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TREE-COVER RATIO OF GRAPHS WITH ASYMPTOTIC CONVERGENCE IDENTICAL TO THAT OF THE SECRETARY PROBLEM

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ABSTRACT. In this paper, we introduce a ratio involving spanning trees and vertex coverings, through a graph-theoretical gambling problem, involving the asymptotic convergence of $1/e$, identical to that of the secretary problem. This constant is the probability of selecting the best applicant in the secretary problem, as well as the radius of convergence of trees and convergent solutions to differential equations. We adopt the spanning tree and vertex cover aspects of this ratio to define the idea of a tree-covering-ratio of graphs. We discuss the asymptotic convergence of this ratio for classes of graphs, which may have application in ideal communication situations involving spanning trees and vertex coverings of extreme networks, and introduce the idea of a tree-cover area by integrating this tree-cover ratio.

1. INTRODUCTION

We shall use the graph theoretical notation of Harris and Hirst [7]; where our graphs are simple and connected with order n and size m.

Spanning trees. The graph-theoretical concept of spanning trees can be found in many real world applications, especially in social networking scenarios. For example, research by Bearman, Moody and Stovel [4] involves work on sexual networks in an American high school which suggest that sexual networks are characterized by long chains or "spanning trees"; meaning that a large part of the school had sexual contact with each another.

Vertex cover. The importance of minimum vertex coverings of graphs occurs often in real life applications involving extreme networks with a large number of nodes (see the parameterized vertex cover problem in [3] and [6]).

Ratios. Ratios such as expanders, Raleigh quotient(see [8]), the central ratio of a graph (see [1]) and eigen-pair ratio of classes of graphs (see [10]),independence and Hall ratios (see [5]), have attracted interest.

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Spanning trees and vertex cover. In this paper, we combine the two ideas of spanning trees and (minimum) vertex cover to introduce the idea of a tree-cover ratio of a graph which arises from a graph-theoretical problem, derived from a hypothetical gambling scenario, with its ratio converging to $1/e$. This constant occurs in other convergent problems such as: the secretary problem (where the best applicant is selected with the probability $1/e$(see [2]); the radius of convergent of trees(see [9]), and the convergence of the solution of differential equation discussed below. The importance of large number of vertices, which occurs in (extreme) networks, allowed for the investigation of asymptotic convergence of this tree-cover ratio for different classes of graphs. The idea of area is also introduced which involves the Riemann integral of this cover tree-ratio.

1.1. **A graph theoretical variation of the secretary problem- a gambling problem with social decision making.** The following problems involve the convergence to the constant $1/e$.

The secretary problem. The policy for the secretary problem is a stopping rule. Under it, the interviewer rejects the first $(r-1)$ applicants (let applicant M be the best applicant among these $r-1$ applicants), and then selects the first subsequent applicant that is better than applicant M. It can be shown that the optimal strategy lies in this class of strategies. For an arbitrary cutoff r, the probability that the best applicant is selected is:

$$P(r) = \frac{r-1}{n} \sum_{i=r}^{n} \frac{1}{i-1}. \tag{1.1}$$

This sum is obtained by noting that if applicant i is the best applicant, then it is selected if and only if the best applicant among the first $i-1$ applicants (the second best overall) is among the first $r-1$ applicants that were rejected.

Hence, letting $n \to \infty$, such that x can be written as

$$x = \lim_{n \longrightarrow \infty} \left(\frac{r}{n}\right).$$

By using $t = \frac{i}{n}$, such that $dt = \frac{1}{t} di$, the sum in (1.1) above can be approximated by the integral

$$P(x) = x \int_{x}^{1} \frac{1}{t} dt\,.$$

This gives rise to the differential equation:

$$P'(x) = -1 - \ln x\,.$$

By setting this derivative to zero and solving for x, the value tend to $1/e$. Thus, the optimal cutoff tend to $1/e$ as n increases, and the best applicant is selected with probability $1/e$.

Radius of convergence. The constant 1/e is also associated with trees- it is the radius of convergence of trees (see [3]).

Convergence of solution of differential equations. The differential equation $\frac{dP(x)}{dx} = -1 - \ln x$, was used in the secretary problem to determine the convergent solution of 1/e. This constant occurs as convergent solution to the following differential equations.

(1) First Order Variable Separable with solution converging to $1/e$.

