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# ON A RELATION BETWEEN OVERCONVERGENCE AND SUMMABILITY OF POWER SERIES

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ABSTRACT. In this paper, some relationships between the overconvergence of power series and the existence of an elongation of the sequence of the partial sums of the series, whose some regular matrix transformations converge, are investigated.

## 1. INTRODUCTION

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence 1 and partial sums $S_n(z) = \sum_{k=0}^{n} a_k z^k$. It is well known that the sequence of the partial sums $\{S_n\}_{n=0}^{\infty}$ is uniformly convergent to a holomorphic function $f$ on each compact subset of the unit disk $\mathbb{D} := \{z : |z| < 1\}$ and divergent for all $z$, where $|z| > 1$. It is also known that it can be constructed such a power series with the property that a certain subsequence of $\{S_n\}$ converges to $f$ on the open sets from exterior of the unit disk where the function $f$ is regular [10]. This is the phenomenon of overconvergence. The following theorem gives a relation between the overconvergence of the power series and the existence of an elongation of $\{S_n\}$ the sequence of the partial sums whose arithmetic means converge.

**Theorem 1.1** (Drobot [1]). *Let $U$ be an open neighborhood of a point $z_1$ such that $|z_1| > 1$. Then, the sequence of the partial sums of the power series with radius of convergence 1, can be elongated to become $(C, 1)$ summable in $U$ if and only if it is overconvergent on $U$.*

In this paper, our aim is to investigate the equivalence between the overconvergence of the power series and the existence of an elongation of $\{S_n\}$ the sequence of the partial sums whose some regular matrix transformations converge.

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## 2. Preliminaries

**2.1. Overconvergence of Power Series.** Let

$$\sum_{n=0}^{\infty} a_n z^n \tag{2.1}$$

be a power series with radius of convergence 1, i.e.

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 1.$$

Let $S_n$ be the $n$th partial sum of the series (2.1). It is well known that the sequence of the partial sums $\{S_n\}_{n=0}^{\infty}$ is uniformly convergent to a holomorphic function $f$ on each compact subset of the unit disk $\mathbb{D} := \{z : |z| < 1\}$ and divergent for all $z$, where $|z| > 1$. It is also known that it can be constructed such a power series with the property that a certain subsequence of $\{S_n\}$ converges to $f$ on the open sets from exterior of the unit disk where the function $f$ is regular [10]. This is the phenomenon of overconvergence.

A power series in (2.1) is called *overconvergent* if there exists an open set $U \subset \{z : |z| \geq 1\}$ and a monotone increasing sequence of positive integers $\{n_k\}$ such that $\{S_{n_k}\}$ converges compactly on $U$. In the special case that also $U \cap \{z : |z| \leq 1\} \neq \emptyset$ then $\{S_{n_k}\}$ generates an analytic continuation of the sum of the series in (2.1) on the unit disk.

The concept of overconvergence was discovered by Porter [10]. After then, it was investigated by Ostrowski [6, 7, 8, 9] and used to characterize the best approximation of polynomials to analytic functions by Walsh [11]. For details, we refer to Hille's book [12, Sec. 16.7].

**2.2. Elongation of Sequences.** Let $m = \{m_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of positive integers. It is called that a sequence $\{S_n\}_{n=0}^{\infty}$ is being *elongated* with respect to the sequence $m = \{m_n\}_{n\in\mathbb{N}}$ if for each $n$ the term $S_n$ is listed $m_n-$times, i.e. if it is written by the following way:

$$(\underbrace{S_1, S_1, ..., S_1}_{m_1-\text{times}}, \underbrace{S_2, S_2, ..., S_2}_{m_2-\text{times}}, ..., \underbrace{S_n, S_n, ..., S_n}_{m_n-\text{times}}, ...). \tag{2.2}$$

The sequence (2.2) is called $m-$elongation of $\{S_n\}$. It is obvious that the sequence $\{S_n\}$ is convergent if and only if any $m-$ elongation of $\{S_n\}$ is convergent with the same limit.

