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NEW CLASS FOR CERTAIN ANALYTIC FUNCTIONS CONCERNED WITH THE STRIP DOMAINS

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ABSTRACT. By considering a certain univalent function in the open unit disk $\mathbb{U}$ which maps $\mathbb{U}$ onto the strip domain w with $\alpha < w < \beta$ $(\alpha < 1,\ \beta > 1)$, some new class for certain analytic functions is discussed.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. The subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in $\mathbb{U}$ is denoted by $\mathcal{S}$.

A function $f(z) \in \mathcal{A}$ is said to be starlike of order α in $\mathbb{U}$ if it satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U})$$

for some real number α with $0 \leqq \alpha < 1$. This class is denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{S}^*(0) = \mathcal{S}^*$. The class $\mathcal{S}^*(\alpha)$ was introduced by Robertson [2]. It is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^* \subset \mathcal{S}$.

Furthermore, let $\mathcal{M}(\beta)$ be the class of functions $f(z) \in \mathcal{A}$ which satisfy

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) < \beta \qquad (z \in \mathbb{U})$$

for some real number β with $\beta > 1$. The class $\mathcal{M}(\beta)$ was investigated by Uralegaddi, Ganigi and Sarangi [4].

Let $p(z)$ and $q(z)$ be analytic in $\mathbb{U}$. Then the function $p(z)$ is said to be subordinate to $q(z)$ in $\mathbb{U}$, written by

$$p(z) \prec q(z) \qquad (z \in \mathbb{U}), \tag{1.1}$$

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if there exists a function $w(z)$ which is analytic in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1 \quad (z \in \mathbb{U})$, and such that $p(z)=q\big(w(z)\big) \quad (z \in \mathbb{U})$. From the definition of the subordinations, it is easy to show that the subordination (1.1) implies that

$$p(0)=q(0) \quad \text{and} \quad p(\mathbb{U}) \subset q(\mathbb{U}). \tag{1.2}$$

In particular, if $q(z)$ is univalent in $\mathbb{U}$, then the subordination (1.1) is equivalent to the condition (1.2).

Remark 1.1. *Let $p(z)$ and $q(z)$ be analytic in $\mathbb{U}$. Then the subordination (1.1) implies that*

$$|p'(0)| \leqq |q'(0)|,$$

and $|p'(0)|=|q'(0)|$ if and only if $p(z)=q(e^{i\theta}z)$ for some real number θ with $0 \leqq \theta < 2\pi$ (cf. [1]).

Motivated by the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{M}(\beta)$, we define new class for certain analytic functions. Let $\mathcal{S}(\alpha,\beta)$ denote the class of functions $f(z) \in \mathcal{A}$ which satisfy the inequality

$$\alpha < \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \beta \qquad (z \in \mathbb{U}) \tag{1.3}$$

for some real number α $(\alpha < 1)$ and some real number β $(\beta > 1)$. Note that α is not necessary to be positive in the class $\mathcal{S}(\alpha,\beta)$.

Remark 1.2. *Let $f(z) \in \mathcal{S}(\alpha,\beta)$. If $\alpha \geqq 0$, then $f(z)$ is starlike in $\mathbb{U}$, which implies that $f(z)$ is univalent in $\mathbb{U}$.*

In order to discuss our new class $\mathcal{S}(\alpha,\beta)$, we need to consider a certain univalent function in $\mathbb{U}$ which maps $\mathbb{U}$ onto the strip domain w with $\alpha < \operatorname{Re} w < \beta$.

Theorem 1.1. *Let α and β be real numbers with $\alpha < 1$ and $\beta > 1$. Then the function $S_{\alpha,\beta}(z)$ defined by*

$$S_{\alpha,\beta}(z) = 1 + \frac{\beta-\alpha}{\pi} i \log\left(\frac{1-e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}}z}{1-e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}}z}\right) \qquad (z \in \mathbb{U}) \tag{1.4}$$

is analytic and univalent in $\mathbb{U}$ with $S_{\alpha,\beta}(0)=1$. In addition, $S_{\alpha,\beta}(z)$ maps $\mathbb{U}$ onto the strip domain w with $\alpha < \operatorname{Re} w < \beta$.

