ADV MATH SCI JOURNAL

Advances in Mathematics: Scientific Journal **3** (2014), no.1, 23–31
ISSN 1857-8365 UDC: 517.956.225:514.747

SYMMETRY ANALYSIS AND INVARIANTS SOLUTIONS OF LAPLACE EQUATION ON SURFACES OF REVOLUTION

AMINU M. NASS

ABSTRACT. One of the applications of symmetry method of differential equation is finding the invariants solution of linear and non-linear differential equations. In this paper we considered Laplace equation on surfaces of revolution and discuss the symmetry algebra based on classical Lie symmetry theory. Symmetry reductions are applied in order to obtain new harmonic functions on surfaces of revolution using the Lie point symmetries.

1. INTRODUCTION

Most of the real life problems in the field of applied mathematics are modeled with differential equations in various form. The difficulty of the differential equation depends on the nature of the problem and the accuracy of the model used. But generally, physicist conclude nature is non-linear. Nowadays must researchers involve in solving mathematical physics problems, focus on finding the solution of differential equation involved. But up to now there is no unique method of finding the analytical solution of differential equations. The method of studying differential equations using their symmetries was introduced by Sophus Lie, who also founded the theory of infinitesimal transformations and Lie groups. Lie's classical approach is based on finding a symmetry group associated with the differential equation. This is a local Lie group of point transformations on the space of independent and dependent variables of differential equation that maps solutions to solutions. The classical method of Lie allows computing the symmetry group associated to a given differential equation. This symmetry group can further be used for many important applications in the context of differential equations. For instance, for

- Determination of group invariant or similarity solutions
- Reduction of order of ordinary differential equations
- Reduction of partial differential equations (PDEs)(reduction in the number of independents variables)
- Construction of new solutions from old solutions.

2010 *Mathematics Subject Classification.* 44A10, 33E12.
Key words and phrases. Laplace equation, Surfaces of revolution, Lie symmetry, Invariant solutions.

Hence, Lie symmetry method is a powerful method for analyzing and finding the solutions of PDEs. A large amount of literature about the classical Lie symmetry theory, its applications and its extensions is available e.g. [1, 2, 3, 4, 7, 10, 13, 15, 5, 6, 9] The motivation of writing this paper come from the fact that, the symmetry properties and reductions of most of the important equations of mathematical physics, with flat and non-flat background metric, have been well investigated e.g. some studies of wave and heat equation, using symmetries, on specific cases of surfaces of revolution such as sphere, torus, cone and hyperbolic space have also been carried out recent in papers [1, 2, 3, 4, 14]. We extend the same method to study symmetry and use them to find harmonic functions on surfaces of revolution. Which are important in the field of science.

2. Preliminaries

In this section we give some basic definitions from differential geometry [11], [12], [8]. Surface of revolution obtained by rotating a plane curve about an axis a form a large class of surfaces.

Definition 2.1. *A parameterized surface $X : D \to \mathbb{R}^3$ is a smooth function of an open set $D \subset \mathbb{R}^2$ into $\mathbb{R}^3$, defined by*

$$X(u,v) = (X_1(u,v), X_2(u,v), X_3(u,v)).$$

Definition 2.2. *Let $X(x,y) : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular parametrization of surface then, the Riemannian metric or first fundamental form of the patch X is defined by*

$$g = ds^2 = Edx^2 + 2Fdxdy + Gdy^2$$

with coefficient of the first fundamental form defined by

$$E = X_x{\cdot}X_x, \qquad F = X_x{\cdot}X_y, \qquad G = X_y{\cdot}X_y. \tag{2.1}$$

Definition 2.3. *Matrix of First Fundamental form:*
Let

$$g_{11} = E = X_x{\cdot}X_x, \qquad g_{12} = F = X_x{\cdot}X_y, \qquad g_{22} = G = X_y{\cdot}X_y. \tag{2.2}$$

