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SOME PROPERTIES FOR GENERAL INTEGRAL OPERATORS

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ABSTRACT. For analytic functions $f_j(z)$ in the open unit disk $\mathbb{U}$ with $f_j(0) = 0$ and $f_j'(0) = 1$, two general integral operators $F_\beta(z)$ and $G_\beta(z)$ are introduced. In view of the results due to S. Owa, J. Nishiwaki and N. Niwa (Int. J. Open Problems Compt. Math. **1**(2008), 1 - 7), new classes $\mathcal{T}_\delta^*(\alpha), \mathcal{S}_\delta^*(\alpha), \mathcal{K}_\delta^*(\alpha)$, and $\mathcal{C}_\delta^*(\alpha)$ are considered. The object of the present paper is to discuss some properties for the general integral operators $F_\beta(z)$ and $G_\beta(z)$ with the above classes.

1. INTRODUCTION

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For analytic functions $f(z)$ and $g(z)$ in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$, written by $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1\,(z \in \mathbb{U})$ such that $f(z) = g(w(z))$. In particular, if $g(z)$ is univalent in $\mathbb{U}$, then the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ (see Duren [2], or Miller and Mocanu [3]). Using subordinations, Owa, Nishiwaki and Niwa [4] have defined the following subclasses $\mathcal{S}_\delta(\alpha)$ and $\mathcal{T}_\delta(\alpha)$ of $\mathcal{A}$.
A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}_\delta(\alpha)$ if it satisfies

$$(f'(z))^\delta \prec \frac{\alpha(1-z)}{\alpha - z} \qquad (z \in \mathbb{U}) \tag{1.2}$$

for some real $\alpha > 1$ and $\delta > 0$. Also, a function $f(z) \in \mathcal{A}$ is said to be a member of the class $\mathcal{T}_\delta(\alpha)$ if it satisfies

$$\left(\frac{1}{f'(z)}\right)^\delta \prec \frac{\alpha(1-z)}{\alpha - z} \qquad (z \in \mathbb{U}) \tag{1.3}$$

for some real $\alpha > 1$ and $\delta > 0$.
For the classes $\mathcal{S}_\delta(\alpha)$ and $\mathcal{T}_\delta(\alpha)$, Owa, Nishiwaki and Niwa [4] have derived

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Theorem 1.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) < \frac{\alpha - 1}{2\delta(\alpha+1)} \qquad (z \in \mathbb{U}) \tag{1.4}$$

for some real $\alpha > 1$ and $\delta > 0$, then $f(z) \in \mathcal{S}_\delta(\alpha)$.

Theorem 1.2. *If $f(z) \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) > \frac{1 - \alpha}{2\delta(\alpha+1)} \qquad (z \in \mathbb{U}) \tag{1.5}$$

for some real $\alpha > 1$ and $\delta > 0$, then $f(z) \in \mathcal{T}_\delta(\alpha)$.

2. General integral operator $F_\beta(z)$

Considering Theorem 1.1 and Theorem 1.2, we introduce the following two classes $\mathcal{S}_\delta^*(\alpha)$ and $\mathcal{T}_\delta^*(\alpha)$.
A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}_\delta^*(\alpha)$ if it satisfies the inequality (1.4) for some real $\alpha > 1$ and $\delta > 0$. Also, we say that a function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{T}_\delta^*(\alpha)$ if it satisfies the inequality (1.5) for some real $\alpha > 1$ and $\delta > 0$. Thus note that $\mathcal{S}_\delta^*(\alpha) \subset \mathcal{S}_\delta(\alpha)$ and $\mathcal{T}_\delta^*(\alpha) \subset \mathcal{T}_\delta(\alpha)$.
For analytic functions $f_j(z)$ given by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (j = 1, 2, 3, \cdots), \tag{2.1}$$

we define the general integral operator $F_\beta(z)$ by

$$F_\beta(z) = \int_0^z \left(\prod_{j=1}^{n} \left(f_j'(t)\right)^{\beta_j}\right) dt \tag{2.2}$$

with $\beta_j > 0$ $(j = 1, 2, 3, \cdots)$ and

$$\sum_{j=1}^{n} \beta_j = \beta.$$

This general integral operator $F_\beta(z)$ was first introduced by Breaz, Owa and Breaz in [1]. Now, we derive