Proof. Note that the function $S_{\alpha,\beta}(z)$ defined by (1.4) can be written as follows:

$$S_{\alpha,\beta}(z) = \frac{\alpha+\beta}{2} + \frac{\beta-\alpha}{\pi} i \log\left(\frac{ie^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}} + (-i)e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}}\, e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}}z}{1-e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}}z}\right). \tag{1.5}$$

From the equality (1.5), a simple check gives us that $S_{\alpha,\beta}(z)$ maps $\mathbb{U}$ onto the strip domain w with $\alpha < \operatorname{Re} w < \beta$. Moreover, it is easy to see that the function $S_{\alpha,\beta}(z)$ is analytic and univalent in $\mathbb{U}$ with $S_{\alpha,\beta}(0)=1$. □

Remark 1.3.

$$S_{\alpha,\beta}(z) = 1 + \frac{\beta-\alpha}{\pi} i \log\left(\frac{1-e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}}z}{1-e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}}z}\right) = 1 + \sum_{n=1}^{\infty} B_n z^n,$$

where

$$B_n = \frac{2(\beta-\alpha)}{n\pi} \sin \frac{n\pi(1-\alpha)}{\beta-\alpha} \qquad (n = 1, 2, \cdots). \tag{1.6}$$

Specially, we note that the coefficient B_n defined by (1.6) is the real number. Since

$$\lim_{\beta \to +\infty} B_n = \lim_{\beta \to +\infty} \left\{ 2(1-\alpha) \frac{\sin \frac{n\pi(1-\alpha)}{\beta-\alpha}}{\frac{n\pi(1-\alpha)}{\beta-\alpha}} \right\} = 2(1-\alpha),$$

a simple check gives us that

$$S_{\alpha,\beta}(z) = 1 + \sum_{n=1}^{\infty} 2(1-\alpha)z^n = \frac{1+(1-2\alpha)z}{1-z} \qquad (\beta \to +\infty),$$

which implies that $S_{\alpha,\beta}(z) \quad (\beta \to +\infty)$ maps $\mathbb{U}$ onto the right half-plane w with $\operatorname{Re} w > \alpha$.

On the other hand, it is easy to see that

$$\lim_{\alpha \to -\infty} B_n = \lim_{\alpha \to -\infty} \left\{ \frac{2(\beta-\alpha)}{n\pi} \sin \left(\frac{n\pi(1-\beta)}{\beta-\alpha} + n\pi \right) \right\}$$

$$= \lim_{\alpha \to -\infty} \left\{ \frac{2(\beta-\alpha)}{n\pi} (-1)^n \sin \frac{n\pi(1-\beta)}{\beta-\alpha} \right\}$$

$$= \lim_{\alpha \to -\infty} \left\{ 2(\beta-1)(-1)^{n-1} \frac{\sin \frac{n\pi(1-\beta)}{\beta-\alpha}}{\frac{n\pi(1-\beta)}{\beta-\alpha}} \right\} = 2(\beta-1)(-1)^{n-1}.$$

Therefore, we find that

$$S_{\alpha,\beta}(z) = 1 + \sum_{n=1}^{\infty} 2(\beta-1)(-1)^{n-1}z^n = \frac{1-(1-2\beta)z}{1+z} \qquad (\alpha \to -\infty),$$

which implies that $S_{\alpha,\beta}(z) \quad (\alpha \to -\infty)$ maps $\mathbb{U}$ onto the left half-plane w with $\operatorname{Re} w < \beta$.

We give some example for $f(z) \in \mathcal{S}(\alpha, \beta)$ as follows.