Then it is often convenient to put the metric as

$$g = ds^2 = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2$$

where the symmetric matrix form is defined by

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = g_{ij}.$$

The inverse of g is

$$g^{-1} = \frac{1}{det(g)} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix} = g^{ij},$$

with

$$det(g) = g_{11}g_{22} - g_{12}^2.$$

Definition 2.4. *For any metric g, the Laplacian on (S, g) is defined by*

$$\Delta u = \frac{1}{\sqrt{| g |}} \frac{\partial}{\partial x^i} (\sqrt{| g |} g^{ij} \frac{\partial}{\partial x^j}(u))$$

To formulated the problem we consider a surfaces of revolution generated by revolving a unit speed profile curve with metric given by

$$g = dx^2 + e^{2f(x)} dy^2. \tag{2.3}$$

Therefore, using the definition of Laplacian on Riemannian manifolds (2.4), Laplace equation on surfaces of revolution with metric (2.3) is given by

$$f'(x)u_x + u_{xx} + e^{-2f(x)} u_{yy} = 0\,. \tag{2.4}$$

3. Objectives of the research

We consider Laplace equation

$$f'(x)u_x + u_{xx} + e^{-2f(x)} u_{yy} = 0\,,$$

on surfaces of revolution parameterized by:

$$X(x, y) = (v(x), e^{f(x)} \cos(y), e^{f(x)} \sin(y))\,. \tag{3.1}$$

The main objectives to be achieved are:

(1) Obtain the determining equations for the symmetries of (2.4).
(2) Finding the Lie symmetries algebra of (2.4).
(3) Finally use the symmetries to obtain the symmetry reduction and find the exact solutions.

4. Symmetries algebra of Laplace equation on surfaces of revolutions

To obtain a Lie symmetries of Laplace equation

$$f'(x)u_x + u_{xx} + e^{-2f(x)} u_{yy} = 0$$

on the surface of revolution, we consider a one parameter lie group of infinitesimal transformations in (x, y, u) given by:

$$x^* = x + \epsilon\xi(x, y, u) + O(\epsilon)\,;$$
$$y^* = y + \epsilon\tau(x, y, u) + O(\epsilon)\,;$$
$$u^* = u + \epsilon\phi(x, y, u) + O(\epsilon)\,;$$

where ϵ is the parameter of the group, therefore the corresponding generator of the Lie algebra is of the form:

$$X = \xi(x, y, u)\frac{\partial}{\partial x} + \tau(x, y, u)\frac{\partial}{\partial y} + \phi(x, y, u)\frac{\partial}{\partial u}\,. \tag{4.1}$$

If $X^{[2]}$ represent the second prolongation of X, then using the invariance criterion:

$$X^{[2]}(f'(x)u_x + u_{xx} + e^{-2f(x)} u_{yy})|_{u_{xx}=-e^{-2f(x)}u_{yy}-f'(x)u_x} = 0\,, \tag{4.2}$$

and comparing the coefficient of u and its derivative gives the following system of eight determining equations:

$e_1: \phi_x f_x + \phi_{xx} + \phi_{yy} e^{-2f(x)} = 0;$
$e_2: \xi f_{xx} + \xi_x f_x + 2\phi_{ux} - \xi_{xx} - \xi_{yy} e^{-2f(x)} = 0;$
$e_3: 2e^{-2(x)}\phi_{uy} - \tau_x f_x - \tau_{xx} - \tau_{yy} e^{-2(x)} = 0;$
$e_4: \tau_x + e^{-2(x)}\xi_y = 0;$
$e_5: \phi_{uu} = 0;$
$e_6: \xi_u = 0;$
$e_7: \tau_u = 0;$
$e_8: -\xi f_x + \xi_x - \tau_y = 0.$

Using $(e_4)_x - e^{-2f(x)}(e_8)_y$ gives

$e_9: -e^{-2(x)}\xi_y f_x + \tau_{xx} + e^{-2(x)}\tau_{yy} = 0.$

Putting e_9 in e_3 gives

$e_{10}: \phi_{uy} = 0.$

From $(e_4)_y + (e_8)_x$ gives:

$e_{11}: -\xi f_{xx} + \xi_{xx} + \xi_{yy} e^{-2(x)} - \xi_x f_x = 0.$

Putting (e_{11}) in e_2 gives:

$e_{12}: \phi_{ux} = 0.$

Therefore, from e_5, e_{10} and e_{12} we can conclude that:

$$\phi = au + g(x, y). \tag{4.3}$$

So the symmetry algebra of Laplace equation

$$f'(x)u_x + u_{xx} + e^{-2f(x)}u_{yy} = 0,$$

on surfaces of revolution is infinite dimensional generated by

$$X = \xi(x, y)\frac{\partial}{\partial x} + \tau(x, y)\frac{\partial}{\partial y} + \phi(x, y, u)\frac{\partial}{\partial u},$$

where $\tau = \tau(x, y)$ is a harmonic function on M^2 satisfying:

$$e^{-2f(x)}\tau_{yy} + \tau_{xx} + f_x\tau_x = 0, \tag{4.4}$$

and the function $\xi = \xi(x, y)$ is given by the following relations:

$$\xi_x - \xi f_x - \tau_y = 0; \tag{4.5}$$

$$\xi_y e^{-2f(x)} + \tau_x = 0. \tag{4.6}$$

Finally, we can say that any solution of (4.4) can gives a symmetry algebra of Laplace equation on surfaces of revolution and the result can be summarized as in the following theorem.

Theorem 4.1. *Let M^2 be a surface of revolution with parametrization*

$$X(x, y) = (v(x), e^{f(x)}\cos(y), e^{f(x)}\sin(y)),$$

then the symmetry algebra of Laplace equation

$$f'(x)u_x + u_{xx} + e^{-2f(x)}u_{yy} = 0$$

on M^2 is infinite dimensional algebras generated by

$$X = \xi(x,y)\frac{\partial}{\partial x} + \tau(x,y)\frac{\partial}{\partial y} + \phi(x,y,u)\frac{\partial}{\partial u},$$

where $\tau(x,y)$ is a harmonic function on M^2 satisfying the relation

$$e^{-2f}\tau_{yy} + \tau_{xx} + f_x\tau_x = 0.$$

The function $\xi(x,y)$ is given by the relations

$$-\xi f_x + \xi_x - \tau_y = 0;$$

$$e^{-2f}\xi_y + \tau_x = 0,$$

and the function $\phi(x,y,u)$ is given by $\phi = au + g(x,y)$.

5. Symmetries Algebra of Laplace equation on surfaces of revolution for simpler forms of $\tau(x,y)$

In this section we provide example of symmetries algebras of Laplace equation on surfaces of revolution using simple solutions of (4.4).

5.1. For $\tau = constant = k$.

Here we have two cases.

5.1.1. *For $\tau = constant = k$ and $\xi = 0$.* The symmetries algebras are:

$$X_1 = \frac{\partial}{\partial y}, \qquad X_2 = u\frac{\partial}{\partial u}, \qquad X_g = g(x,y)\frac{\partial}{\partial u}.$$

5.1.2. *For $\tau = constant = k$ and $\xi \neq 0$.* Using (4.6) $\xi = \xi(x)$ and from (4.5) we have:

$$-\xi f_x + \xi_x = 0. \tag{5.1}$$

Solving (5.1) gives $\xi(x) = k_2 e^{f(x)}$. Therefore the symmetries algebras are

$$X_1 = e^{f(x)}\frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = u\frac{\partial}{\partial u}, \quad X_g = g(x,y)\frac{\partial}{\partial u}.$$

5.2. For $\tau = \tau(y)$ and $\xi \neq 0$.

From (4.4) we have:

$$\tau = k_1 y + k_2.$$

Using (4.6) and (4.5) we have:

$$\xi = k_1 e^{f(x)}\int e^{-f(x)}dx + k_3 e^{f(x)}. \tag{5.2}$$

Therefore the symmetries algebras are:

$$X_1 = (y\frac{\partial}{\partial y} + e^{f(x)}\int e^{-f(x)}dx\frac{\partial}{\partial x}), \qquad X_2 = \frac{\partial}{\partial y},$$

$$X_3 = e^{f(x)}\frac{\partial}{\partial x} \qquad X_4 = u\frac{\partial}{\partial u}, \qquad X_g = g(x,y)\frac{\partial}{\partial u}.$$