Theorem 1.1. *The cutoff number $1/e$ of the secretary problem is also the convergence of the solution of the following separable variable differential equations:*
(i.) $\frac{dy}{dx} = e^{-x}$; $y(1) = 0$
(ii.) $\frac{dy}{dx} = -nx^{n-1}y\,;\, n \in \mathbb{R}\,;\, n \neq 0\,, y(1) = 0$

Proof. (i.) $\frac{dy}{dx} = e^{-x};\ y(1) = 0.$

$$\frac{dy}{dx} = e^{-x} \Longrightarrow y = \int e^{-x}dx = -e^{-x} + c = y(x);$$
$$\Longrightarrow y(x) = -e^{-x} + e^{-1} \quad \text{which converges to} \quad e^{-1}.$$

(ii.) $\frac{dy}{dx} = -nx^{n-1}y\,;\, n \in \mathbb{R}\,;\, n \neq 0, y(1) = 0$

$$\frac{dy}{dx} = -nx^{n-1}y \Longrightarrow \ln y = -x^n + c \Longrightarrow y = -e^{-x^n} + c;$$
$$y(1) = -e^{-1} + c = 0 \Longrightarrow c = e^{-1}\,;$$
$$\Longrightarrow y = -e^{-x^n} + e^{-1} \quad \text{converges to} \quad e^{-1}\,.$$

□

(2) Integrating factor

Theorem 1.2. *The constant e^{-1} is the convergent solution of the following differential equation:*

$$\frac{dy}{dx} + \frac{y}{x^2(e-1/x)} = -\frac{1}{x^n}; n > 1, n \in \mathbb{R}; y(1) = (e-1)^{-1}\Big(1 - \frac{1}{n} + \frac{e}{n-1}\Big)$$

Proof. The integrating factor of this differential equation is

$$e^{\int \frac{dx}{x^2(e-\frac{1}{x})}} = e^{\ln(e-\frac{1}{n})} = e - \frac{1}{n}\,,$$

Thus:

$$\frac{d}{dx}\Big[y(e - \frac{1}{x})\Big] = -\frac{1}{x^n}(e - \frac{1}{n})$$
$$\Longrightarrow y(x) = \frac{e}{(n-1)x^{n-1}}(e - \frac{1}{x})^{-1} - \frac{1}{nx^n}(e - \frac{1}{x})^{-1} + c(e - \frac{1}{x})^{-1}$$
$$\Longrightarrow y(1) = (e-1)^{-1}\Big(1 - \frac{1}{n} + \frac{e}{n-1}\Big)$$
$$\Longrightarrow c = 1 \quad \text{and} \quad y(x) \quad \text{converges to} \quad e^{-1}.$$

□

We now provide a graph theoretical variation of the secretary problem with convergent ratio identical to the cut-off number $1/e$, and use it to motivate for the definition of a tree-cover ratio of classes of graphs.

1.2. **Gambling problem with social decision making and guaranteed win.** We have n gamblers each coming to the casino with 1 million dollars each. We assume these individuals do not know each other, and they agree to the conditions of the game determined by the casino. The casino guarantees that a pair will leave with 2 million dollars each, and selects 1 participant randomly, say n_i.

This n_i is given 2 million (so he/she has a total of 3 million dollars) by the casino and n_i must decide who to share the 3 million dollars with by social interaction with the other $n-1$ participants. This is done with exactly one spanning tree which he/she arbitrarily selects. Only n_i and the casino knows who has been selected. This participant must decide who he/she "likes" the most through the spanning tree. The casino then selects, randomly, an individual (other than n_i) from the remaining $n-1$ participants. This individual, say n_k, must decide, through communication involving all possible spanning trees determined by the $n-1$ participants, if he/she has been chosen by n_i.

A correct guess, i.e a perfect match (in terms of the individual chosen by n_i) means both n_i and n_k walk away with 2 million dollars each and the game is over. If n_k is correct (in terms of not being chosen by n_i), he keeps his million, remains in the game but cannot play to win and is an inactive participant, and then the casino selects the next participant. Otherwise, if n_k is wrong (he believes he has been chosen by n_k, but was not), he loses a million and the casino then selects a next participant (with n_k remaining as part of the communication spanning trees but cannot be chosen again-an inactive participant).

The last case is when n_k is wrong (he believes he has not been chosen when he has been chosen by n_i). In this case, since there must be a perfect match, the casino makes the changes as per $7(iv)$ below.

Conditions:

(1) All n individuals are communicatively linked by edges of a complete graph, and have not known each other before the game.

(2) Every individual can communicate with the others with a two way directed edge-i.e we have a complete digraph, G, representing their connections.

(3) Individual n_i is selected at random by the casino. This individual then interacts with all the $n-1$ others by selecting any one of the possible spanning trees: either directly to each of the individuals (a star graph= a spanning tree), connecting from n_i to the remaining $n-1$ vertices. Or, for example, via all possible paths to n_j, i.e through discussing with the individuals along each path to n_j.