**2.3. A-Summability.** Let $A = (a_{n,k})(n, k = 1, 2, 3, ...)$ be an infinite matrix of real (or complex) numbers. A sequence $(S_n)$ of real (or complex) numbers is said to be *summable* to a number $S$ by the method $A = (a_{n,k})$, shortly *$A$-summable* to $S$, if the limit relation

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} S_k = S$$

holds, and it is written as $A - lim_{n\to\infty} S_n = S$. The matrix $A = (a_{n,k})$ is called regular if it transforms convergent sequences to convergent sequences with the same limit.

It is well known that the matrix $A = (a_{n,k})$ is regular if and only if it satisfies the following conditions (see [13, p. 142], also [14]):

(i) There exists a constant $M > 0$ such that $\sum_{k=1}^{\infty} |a_{n,k}| \leq M$, for each $n = 1, 2, 3, \ldots$;

(ii) For each positive integer $k$,

$$\lim_{n\to\infty} a_{n,k} = 0; \tag{2.3}$$

(iii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = 1$.

*Cesaro Matrices*: The Cesaro means $C_\alpha$ of order $\alpha \geq 1$ which transform the given sequence $\{S_n\}$ into the sequence

$$\sigma_n^\alpha = \frac{1}{\binom{n+\alpha}{n}} \sum_{k=0}^{n} \binom{n-k+\alpha-1}{n-k} S_k,$$

have the matrix representation $\left(a_{n,k}^\alpha\right)$ defined by

$$a_{n,k}^\alpha = \begin{cases} \frac{\binom{n-k+\alpha-1}{n-k}}{\binom{n+\alpha}{n}} & \text{if} \quad k \leq n \\ 0 & \text{if} \quad k > n. \end{cases}$$

For $\alpha = 1$ we obtain the arithmetic means of $\{S_n\}$ as

$$\sigma_n = \sigma_n^1 = \frac{1}{n+1} \sum_{k=0}^{n} S_k.$$

The matrices $\left(a_{n,k}^\alpha\right)$ are called the Cesaro matrices of order $\alpha$; and they are regular for all positive integers $\alpha$ [14, p.33].

*Riesz Matrices*: Suppose that $\{p_n\}$ is a sequence of non-negative numbers which are not all 0 and put

$$P_n = p_0 + p_1 + \ldots + p_n; \qquad p_0 > 0.$$

The transformation of $\{S_n\}$ given by

$$R_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k S_k$$

is called as the Riesz mean $(R, p_n)$. The matrix representation of the $(R, p_n)$ mean is defined as follows

$$r_{n,k} = \begin{cases} \frac{p_k}{P_n} & \text{if} \quad k \leq n \\ 0 & \text{if} \quad k > n. \end{cases}$$

If we take $p_n = 1$ for all $n \in \mathbb{N}$, then $(R, p_n)$ coincide with the Cesaro means $C_1$. Besides, $(R, p_n)$ is regular if and only if $lim_{n\to\infty} P_n = \infty$ [14].

## 3. Main Results

Let us consider a power series

$$f(z)=\sum_{n=0}^{\infty}a_n z^n \qquad \text{with} \qquad \limsup_{n\to\infty}|a_n|^{\frac{1}{n}}=1 \tag{3.1}$$

and denote its partial sums as follows

$$S_n(z)=\sum_{k=0}^{n}a_k z^k \tag{3.2}$$

Let $A=(a_{n,k})$ be any regular matrix transformation. The matrix $A$ transforms the sequence $\{S_n(z)\}$ defined by (3.2) into the sequence $\{A_n(z)\}$, where

$$A_n(z)=\sum_{k=1}^{\infty}a_{n,k}S_k(z).$$

By the regularity of the matrix $A$, the sequence $\{A_n\}$ converges compactly to $f$ on $\mathbb{D}$.In addition, it may happen that some subsequences of $\{A_n\}$ converge in $|z|>1$ (for an example see [3]).