Example 1.1. *Let us consider the function $f(z)$ given by*

$$f(z) = z \exp \left\{ \frac{\beta-\alpha}{\pi} i \int_0^z \frac{1}{t} \log \left(\frac{1 - e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}} t}{1 - e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}} t} \right) dt \right\} \tag{1.7}$$

$$= z + \frac{2(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha} z^2 + \cdots \qquad (z \in \mathbb{U})$$

with $\alpha < 1$ and $\beta > 1$. Then we have

$$\frac{zf'(z)}{f(z)} = 1 + \frac{\beta-\alpha}{\pi} i \log \left(\frac{1 - e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}} z}{1 - e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}} z} \right) = S_{\alpha,\beta}(z) \qquad (z \in \mathbb{U}).$$

According to Theorem 1.1. it is clear that the function $f(z)$ given by (1.7) satisfies the inequality (1.3), which implies that $f(z) \in \mathcal{S}(\alpha, \beta)$.

Applying the function $S_{\alpha,\beta}(z)$ defined by (1.4), we give a necessary and sufficient condition for $f(z) \in \mathcal{A}$ to belong to the class $\mathcal{S}(\alpha, \beta)$.

Lemma 1.1. *Let $f(z) \in \mathcal{A}$. Then $f(z) \in \mathcal{S}(\alpha, \beta)$ if and only if*

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{\beta - \alpha}{\pi} i \log \left(\frac{1 - e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}} z}{1 - e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}} z} \right) = S_{\alpha,\beta}(z) \qquad (z \in \mathbb{U}), \tag{1.8}$$

where $\alpha < 1$ and $\beta > 1$.

By using the subordination (1.8), we discussed some properties for $f(z) \in \mathcal{S}(\alpha, \beta)$.

2. Some results

Rogosinski [3] proved the coefficient estimates for subordinate functions.

Lemma 2.1. *Let $q(z) = \sum_{n=1}^{\infty} B_n z^n$ be analytic and univalent in $\mathbb{U}$, and suppose that $q(z)$ maps $\mathbb{U}$ onto a convex domain. If $p(z) = \sum_{n=1}^{\infty} A_n z^n$ is analytic in $\mathbb{U}$ and satisfies the following subordination*

$$p(z) \prec q(z) \qquad (z \in \mathbb{U}),$$

then

$$|A_n| \leqq |B_1| \qquad (n = 1, 2, \cdots).$$

Applying Lemma 2.1, we deduced some coefficient estimate for $f(z) \in \mathcal{S}(\alpha, \beta)$ bellow.

Theorem 2.1. *If the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}(\alpha, \beta)$, then*

$$|a_n| \leqq \prod_{k=2}^{n} \frac{k - 2 + \frac{2(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}}{k-1} \qquad (n = 2, 3, \cdots).$$

Proof. According to the assertion of Lemma 1.1 the function $f(z)$ satisfies the subordination (1.8). Let us define the function $p(z)$ by

$$p(z) = \frac{zf'(z)}{f(z)} \qquad (z \in \mathbb{U}). \tag{2.1}$$

Then, the subordination (1.8) can be written as follows :

$$p(z) \prec S_{\alpha,\beta}(z) \qquad (z \in \mathbb{U}), \tag{2.2}$$

where $S_{\alpha,\beta}(z)$ is defined by (1.4). Note that the function $S_{\alpha,\beta}(z)$ is convex in $\mathbb{U}$, and has the following form

$$S_{\alpha,\beta}(z) = 1 + \sum_{n=1}^{\infty} B_n z^n,$$

where B_n is defined by (1.6). If we let

$$p(z) = 1 + \sum_{n=1}^{\infty} A_n z^n,$$

then by Lemma 2.1, we see that the subordination (2.2) implies that

$$|A_n| \leqq B_1 \qquad (n = 1, 2, \cdots), \tag{2.3}$$

where

$$B_1 = \frac{2(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}. \tag{2.4}$$