Theorem 2.1. *If $f_j(z) \in \mathcal{S}_{\delta_j}^*(\alpha_j)$ for each $j = 1, 2, 3, \cdots, n$, then*

$$\operatorname{Re}\left(\frac{zF_\beta''(z)}{F_\beta'(z)}\right) < \frac{(1-\alpha)\beta}{2\delta(\alpha+1)} \qquad (z \in \mathbb{U}), \tag{2.3}$$

where

$$\frac{1-\alpha}{2\delta(\alpha+1)} = \max_{1 \leqq j \leqq n} \frac{1-\alpha_j}{2\delta_j(\alpha_j+1)}$$

and $\sum_{j=1}^{n} \beta_j = \beta$. This implies that $F_\beta(z) \in \mathcal{S}_{\frac{\delta}{\beta}}^*(\alpha)$.

Proof. From the definition (2.2), we see that

$$\operatorname{Re}\left(\frac{zF_\beta''(z)}{F_\beta'(z)}\right) = \sum_{j=1}^{n} \operatorname{Re}\left(\beta_j \frac{zf_j''(z)}{f_j'(z)}\right)$$

$$< \sum_{j=1}^{n} \frac{(1-\alpha_j)\beta_j}{2\delta_j(\alpha_j+1)} \leqq \frac{1-\alpha}{2\delta(\alpha+1)}\left(\sum_{j=1}^{n}\beta_j\right) = \frac{(1-\alpha)\beta}{2\delta(\alpha+1)}$$

for $z \in \mathbb{U}$. This completes the proof of the theorem. □

Corollary 2.1. *If $f_j(z) \in \mathcal{S}_\delta^*(\alpha)$ for all $j = 1, 2, 3, \cdots, n$, then $F_\beta(z) \in \mathcal{S}_{\frac{\delta}{\beta}}^*(\alpha)$.*

Example 2.1. *Let us consider the functions $f_j(z)$ $(j = 1, 2, 3, \cdots, n)$ which satisfy*

$$\prod_{j=1}^{n} \left(f_j'(z)\right)^{\beta_j} = (1-z)^{p-1} \qquad (z \in \mathbb{U})$$

with $p = \dfrac{(1-\alpha)\beta}{\delta(\alpha+1)} + 1$. Then we have that

$$F_\beta(z) = \frac{1}{p}\left(1-(1-z)^p\right)$$

and

$$\operatorname{Re}\left(\frac{zF_\beta''(z)}{F_\beta'(z)}\right) = \operatorname{Re}\left(\frac{(1-p)z}{1-z}\right) < \frac{(1-\alpha)\beta}{2\delta(\alpha+1)} \qquad (z \in \mathbb{U}).$$

Theorem 2.2. *If $f_j(z) \in \mathcal{T}_{\delta_j}^*(\alpha_j)$ for each $j = 1, 2, 3, \cdots, n$, then*

$$\operatorname{Re}\left(\frac{zF_\beta''(z)}{F_\beta'(z)}\right) > \frac{(\alpha-1)\beta}{2\delta(\alpha+1)} \qquad (z \in \mathbb{U}), \tag{2.4}$$

where

$$\frac{\alpha-1}{2\delta(\alpha+1)} = \min_{1\leqq j\leqq n} \frac{\alpha_j-1}{2\delta_j(\alpha_j+1)}$$

and $\beta = \sum_{j=1}^{n}\beta_j$. This implies that $F_\beta(z) \in \mathcal{T}_{\frac{\delta}{\beta}}^*(\alpha)$.