One of the main results which will be proved in the next section is the following:

**Theorem 3.1.** *Let $A=(a_{n,k})$ be any regular matrix transformation and $U$ be an open neighborhood of a point $z_1$ such that $|z_1|>1$. If the power series (3.1) is overconvergent to a limit function $F$, then there exists an elongation of the sequence (3.2) which is compactly $A-$summable in $U$ to the function $F$.*

The other result gives a partially converse of the above-mentioned theorem.

**Theorem 3.2.** *Let $(R,p_n)$ be any regular Riesz method. Suppose that (3.1) has an analytic continuation and that there exists an elongation of the sequence (3.2) such that its sequence of $(R,p_n)$ means converges compactly in an open set $U$ outside the unit disk. Then, the power series (3.1) is overconvergent.*

Drobot [1] and Gharibyan and Luh [3] proved, separately, these theorems for Cesaro means of order $\alpha=1$ so that for all the Cesaro methods $C_\alpha$, $\alpha\geq 1$, instead of the regular matrix transformation $A=(a_{n,k})$. Luh and Stepanyan [2] interested in the same problem for power series with radius of convergence zero using Cesaro methods. Especially, for the same problem, Luh and Nieß in [4] considered Faber series instead of power series. They investigated the equivalence between the overconvergence of a Faber series and the existence of an elongation of the partial sums of the Faber series whose Cesaro summable.

## 4. Proofs of Main Results

4.1. **Proof of Theorem 3.1.** Before giving the proof of Theorem 3.1, we first prove a general theorem on the $A-$summability for arbitrary sequences $\{f_n\}$ of functions which are defined and compactly bounded on an open set $U\subset\mathbb{C}$ (i.e. for every compact set $B\subset U$ and every $n\in\mathbb{N}$ there exists a constant $K_n$, such that $|f_n(z)|\leq K_n$ for all $z\in B$.)

**Theorem 4.1.** *Let $A = (a_{n,k})$ be any regular matrix transformation, $U \subset \mathbb{C}$ be an open set and suppose that $\{f_n\}$ is a sequence of functions which are defined and compactly bounded on $U$. If there exists a subsequence $\{f_{n_k}\}$ which is compactly convergent on $U$ to a limit function $F$, then there exists an elongation of $\{f_n\}$ which is compactly $A-$summable on $U$ to the function $F$.*

*Proof.* The open set $U$ can be exhausted by a sequence $\{B_n\}$ of compact sets with the property that $B_n \subset B_{n+1} \subset U$ for all $n \in \mathbb{N}$ and that for every compact set $B \subset U$ there exists a positive integer $n_0 = n_0(B)$ with $B \subset B_{n_0}$ (see [15, p.285]).

Assume that there exists a monotone increasing sequence $\{n_k\}$ of positive integers such that $\{f_{n_k}\}$ is compactly convergent on $U$ to the limit function $F$. Let $\{\epsilon_k\}$ be a sequence of positive integers which will be determined later. We now elongate the sequence $\{f_n\}$ to the sequence $\{\widetilde{f_n}\}$ where the terms $f_{n_k}$ for $k \geq 1$ are listed $\epsilon_k + 1$ times while the others remain unchanged, i.e.

$$\{\widetilde{f_n}\} = (f_1, f_2, \dots, f_{n_1 - 1}, \underbrace{f_{n_1}, f_{n_1}, \dots, f_{n_1}}_{\epsilon_1 + 1 - \text{times}}, f_{n_1+1}, \dots, f_{n_k - 1}, \underbrace{f_{n_k}, f_{n_k}, \dots, f_{n_k}}_{\epsilon_k + 1 - \text{times}}, \dots).$$

