} $$

Next, we consider the curvature properties of the connections ${}^1\Gamma^\nu_{\alpha\beta}$. Using (2.7) and (3.6) we compute the following adapted components of ${}^1R_{\alpha\beta\sigma.}{}^{\nu}$:

$$ {}^1R_{ij\bar{k}.}{}^{s} = {}^1R_{\bar{i}\bar{j}k.}{}^{\bar{s}} = 0, $$

$$ {}^1R_{\bar{i}\bar{j}k.}{}^{s} = 2\partial_{[\bar{i}}\ S^k_{\bar{j}]s} + 2S^k_{[\bar{i}|p|}\ S^p_{\bar{j}]s}, \quad {}^1R_{ij\bar{k}.}{}^{\bar{s}} = 2\partial_{[i}\ S^{\bar{k}}_{j]\bar{s}} + 2S^{\bar{k}}_{[i|\bar{p}|}\ S^{\bar{p}}_{j]\bar{s}}\,. $$

We shall consider the following example.

Let A_N be a space with a Weyl connection $\Gamma^{\nu}_{\alpha\beta}$, fundamental tensor $g_{\alpha\beta}$ and an additional 1-form (covector) ω_σ which satisfy the condition $\nabla_\sigma g_{\alpha\beta} = 2\omega_\sigma g_{\alpha\beta}$. According to [2], we have:

$$\Gamma^{\nu}_{\alpha\beta} = \{^{\nu}_{\alpha\beta}\} - (\omega_\alpha \delta^{\nu}_{\beta} + \omega_\beta \delta^{\nu}_{\alpha} - \omega_\sigma g^{\sigma\nu} g_{\alpha\beta}) \, ,$$

where $g_{\alpha\sigma} g^{\sigma\beta} = \delta^{\beta}_{\alpha}$, and $\{^{\nu}_{\alpha\beta}\}$ are the Christoffel symbols of $g_{\alpha\beta}$.

For the curvature tensor $R_{\alpha\beta\sigma.}^{\ \ \ \ \nu}$ of $\Gamma^{\nu}_{\alpha\beta}$ it is valid:

$$R_{\alpha\beta\sigma.}^{\ \ \ \ \sigma} = -2N\nabla_{[\alpha}\omega_{\beta]} \, . \tag{3.7}$$

Let the space A_N contain a composition $X_n \times X_m$ of the type (ch, ch), i.e. $\Gamma^{\nu}_{\alpha\beta}$ satisfy conditions (2.4). We consider the connections ${}^1\Gamma^{\nu}_{\alpha\beta}$ defined by (3.1). Using (2.7) and (3.1) we get:

$${}^1R_{\alpha\beta\sigma.}^{\ \ \ \ \sigma} = R_{\alpha\beta\sigma.}^{\ \ \ \ \sigma} + 2N\nabla_{[\alpha} S^{\sigma}_{\beta]\sigma} \, . \tag{3.8}$$

Now, let the deformation tensor $S^{\nu}_{\alpha\beta}$ in (3.1) be given by:

$$S^{\nu}_{\alpha\beta} = \omega_\alpha a^{\nu}_{\beta} \, . \tag{3.9}$$

For the adapted coordinates we obtain:

$$\begin{gathered} S^{k}_{ij} = \omega_i \delta^{k}_{j}, \quad S^{\bar{k}}_{\bar{i}\bar{j}} = \omega_{\bar{i}} \delta^{\bar{k}}_{\bar{j}}, \quad S^{k}_{\bar{i}j} = \omega_{\bar{i}} \delta^{k}_{j}, \quad S^{\bar{k}}_{i\bar{j}} = \omega_i \delta^{\bar{k}}_{\bar{j}}, \\ S^{k}_{i\bar{j}} = S^{\bar{k}}_{\bar{i}j} = S^{\bar{k}}_{ij} = S^{k}_{\bar{i}\bar{j}} = 0 \, . \end{gathered} \tag{3.10}$$

Then, the connections (3.1) with $S^{\nu}_{\alpha\beta}$ given by (3.9) satisfy the necessary and sufficient conditions (3.2) of Theorem 3.1. In the adapted coordinates, by (3.7), (3.8) and (3.10) we have:

$${}^1R_{ij\sigma.}^{\ \ \ \sigma} = {}^1R_{\bar{i}\bar{j}\sigma.}^{\ \ \ \sigma} = {}^1R_{\bar{i}j\sigma.}^{\ \ \ \sigma} = 0 \, ,$$

and it follows that ${}^1R_{\alpha\beta\sigma.}^{\ \ \ \ \sigma} = 0$.

3.2. **Transformations of connections on spaces of geodesic compositions.** Let A_N be a space of a (g, g)-composition $X_n \times X_m$, and consider again the connections ${}^1\Gamma^{\nu}_{\alpha\beta}$ defined by (3.1).