5.3. **For $\tau = \tau(x)$ and $\xi \neq 0$.** From (4.4) we have:

$$\tau(x) = k_1 \int e^{-f(x)} dx + k_2\,. \tag{5.3}$$

Substituting (5.3) in (4.6) gives:

$$\xi_y(x,y) + k_1 e^{f(x)} = 0\,. \tag{5.4}$$

Solving (5.4) and (4.5) simultaneously and using (5.3) gives:

$$\xi = -k_1 y e^{f(x)} + k_3 e^{f(x)}\,. \tag{5.5}$$

Therefore the symmetries are:

$$X_1 = (\int e^{-f(x)} \frac{\partial}{\partial y} - y e^{f(x)} \frac{\partial}{\partial x}), \qquad X_2 = \frac{\partial}{\partial y}$$

$$X_3 = e^{f(x)} \frac{\partial}{\partial x} \qquad X_u = u \frac{\partial}{\partial u}, \qquad X_g = g(x,y) \frac{\partial}{\partial u}\,.$$

6. Symmetry reductions and invariant solutions

In this section we give symmetry reduction of Laplace equation on surfaces of revolution and solve the reduced PDE to find harmonic functions on surfaces of revolution using the standard method of similarity of variables.

6.1. Reductions and exact solutions for $\tau = constant = k$, $\xi = 0$.

6.1.1. *Subalgebra* $\langle \frac{1}{a} X_1 + \frac{1}{b} X_2 \rangle$. Similarity variables are:

$$z(x,y) = x, \qquad u = V(z) e^{\frac{ay}{b}}\,. \tag{6.1}$$

Substituting (6.1) in (2.4) we have the reduction as:

$$V''(z) + f'(z) V'(z) + \frac{a^2}{b^2} e^{-2f(z)} V(z) = 0\,. \tag{6.2}$$

This ODE has a symmetry $\overline{X} = V \frac{\partial}{\partial V}$ which reduces (6.2) to Riccati equation:

$$\frac{dw(t)}{dt} = -w^2(t) - w(t) \frac{df(t)}{dt} - \frac{a^2}{b^2} e^{-2f(t)}\,. \tag{6.3}$$

Solving (6.3) we have:

$$w(t) = -\frac{\tan\left(\frac{a \int e^{-f(z)} dz + c_1 b}{b}\right) a}{b e^{f(t)}}\,,$$

and changing variables we have:

$$V(z) = \cos\left(\frac{a \int e^{-f(z)} dz + c_1 b}{b}\right) c_2\,.$$

Therefore the solution is:

$$u(x) = e^{\frac{ay}{b}} \cos\left(\frac{a \int e^{-f(x)} dx + c_1 b}{b}\right) c_2\,.$$

6.1.2. *Subalgebra* $\langle \frac{1}{a}X_1 + \frac{1}{b}X_{g=(cy+d)} \rangle$. Similarity variables are:

$$z(x,y) = x, \qquad u = V(z) + \frac{a}{b}(\frac{cy^2}{2} + yd)\,. \tag{6.4}$$

Substituting (6.4) in (2.4) we have the reduction as:

$$V''(z) + V'(z)f'(z) + \frac{ac}{b}e^{-2f(z)} = 0\,. \tag{6.5}$$

Solving this second order ODE by letting $q(x) = V^{'}(x)$ and using integrating factor we have:

$$V\left(z\right) = \int \left(-\frac{\left(ad\left(\int \mathrm{e}^{-f(z)}\ \mathrm{d}z\right) - c_1 b\right)\mathrm{e}^{-f(z)}}{b} \right)\ \mathrm{d}z + c_2\,.$$