If we denote the $n$th term of the transformation of the elongated sequence $\{\widetilde{f_n}\}$ under the matrix $A$ by $A_n$, then

$$\begin{aligned} A_n(z) &:= \sum_{\nu=1}^{\infty} a_{n,\nu} \widetilde{f_\nu}(z) = \sum_{\nu=1}^{n_1} a_{n,\nu} f_\nu(z) + \sum_{\nu=n_1+1}^{n_1+\epsilon_1} a_{n,\nu} f_{n_1}(z) \\ &+ \sum_{\nu=n_1+1}^{n_2} a_{n,\nu+\epsilon_1} f_\nu(z) + \sum_{\nu=n_2+1}^{n_2+\epsilon_2} a_{n,\nu+\epsilon_1} f_{n_2}(z) + \dots \\ &+ \sum_{\nu=n_{k-1}+1}^{n_k} a_{n,\nu+\beta_{k-1}} f_\nu(z) + \sum_{\nu=n_k+1}^{n_k+\epsilon_k} a_{n,\nu+\beta_{k-1}} f_{n_k}(z) + \dots \\ &= \sum_{k=1}^{\infty} \sum_{\nu=n_{k-1}+1}^{n_k} a_{n,\nu+\beta_{k-1}} \left(f_\nu(z) - f_{n_k}(z)\right) + \sum_{\nu=1}^{\infty} a_{n,\nu} g_\nu(z), \end{aligned}$$

where $\beta_k = \sum_{\nu=1}^{k} \epsilon_\nu$ and $\{g_\nu\}$ is an elongation of $\{f_{n_\nu}\}$, that is

$$\{g_\nu\} = (\underbrace{f_{n_1}, f_{n_1}, \dots, f_{n_1}}_{n_1+\epsilon_1 - \text{times}}, \underbrace{f_{n_2}, f_{n_2}, \dots, f_{n_2}}_{n_2 - n_1 + \epsilon_2 - \text{times}}, \dots, \underbrace{f_{n_k}, f_{n_k}, \dots, f_{n_k}}_{n_k - n_{k-1} + \epsilon_k - \text{times}}, \dots).$$

Since the subsequence $\{f_{n_k}\}$ of $\{f_n\}$ converges compactly to the function $F$ on $U$, then the $m-$elongation of $\{f_{n_k}\}$, $\{g_k\}$ is so, where $m = \{m_k\} = \{n_k - n_{k-1} + \epsilon_k\}$. By the regularity of the matrix $A = (a_{n,k})$, the sequence $\{h_n\}$ defined by

$$h_n(z) := \sum_{\nu=1}^{\infty} a_{n,\nu} g_\nu(z)$$

converges compactly to the function $F$ on $U$. If we prove that

$$\sum_{k=1}^{\infty} \sum_{\nu=n_{k-1}+1}^{n_k} a_{n,\nu+\beta_{k-1}} \left(f_\nu(z) - f_{n_k}(z)\right)$$

tends to zero, when $n \to \infty$, the assertion follows. Let $\{\gamma_n\}$ be a sequence of nonnegative numbers which tends to zero. Let us now choose the natural numbers $\epsilon_k$, $k \in \mathbb{N}$, such that for each $k > 1$ the inequality

$$\sum_{\nu=n_{k-1}+1}^{n_k} |a_{n,\nu+\beta_{k-1}}| \leq \frac{\gamma_n}{M_k 2^k} \tag{4.1}$$

holds, where

$$M_k = \max_{n_{k-1}+1 \leq \nu \leq n_k} \sup_{z \in B_k} |f_\nu(z)|.$$

Consequently, by the inequality (4.1) it is obtained that

$$\begin{aligned} |\sum_{k=1}^{\infty} \sum_{\nu=n_{k-1}+1}^{n_k} a_{n,\nu+\beta_{k-1}} \left(f_\nu(z) - f_{n_k}(z)\right)| &\leq \sum_{k=1}^{\infty} 2M_k \sum_{\nu=n_{k-1}+1}^{n_k} |a_{n,\nu+\beta_{k-1}}| \\ &\leq \sum_{\nu=1}^{n_1} 2M_1 |a_{n,\nu}| + \sum_{k=2}^{\infty} \frac{\gamma_n}{2^{k-1}} \\ &= \sum_{\nu=1}^{n_1} 2M_1 |a_{n,\nu}| + \gamma_n. \end{aligned}$$

By the fact that $\gamma_n$ tends to zero, when $n \to \infty$ and by (2.3), we have the desired result. The proof is complete. □

The following theorem gives overconvergence phenomena on the sequence of the partial sums of a power series.