(4) Once n_i has found, through a spanning tree, the individual n^* he wants to share the money with, it is kept to this individual. The probability of finding this individual will be:

$$P = \frac{1}{t(K_n)} = \frac{1}{n^{n-2}} \tag{1.2}$$

Where $t(K_n) = n^{n-2}$ is the possible number of spanning trees of the complete graph.

(5) We now remove n_i and work with the complete subgraph, H of G, induced by the remaining $n-1$ active individuals (vertices, which is a covering set of G), and select a n_k^1 (as a leader) randomly, such that each of the $n-1$ individual in this subgraph have interacted with n_i through some spanning tree.

(6) This individual leader n_k^1 interacts with the remaining $n-2$ vertices using all possible spanning trees on the set of $n-1$ vertices and decides if he has or has not been chosen by n_i (conforming or contradicting n_i's choice).

(7) This individual n_k^1 must go through all possible spanning trees $t(K_{n-1})$ before making a decision.

(i.) If n_k^1 decides through social interaction with others that he is chosen/not chosen by n_i and is correct (a perfect match), the pair n_i and n_k^1 walk away with 2 million dollars each and the game ends.

(ii.) If n_k^1 is correct (a match) in the "n_i has not chosen me" case, then the contestant keeps his million and the casino selects the next participant other than n_k^1 (n_k^1 cannot be chosen again but remains in the game as a communicator or inactive participant or vertex).

(iii.) If n_k^1 is wrong (a non- match) by saying that he has been chosen, when in fact he has not been chosen. He loses the million to the casino and the casino proceed randomly to the next active vertex n_k^2 in the subgraph H, excluding n_k^1.

(iv.) If n_k^1 is wrong by saying that he has not been chosen by n_i, when in fact he has been chosen by n_i, i.e. $n_k^1 = n^*$. If this is the last contestant, then the casino declares a perfect match and the game ends. Otherwise, he keeps his million and the casino swaps him with an arbitrary active participant n_k^j (n_k^j becomes inactive but keeps his million), and the casino proceed randomly to the next active vertex n_k^2 in the subgraph H (the participant now know that he is n^* so will eventually be a perfect matched with n_i). The new leader n_k^2 selected randomly by the casino, interacts with the other active individuals in the same way n_k^1 did and decides if he is chosen or not by n_i.

(v) The game stops when a perfect match is found. If no perfect match has been found after $n-2$ contestants in the set of $n-1$ contestants, then the last contestant allows for a perfect match by default.

For each of the $n-1$ vertices in H, we have $(n-1)^{n-3}$ spanning trees, so that the probability of arriving at a perfect match of n_i with n^* will be according to the theorem below:

Theorem 1.3. *The probability of arriving at a perfect match of n_i with n^* through spanning trees in the gambling problem above is:*

$$\left(\frac{n-1}{n}\right)^{n-2} = \frac{|S|t(H(S))}{t(K_n)} \tag{1.3}$$

Proof. This probability of a perfect match through the spanning trees is given as: (probability of selecting n^*) $\times$ (probability of n_k^1 in H having a perfect match with n_i through $t(K_{n-1})$ spanning trees OR n_k^2 having a perfect match with n_i through $t(K_{n-1})$ spanning trees OR ... OR n_k^{n-1} having a perfect match with n_i through spanning trees).

Mathematically, this can be expressed as:

$$\begin{aligned}\frac{1}{t(K_n)} \times (n-1)t(K_{n-1}) &= \frac{(n-1)(n-1)^{n-3}}{n^{n-2}} \\ &= \left(\frac{n-1}{n}\right)^{n-2} \\ &= \frac{|S|t(H(S))}{t(K_n)},\end{aligned}$$

where S is the number of vertices in the minimum vertex covering set of K_n and $H(S)$ is the subgraph induced by these vertices. □

Corollary 1.1. *The probability ratio* $\left(\frac{n-1}{n}\right)^{n-2} = \frac{|S|t(H(S))}{t(K_n)}$ *of the gambling problem converges to* e^{-1}*. (Same is the probability of selecting the best applicant in the secretary problem; the radius of convergence of the trees and the convergence of the solution of the differential equations in Theorem 1.1 and Theorem 1.2.*

Proof. Let

$$q = \left(\frac{n-1}{n}\right)^{n-2}$$
$$\implies lnq = \frac{ln(1-\frac{1}{n})}{\frac{1}{n-2}}$$

As $n \to \infty$, $lnq \to -1$. Hence, $q \to e^{-1}$ as $n \to \infty$. □

The ratio $\frac{|S|t(H(S))}{t(K_n)}$ involving spanning trees and vertex cover, S, with its convergence property, therefore provides the motivation for the definition of the tree-cover ratio and asymptotes of classes of graphs presented below.