Now, the equality (2.1) implies that

$$zf'(z) = p(z)f(z).$$

Then, the coefficients of z^n in both sides lead to

$$(n-1)a_n = A_{n-1} + A_{n-2}a_2 + \cdots + A_1 a_{n-1} \qquad (n = 2, 3, \cdots). \tag{2.5}$$

A simple calculation combined with the inequality (2.3) yields that

$$\begin{aligned} |a_n| &= \frac{1}{n-1}\left|A_{n-1} + A_{n-2}a_2 + \cdots + A_1 a_{n-1}\right| \\ &\leqq \frac{1}{n-1}\left(|A_{n-1}| + |A_{n-2}||a_2| + \cdots + |A_1||a_{n-1}|\right) \\ &\leqq \frac{B_1}{n-1}\sum_{k=2}^{n} |a_{k-1}| \qquad (|a_1| = 1), \end{aligned}$$

where B_1 is given in (2.4). To prove the assertion of the theorem, we need show that

$$X_n \equiv \frac{B_1}{n-1}\sum_{k=2}^{n} |a_{k-1}| \leqq \prod_{k=2}^{n} \frac{k-2+B_1}{k-1} \qquad (n = 2, 3, \cdots). \tag{2.6}$$

We now use the mathematical induction for the proof of the theorem. Since

$$X_2 = B_1|a_1| = B_1 \qquad (|a_1| = 1),$$

it is clear that the assertion is holds true for $n = 2$.
We assume that the proposition is true for $n = m$. Then, some calculation gives us that

$$\begin{aligned} X_{m+1} &= \frac{B_1}{(m+1)-1}\sum_{k=2}^{m+1} |a_{k-1}| = \frac{B_1}{m}\left(\sum_{k=2}^{m} |a_{k-1}| + |a_m|\right) \\ &\leqq \frac{B_1}{m}\left(1 + \frac{B_1}{m-1}\right)\sum_{k=2}^{m} |a_{k-1}| = \frac{m-1+B_1}{m}\,\frac{B_1}{m-1}\sum_{k=2}^{m} |a_{k-1}| \\ &\leqq \frac{m-1+B_1}{m}\prod_{k=2}^{m} \frac{k-2+B_1}{k-1} = \prod_{k=2}^{m+1} \frac{k-2+B_1}{k-1}, \end{aligned}$$

which implies that the inequality (2.6) is true for $n = m + 1$.
By the mathematical induction, we prove that

$$|a_n| \leqq \prod_{k=2}^{n} \frac{k-2+B_1}{k-1} \qquad (n = 2, 3, \cdots),$$

where B_1 is given in (2.4). This completes the proof of Theorem 2.1. □

We next give sharp bounds on the second and third coefficients for $f(z) \in \mathcal{S}(\alpha,\beta)$. To obtain some sharp coefficient estimates, we need the following lemma due to Rogosinski [3].

Lemma 2.2. *Let* $p(z) = \sum_{n=1}^{\infty} A_n z^n$ *and* $q(z) = \sum_{n=1}^{\infty} B_n z^n$ *be analytic in* $\mathbb{U}$. *If* $p(z) \prec q(z)$ $(z \in \mathbb{U})$, *then*

$$\sum_{k=1}^{m} |A_k|^2 \leqq \sum_{k=1}^{m} |B_k|^2 \qquad (m = 1, 2, \cdots).$$

By Remark 1.1 and Lemma 2.2, we obtained

Theorem 2.2. *If the function* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}(\alpha,\beta)$, *then*

$$|a_2| \leqq \frac{2(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}, \tag{2.7}$$

and

$$|a_3| \leqq \frac{\beta-\alpha}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha} \left(\cos \frac{\pi(1-\alpha)}{\beta-\alpha} + \frac{2(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha} \right). \tag{2.8}$$

Moreover, the equality holds in either inequality if and only if

$$f(z) = z \exp \left\{ \int_0^z \frac{S_{\alpha,\beta}\left(e^{i\theta} t\right) - 1}{t} \, dt \right\}, \tag{2.9}$$

for some real number θ $(0 \leqq \theta < 2\pi)$, *where* $S_{\alpha,\beta}(z)$ *is defined by (1.4).*