Theorem 3.2. *Let the composition $X_n \times X_m$ be geodesic, i.e. condition (2.5) holds. Then,*

$$a^{\sigma}_{\alpha} \ {}^1\nabla_\beta a^{\nu}_{\sigma} + a^{\sigma}_{\beta} \ {}^1\nabla_\sigma a^{\nu}_{\alpha} = 0 \tag{3.11}$$

if and only if in the adapted to the composition coordinates the deformation tensor $S^{\nu}_{\alpha\beta}$ satisfies the conditions:

$$S^{k}_{\bar{i}\bar{j}} = S^{\bar{k}}_{ij} = 0. \tag{3.12}$$

Proof. Using (2.1) and (3.1) we have:

$$a^{\sigma}_{\alpha} \ {}^1\nabla_\beta a^{\nu}_{\sigma} + a^{\sigma}_{\beta} \ {}^1\nabla_\sigma a^{\nu}_{\alpha} = a^{\sigma}_{\alpha} \nabla_\beta a^{\nu}_{\sigma} + a^{\sigma}_{\beta} \nabla_\sigma a^{\nu}_{\alpha} + L^{\nu}_{\alpha\beta}, \tag{3.13}$$

where

$$L^{\nu}_{\alpha\beta} = S^{\nu}_{\beta\alpha} - S^{\rho}_{\beta\sigma} a^{\sigma}_{\alpha} a^{\nu}_{\rho} + S^{\nu}_{\sigma\rho} a^{\rho}_{\alpha} a^{\sigma}_{\beta} - S^{\rho}_{\sigma\alpha} a^{\nu}_{\rho} a^{\sigma}_{\beta}. \tag{3.14}$$

In the adapted coordinates, by (3.14) we evaluate the components of $L^{\nu}_{\alpha\beta}$ as follows:

$$L^{\bar{k}}_{ij} = 4S^{\bar{k}}_{ji}, \quad S^{l}_{\bar{i}\bar{j}} = 4S^{k}_{\bar{j}\bar{i}}, \qquad (3.15)$$
$$L^{k}_{ij} = L^{k}_{\bar{i}j} = L^{k}_{i\bar{j}} = L^{\bar{k}}_{\bar{i}j} = L^{\bar{k}}_{i\bar{j}} = L^{\bar{k}}_{\bar{i}\bar{j}} = 0.$$

Equalities (2.5) and (3.13) imply that if the composition $X_n \times X_m$ is geodesic, then (3.11) holds if and only if $L^{\nu}_{\alpha\beta} = 0$. According to (3.15), the last condition holds if and only if also (3.12) holds, which completes the proof of the theorem. □

If conditions (2.5) and (3.11) holds, using (2.6), (3.1) and (3.12) we obtain:

$$\begin{aligned} &{}^1\Gamma^{k}_{ij} = \Gamma^{k}_{ij} + S^{k}_{ij}, \quad {}^1\Gamma^{k}_{\bar{i}j} = \Gamma^{k}_{\bar{i}j} + S^{k}_{\bar{i}j}, \quad {}^1\Gamma^{k}_{i\bar{j}} = \Gamma^{k}_{i\bar{j}} + S^{k}_{i\bar{j}}, \\ &{}^1\Gamma^{\bar{k}}_{\bar{i}j} = \Gamma^{\bar{k}}_{\bar{i}j} + S^{\bar{k}}_{\bar{i}j}, \quad {}^1\Gamma^{\bar{k}}_{i\bar{j}} = \Gamma^{\bar{k}}_{i\bar{j}} + S^{\bar{k}}_{i\bar{j}}, \quad {}^1\Gamma^{\bar{k}}_{\bar{i}\bar{j}} = \Gamma^{\bar{k}}_{\bar{i}\bar{j}} + S^{\bar{k}}_{\bar{i}\bar{j}}, \\ &{}^1\Gamma^{\bar{k}}_{ij} = {}^1\Gamma^{k}_{\bar{i}\bar{j}} = 0. \end{aligned} \qquad (3.16)$$

Then, by (2.7) and (3.16) for the adapted components of ${}^1R_{\alpha\beta\sigma.}{}^{\nu}$ we get:

$${}^1R_{\bar{i}\bar{j}\bar{k}.}{}^{s} = {}^1R_{ijk.}{}^{\bar{s}} = 0.$$

Now we consider the following example.

Let $\Gamma^{\nu}_{\alpha\beta}$ be an equiaffine connection with main density e. In accordance to [2], $\Gamma^{\nu}_{\alpha\beta}$ is equiaffine if and only if one of the following two equivalent conditions hold:

$$\Gamma^{\nu}_{\alpha\nu} = \partial_{\alpha} \ln e \quad \Longleftrightarrow \quad R_{\alpha\beta\nu.}{}^{\nu} = 0\,. \qquad (3.17)$$

Using (3.8), (3.1) and (3.17) it follows that ${}^1\Gamma^{\nu}_{\alpha\nu} = \text{grad}$ if and only if $S^{\nu}_{\alpha\nu} = \text{grad}$. Also, if $S^{\nu}_{\alpha\nu} = \text{grad}$, then ${}^1R_{\alpha\beta\nu.}{}^{\nu} = 0$.

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