Proof. Since $f_j(z) \in \mathcal{T}_{\delta_j}^*(\alpha_j)$ $(j = 1, 2, 3, \cdots, n)$, we have that

$$\operatorname{Re}\left(\frac{zF_\beta''(z)}{F_\beta'(z)}\right) = \sum_{j=1}^{n} \operatorname{Re}\left(\beta_j \frac{zf_j''(z)}{f_j'(z)}\right)$$

$$> \sum_{j=1}^{n} \frac{(\alpha_j-1)\beta_j}{2\delta_j(\alpha_j+1)} \geqq \frac{\alpha-1}{2\delta(\alpha+1)}\left(\sum_{j=1}^{n}\beta_j\right) = \frac{(\alpha-1)\beta}{2\delta(\alpha+1)}$$

for $z \in \mathbb{U}$. This completes the proof of the theorem. □

Corollary 2.2. *If $f_j(z) \in \mathcal{T}_\delta^*(\alpha)$ for all $j = 1, 2, 3, \cdots, n$, then $F_\beta(z) \in \mathcal{T}_{\frac{\delta}{\beta}}^*(\alpha)$.*

Example 2.2. *Let consider the functions $f_j(z)$ $(j=1,2,3,\cdots,n)$ defined by*

$$\prod_{j=1}^{n}\left(f_j'(z)\right)^{\beta_j}=(1-z)^{2(1-p)}\qquad(z\in\mathbb{U})$$

with $p=\dfrac{(\alpha-1)\beta}{2\delta(\alpha+1)}+1$. *Then we have that*

$$\operatorname{Re}\left(\frac{zF_\beta''(z)}{F_\beta'(z)}\right)=\operatorname{Re}\left(\frac{2(p-1)z}{1-z}\right)>p-1=\frac{(\alpha-1)\beta}{2\delta(\alpha+1)}\qquad(z\in\mathbb{U}).$$

3. General integral operator $G_\beta(z)$

Let us define the subclasses $\mathcal{K}_\delta^*(\alpha)$ and $\mathcal{C}_\delta^*(\alpha)$ of $\mathcal{A}$.
A function $f(z)\in\mathcal{A}$ is said to be in the class $\mathcal{K}_\delta^*(\alpha)$ if it satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)<\frac{\alpha-1}{2\delta(\alpha+1)}+1\qquad(z\in\mathbb{U})\tag{3.1}$$

for some real $\alpha>1$ and $\delta>0$. Also, we say that a function $f(z)\in\mathcal{A}$ is said to be a member of the class $\mathcal{C}_\delta^*(\alpha)$ if it satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\frac{1-\alpha}{2\delta(\alpha+1)}+1\qquad(z\in\mathbb{U})\tag{3.2}$$

for some real $\alpha>1$ and $\delta>0$.
For analytic functions $f_j(z)$ $(j=1,2,3,\cdots,n)$ given by (2.1), we introduce

$$G_\beta(z)=\int_0^z\left(\prod_{j=1}^{n}\left(\frac{f_j(t)}{t}\right)^{\beta_j}\right)dt\tag{3.3}$$

with $\beta_j>0$ $(j=1,2,3,\cdots,n)$ and $\sum_{j=1}^{n}\beta_j=\beta$.
For this general integral operator $G_\beta(z)$, we derive

Theorem 3.1. *If $f_j(z)\in\mathcal{K}_{\delta_j}^*(\alpha_j)$ for each $j=1,2,3,\cdots,n$, then*

$$\operatorname{Re}\left(\frac{zG_\beta''(z)}{G_\beta'(z)}\right)<\frac{(1-\alpha)\beta}{2\delta(\alpha+1)}\qquad(z\in\mathbb{U}),\tag{3.4}$$

where

$$\frac{1-\alpha}{2\delta(\alpha+1)}=\max_{1\leqq j\leqq n}\frac{1-\alpha_j}{2\delta_j(\alpha_j+1)}$$

and $\beta=\sum_{j=1}^{n}\beta_j$. *This means that* $G_\beta(z)\in\mathcal{S}^*_{\frac{\delta}{\beta}}(\alpha)$.