Proof. According to the proof of Theorem 2.1, the function $f(z)$ satisfies the subordination (2.2). Also, it follows from the equality (2.5) that

$$a_2 = A_1$$

and

$$2a_3 = A_2 + A_1 a_2 = A_2 + {A_1}^2.$$

Then by Remark 1.1, we have

$$|p'(0)| \leqq \left|S'_{\alpha,\beta}(0)\right|, \tag{2.10}$$

with equality if and only if $p(z) = S_{\alpha,\beta}\left(e^{i\theta} z\right)$ for some real number θ $(0 \leqq \theta < 2\pi)$. This implies that

$$|a_2| = |A_1| = |p'(0)| \leqq \left|S'_{\alpha,\beta}(0)\right| = B_1 = \frac{2(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}$$

with equality if and only if $f(z)$ is given in (2.9).

We next prove the sharp bound on the third coefficient for $f(z) \in \mathcal{S}(\alpha,\beta)$. According to Lemma 2.2, the subordination (2.2) implies that

$$|A_1|^2 + |A_2|^2 \leqq {B_1}^2 + {B_2}^2.$$

Thus, we find that

$$2|a_3| = \left|A_2 + {A_1}^2\right| \leqq |A_2| + |A_1|^2$$

$$= \frac{2B_2|A_2|}{2B_2} + |A_1|^2 \leqq \frac{{B_2}^2 + |A_2|^2}{2B_2} + |A_1|^2$$

$$= \frac{{B_2}^2 + \left(|A_1|^2 + |A_2|^2\right) + (2B_2 - 1)|A_1|^2}{2B_2}$$

$$\leqq \frac{{B_2}^2 + \left({B_1}^2 + {B_2}^2\right) + (2B_2 - 1){B_1}^2}{2B_2} = B_2 + {B_1}^2$$

$$= \frac{2(\beta-\alpha)}{\pi} \sin\frac{\pi(1-\alpha)}{\beta-\alpha} \left(\cos\frac{\pi(1-\alpha)}{\beta-\alpha} + \frac{2(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right).$$

Note that we have employed the equality (2.10) here, so equality holds if and only if $f(z)$ is given in (2.9). This completes the proof of the theorem. □

Remark 2.1. *According to Theorem 1.1 and Example 1.1, the extremal function $f(z) \in \mathcal{S}(\alpha,\beta)$ is given by (1.7), and its rotation. Note that the function $f(z)$ given by (1.7) has the following form:*

$$f(z) = z\exp\left\{\int_0^z \frac{S_{\alpha,\beta}(t) - 1}{t}\,dt\right\} = z\exp\left\{\sum_{k=1}^{\infty}\frac{B_k}{k}z^k\right\} = z + \sum_{n=2}^{\infty} b_n z^n,$$

where

$$b_n = \sum_{k=1}^{n-1}\frac{1}{k!}\left\{\sum_{\sum_{j=1}^{k} m_j = n-1}\left(\prod_{j=1}^{k}\frac{B_{m_j}}{m_j}\right)\right\}.$$

From this fact, we expect the sharp bounds on the coefficients for $f(z) \in \mathcal{S}(\alpha,\beta)$ as follows:

(2.11)

$$|a_n| \leqq \sum_{k=1}^{n-1}\frac{1}{k!}\left\{\sum_{\sum_{j=1}^{k} m_j = n-1}\left(\prod_{j=1}^{k}\frac{2(\beta-\alpha)}{{m_j}^2\,\pi}\sin\frac{m_j\pi(1-\alpha)}{\beta-\alpha}\right)\right\} \qquad (n = 2, 3, \cdots).$$

Actually, letting $n = 2$ in (2.11), we have the coefficient inequality (2.7). Also, taking $n = 3$ in (2.11), we find the coefficient inequality (2.8).

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