**Theorem 4.2.** *Given a simply connected domain $G_0$ with $\mathbb{D} \subset G_0$, $\bar{\mathbb{D}} \not\subset G_0$, an open set $U_0 \subset G_0^c$ with simply connected components and a function $f$ which is holomorphic on $U_0$. Then there exists a holomorphic function $f_0$ on $G_0$ with power series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{with} \qquad \limsup_{n \to \infty} |a_n|^{\frac{1}{n}} = 1, \qquad S_n(z) = \sum_{k=0}^{n} a_n z^n \tag{4.2}$$

*and a subsequence $\{S_{n_k}\}$ of $\{S_n\}$ converges compactly on the open set $U := G_0 \cup U_0$ to the function*

$$F(z) := \begin{cases} f_0(z) & \text{if} \quad z \in G_0 \\ f(z) & \text{if} \quad z \in U_0. \end{cases}$$

For the proof see [5].

The following theorem gives the proof of Theorem 3.1.

**Theorem 4.3.** *Consider the power series (4.2) from Theorem 4.2. Then there exists an elongation of the sequence of its partial sums which is compactly $A-$summable on $U$ to the function $F$.*

*Proof.* The sequence of partial sums of the considered power series obviously satisfies all assumptions of Theorem 4.1. □

*Remark.* If we consider Cesaro Matrix of order $\alpha$ instead of the matrix $A$ in Theorem 4.1 and Theorem 4.3, we obtain Theorem 1 and Theorem 2 in [3], respectively.

4.2. **Proof of Theorem 3.2.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence 1 and partial sums $S_n(z) = \sum_{k=0}^{n} a_k z^k$. Let $(R, p_n)$ be any regular Riesz matrix transformation, that is $\lim_{n\to\infty} P_n = \infty$, where $P_n = p_0 + p_1 + ... + p_n, p_0 > 0$.

Suppose that there exists an $m = \{m_n\}_{n=0}^{\infty}$ elongation of the sequence $\{S_n(z)\}$ such that its sequence of the Riesz means converges compactly in an open set $U$ outside the unit disk, that is, the sequence $\{R_n(z)\}$ defined by

$$R_n(z) = \frac{1}{P_n} \sum_{k=0}^{n} p_k \widetilde{S}_k(z),$$

where

$$\{\widetilde{S}_k(z)\} = (\underbrace{S_0, S_0, ..., S_0}_{m_0-\text{times}}, \underbrace{S_1, S_1, ..., S_1}_{m_1-\text{times}}, ..., \underbrace{S_n, S_n, ..., S_n}_{m_n-\text{times}}, ...),$$

is convergent uniformly on every compact subset of $U$.

Let us consider that a special subsequence of these $(R, p_n)$ Riesz means has the form

$$\tau_n(z) = \frac{1}{M_n} \sum_{k=0}^{n} \left( \sum_{\nu=\beta_{k-1}}^{\beta_k - 1} p_\nu \right) S_k(z) = \sum_{k=0}^{n} \left(1 - \frac{M_{k-1}}{M_n}\right) a_k z^k,$$

where

$$\beta_k = \sum_{\nu=0}^{k} m_\nu, \qquad \beta_{-1} = 0; \qquad M_n = \sum_{\nu=0}^{\beta_n - 1} p_\nu, \qquad M_{-1} = 0.$$

Thus, if $f$ has an analytic contiunation, the proof of Theorem 3.2 is the same with the proof of Theorem 7 in [3], since $\lim_{n\to\infty} M_n = \infty$. So we did not repeat it here.

*Remark.* If we take $p_n = 1$ for each $n = 0, 1, 2, ...$, we get Theorem 7 in [3] from Theorem 3.2.

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