Proof. Noting that $f_j(z)\in\mathcal{K}_{\delta_j}^*(\alpha_j)$ $(j=1,2,3,\cdots,n)$, we see that

$$\operatorname{Re}\left(\frac{zG_\beta''(z)}{G_\beta'(z)}\right)=\sum_{j=1}^{n}\beta_j\left(\frac{zf_j'(z)}{f_j(z)}-1\right)$$

$$<\sum_{j=1}^{n}\frac{(1-\alpha_j)\beta_j}{2\delta_j(\alpha_j+1)}\leqq\frac{1-\alpha}{2\delta(\alpha+1)}\left(\sum_{j=1}^{n}\beta_j\right)=\frac{(1-\alpha)\beta}{2\delta(\alpha+1)}$$

for $z \in \mathbb{U}$. $\square$

Corollary 3.1. *If $f_j(z) \in \mathcal{K}^*_\delta(\alpha)$ for all $j = 1, 2, 3, \cdots, n$, then $G_\beta(z) \in \mathcal{S}^*_{\frac{\delta}{\beta}}(\alpha)$.*

Example 3.1. *If we take the functions $f_j(z)$ $(j = 1, 2, 3, \cdots, n)$ defined by*

$$\prod_{j=1}^{n} \left(\frac{f_j(z)}{z}\right)^{\beta_j} = (1-z)^{p-1}$$

with $p = \dfrac{(1-\alpha)\beta}{\delta(\alpha+1)} + 1$, then we have that

$$G_\beta(z) = \frac{1}{p}\left(1 - (1-z)^p\right),$$

which implies that

$$\operatorname{Re}\left(\frac{zG''_\beta(z)}{G'_\beta(z)}\right) = \operatorname{Re}\left(\frac{(1-p)z}{1-z}\right) < \frac{(1-\alpha)\beta}{2\delta(\alpha+1)} \qquad (z \in \mathbb{U}).$$

Finally, we prove

Theorem 3.2. *If $f_j(z) \in \mathcal{C}^*_{\delta_j}(\alpha_j)$ for each $j = 1, 2, 3, \cdots, n$, then*

$$\operatorname{Re}\left(\frac{zG''_\beta(z)}{G'_\beta(z)}\right) > \frac{(\alpha-1)\beta}{2\delta(\alpha+1)} \qquad (z \in \mathbb{U}), \tag{3.5}$$

where

$$\frac{\alpha-1}{2\delta(\alpha+1)} = \min_{1 \leqq j \leqq n} \frac{\alpha_j - 1}{2\delta_j(\alpha_j+1)}$$

and $\beta = \sum_{j=1}^{n} \beta_j$. This means that $G_\beta(z) \in \mathcal{T}^*_{\frac{\delta}{\beta}}(\alpha)$.

Proof. Note that, for $f_j(z) \in \mathcal{C}^*_{\delta_j}(\alpha_j)$ $(j = 1, 2, 3, \cdots, n)$, we have that

$$\operatorname{Re}\left(\frac{zG''_\beta(z)}{G'_\beta(z)}\right) = \sum_{j=1}^{n} \beta_j \left(\frac{zf'_j(z)}{f_j(z)} - 1\right)$$

$$> \sum_{j=1}^{n} \frac{(\alpha_j-1)\beta_j}{2\delta_j(\alpha_j+1)} \leqq \frac{\alpha-1}{2\delta(\alpha+1)}\left(\sum_{j=1}^{n} \beta_j\right) = \frac{(\alpha-1)\beta}{2\delta(\alpha+1)}$$

for $z \in \mathbb{U}$. $\square$

Corollary 3.2. *If $f_j(z) \in \mathcal{C}^*_\delta(\alpha)$ for all $j = 1, 2, 3, \cdots, n$, then $G_\beta(z) \in \mathcal{T}^*_{\frac{\delta}{\beta}}(\alpha)$.*

Example 3.2. *Considering the functions $f_j(z)$ $(j = 1, 2, 3, \cdots, n)$ defined by*

$$\prod_{j=1}^{n} \left(\frac{f_j(z)}{z}\right)^{\beta_j} = (1-z)^{2(1-p)}$$

with $p = \dfrac{(\alpha-1)\beta}{\delta(\alpha+1)} + 1$, we have that

$$\operatorname{Re}\left(\frac{zG''_\beta(z)}{G'_\beta(z)}\right) = \operatorname{Re}\left(\frac{2(p-1)z}{1-z}\right) > p - 1 = \frac{(\alpha-1)\beta}{2\delta(\alpha+1)} \qquad (z \in \mathbb{U}).$$

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