Changing the variables gives us harmonic function:

$$u\left(x,y\right) = \int \left(-\frac{\left(ad\left(\int \mathrm{e}^{-f(x)}\ \mathrm{d}x\right) - c_1 b\right)\mathrm{e}^{-f(x)}}{b} \right)\ \mathrm{d}x + c_2 + \frac{a\left(\frac{1}{2}cy^2 + dy\right)}{b}\,.$$

6.2. **Reductions and exact solutions for $\tau = Constant = k$ and $\xi \neq 0$.**

6.2.1. *Subalgebra* $\langle \frac{1}{a}X_1 + \frac{1}{b}X_3 \rangle$. Similarity variables are:

$$z(x,y) = y, \qquad u\left(x,y\right) = V\left(z\right)\mathrm{e}^{\frac{a\left(\int \mathrm{e}^{-f(x)}\ \mathrm{d}x\right)}{b}}\,. \tag{6.6}$$

Substituting (6.6) in (2.4) we have the reduction as:

$$b^2V^{''}(z) + a^2V^{'}(z) = 0\,, \tag{6.7}$$

which gives:

$$V\left(z\right) = C_1 \sin\left(\frac{az}{b}\right) + C_2 \cos\left(\frac{az}{b}\right)\,.$$

After changing the variables we have the solution of (2.4) as:

$$u\left(x,y\right) = \left(C_1 \sin\left(\frac{ay}{b}\right) + C_2 \cos\left(\frac{ay}{b}\right)\right)\mathrm{e}^{\frac{a\left(\int \mathrm{e}^{-f(x)}\ \mathrm{d}x\right)}{b}}\,.$$

6.2.2. *Subalgebra* $\langle \frac{1}{a}X_1 + \frac{1}{b}X_2 \rangle$. Similarity variables are

$$z(x,y) = y - \frac{a}{b}\int e^{-f(x)}dx, \qquad u = V(z)\,. \tag{6.8}$$

Substituting (6.8) in (2.4) we have the reduction as:

$$V^{''}(z) = 0\,, \tag{6.9}$$

which gives:

$$V(z) = Az + B\,.$$

By charging variables we have:

$$u = cy - \frac{ac}{b}\int e^{-f(x)}dx + d\,.$$

6.3. **Reductions and exact solutions for $\tau = \tau(y)$ and $\xi \neq 0$.**

6.3.1. *Subalgebra* $\langle \frac{1}{a}X_2 + \frac{1}{b}X_3 + X_{g=(cy+d)} \rangle$. Similarity variables are:

$$z\,(x,y) = -\frac{a}{b}\int \mathrm{e}^{-f(x)}\ \mathrm{d}x + y, \qquad u(x,y) = V(z) + (cay + ad)\int e^{-f(x)}dx\,.$$

Substituting in (2.4) we have the reduction as:

$$V^{''}(z) = 0\,, \tag{6.10}$$

which gives:

$$V(z) = c_1 z + c_2\,. \tag{6.11}$$

By changing variables we have:

$$u = c_1\left(\frac{-a}{b}\int e^{-f(x)} + y\right) + (acy + ad)\int e^{-f(x)}dx + c_2\,.$$

6.3.2. *Subalgebra* $\langle \frac{1}{a}X_2 + \frac{1}{b}X_3 + \frac{1}{c}X_4 \rangle$. Similarity variables are:

$$z\,(x,y) = \frac{a}{b}\int \mathrm{e}^{-f(x)}\ \mathrm{d}x - y, \qquad u(x,y) = V(z)e^{\frac{by}{c}}\,.$$

Substituting in (2.4) we have the reduction as:

$$(a^2c^2 + b^2c^2)V^{''}(z) - 2b^3V^{'}(z) + b^4V(z) = 0\,. \tag{6.12}$$

Solving the ODE (6.12) and finding the real solution we have:

$$u = (c_1 + c_2)e^{\frac{ab(ay + b\int e^{-f(x)}dx)}{c(a^2+b^2)}}\cos\left(\frac{ab(ay + b\int e^{-f(x)}dx)}{c(a^2+b^2)}\right)\,.$$

6.3.3. *Subalgebra* $\langle \frac{1}{a}X_3 + X_{g=(cy+d)} \rangle$. Similarity variables are:

$$z\,(x,y) = y, \qquad u\,(x,y) = V\,(\mathrm{z}) + (aby + ac)\int \mathrm{e}^{-f(x)}\ \mathrm{d}x\,.$$

Substituting in (2.4) we have the reduction as:

$$V^{''}(z) = 0\,,$$

which gives:

$$V(z) = c_1 z + c_2\,.$$

Therefore the harmonic function is given by:

$$u\,(x,y) = c_1 y + c_2 + (aby + ac)\int \mathrm{e}^{-f(x)}\ \mathrm{d}x\,.$$

7. Conclusion

In this paper we have successfully analyzed the symmetry algebra of Laplace equation on surfaces of revolution. Symmetries algebras found are utilized to find harmonic functions on surfaces of revolution which are of great importance in the field of mathematics such as in electromagnetism, fluid dynamics and astronomy.

Acknowledgment

The author acknowledge the support of King Fahd University of Petroleum and Minerals through there facilities.

References

[1] H.Azad, M. T. Mustafa: *Symmetry analysis of wave equation on sphere*, J. Math. Anal. Appl. **330**, 1180. 5. A. Y., 2012.
[2] Al-dweik, M. T. Mustafa, K. Mpungu: *Symmetry classification of heat equation on surfaces of revolution*, Mogran, 2012.
[3] M. Nadjafikhah, A. H. Zaeim: *Symmetry Analysis of Wave Equation on Hyperbolic Space*, International Journal of Nonlinear Science **11** (1),(2010), 35–43.
[4] A.Jhangeer, I. Naeem: *Similarity variables and reduction of the heat equation on torus*, Commun. Nonlinear Sci. Numer. Simul. **17**(3) (2012) 1251–1257.
[5] N. H. Ibragimov: *Elementary Lie group analysis and ordinary differential equations*, John Wiley and Sons, Chichester, 1999.
[6] P. E. Clarkson, E. L. Mansfield: *Symmetry reductions and exact solutions of a class of nonlinear heat equations*, Phys. D. **70**, 250, 1994.
[7] C. Ou, Q. Huang: *Symmetry reductions and exact solutions of the affine heat equation*, J. Math. Anal. Appl. **346** 521, 2008.
[8] L. P. Eisenhart: *Riemannian geometry*,Princeton University Press, Princeton, 1925.
[9] P. E. Hydon: *Symmetry methods for differential equations*, Cambridge University Press, Cambridge, 2000.
[10] H. Azad, M.T. Mustafa, M. Ziad: *Group Classification, optimal system and optimal reductions of a class of Klein Gordon equations*, Communications in Nonlinear Science and Numerical Simulation, **15** (5), (2010), 1132.
[11] A. Pressley: *Elementry Differential Geometry*, Springer, London, 2003.
[12] A. Gray, E. Abbena, S.Salamon: *Modern differential geometry of curves and surfaces with Mathematica*, 3rd ed., 2006.
[13] H. Bokhari, A. H. Kara, F. D. Zaman: *On the symmetry structure of the Minkowski Metric and a Weyl Re-Scaled Metric*, International Journal of Theoretical Physics, **19** (12), (2007) 1356-1360.
[14] Ahmad, A. H. Bokhari, A. H. Kara, F. D. Zaman: *Symmetry classifications and reductions of some classes of (2+1)-nonlinear heat equation*, Journal of Mathematical Analysis and Applications, DOI:10.1016/j.jmaavol., **339**, (2008), 175–181.
[15] H. Bokhari, T. M. Mustafa, F. Zaman: *An exact solution of a quasilinear Fisher equation in cylindrical coordinates*, Nonlinear Analysis: Theory, Methods and Applications, DOI: 10.1016/j.na, 2007.

Department of General Studies Affiliated College at Hafr Al-Batin
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia
E-mail address: aminunass@kfupm.edu.sa