Definition 1.1. *Let $t(G)$ be the number of spanning trees of a connected graph of order n. Let S be a set of vertices of a minimum vertex cover of G, and H the subgraph of G induced by S. We consider only the two cases (i) either $H(S)$ is connected or (ii) H(S) consist of n isolated vertices. In case (ii), $t(H(S))$ is defined as $t(H(S)) = 1$.*

Then the ratio:

$$tc(G)_s = \frac{|S|t(H(S))}{t(G)}$$

is the **tree-cover ratio** *of G with respect to S.*

Note: If $H(S)$ is disconnected, and does not just consists of $|S|$ isolated vertices, then a spanning forest may be considered involving the components of $H(S)$, but such cases are not considered in this paper.

Definition 1.2. *The importance of graphs with a large number of vertices is well known, If ξ is a class of graph and*

$$tc(G)_s = \frac{|S|t(H(S))}{t(G)} = f(n)$$

for each $G \in \xi$, where n is the size of G, then the horizontal asymptote of $f(n)$ is defined by:

$$Asyp(\xi)_s = \lim_{n \to \infty} f(n)$$

This asymptote is called the **tree-cover asymptote** of ξ, which is an indication of the behaviour of the tree cover ratio when the graph has a large number of vertices, such as in extreme networks.

1.3. **An Ideal communication problem and tree-cover asymptote.** In [9], the communication problem is to select a minimal set of placed sensor devices in a service area so that the entire service area is accessible by the minimal set of sensors. Finding the minimal set of sensors is modeled as a vertex-cover problem, where the vertex-cover set facilitates the communications between the sensors. The tree-cover asymptote may therefore have application where communication is involved in networks with a large number of vertices, i.e. in extreme networks.

If $H(S)$, in the tree-cover definition, is connected, and M represents the vertices of G not in S, then each vertex of M is connected directly by an edge (an out-edge) to a vertex of $H(S)$ which is part of a spanning tree. Thus, the ease of communication between vertices of $H(S)$ and M through the out edges, involving spanning trees, may be represented by this tree-cover ratio- the "ideal" case, involving large number of vertices, being when this tree-cover asymptotic ratio of $1/e$ is the smallest (and positive)- which we believe in the case of complete graphs.

2. Examples of tree-cover ratios and asymptotes

2.1. **Complete graph.** Let K_n be a complete graph on n vertices. Then, a minimum covering set of K_n is any subset of $n-1$ vertices of K_n, and since $t(K_n) = n^{n-2}$; $t(K_{n-1}) = (n-1)^{n-3}$, we have the tree-over ratio of K_n to be:

$$\begin{aligned} tc(K_n)_s &= \frac{|S|t(K_{n-1})}{t(K_n)} = f(n) \\ &= \frac{(n-1)(n-1)^{n-3}}{n^{n-2}} \\ &= \left(\frac{n-1}{n}\right)^{n-2}. \end{aligned}$$

Hence, the tree-cover asymptote for the complete graph is given as:

$$Asyp(K_n)_s = \lim_{n \to \infty} \left(\frac{n-1}{n}\right)^{n-2} = e^{-1}.$$

2.2. **Cycles.** The cycle, C_n on n vertices has $t(C_n) = n$ number of spanning trees, and if n is even, a minimum vertex cover, S, will be the $n/2$ vertices of the disconnected graph induced by every alternate vertex of the cycle, so that $t(H(S)) = 1$ and $|S| = n/2$. Thus,

$$tc(C_n)_s = \frac{(n/2).1}{n} = \frac{1}{2},$$

so that:

$$Asyp(C_n)_s = \frac{1}{2}\,.$$

2.3. **Complete split-bipartite graph.** Let $K_{n/2,n/2}$ be a complete split-bipartite graph on n vertices. Then, $t(K_{n/2,n/2}) = \left(\frac{n}{2}\right)^{n-2}$ is its number of spanning trees, and either partite set can be taken as a minimum vertex cover which yields $t(H(S)) = 1$, so that:

$$tc(K_{n/2,n/2})_s = \frac{n}{2(n/2)^{n-2}} = \left(\frac{2}{n}\right)^{n-3},$$

and

$$Asyp(K_{n/2,n/2})_s = \lim_{n\to\infty}\left(\frac{2}{n}\right)^{n-3} = 0\,.$$

2.4. **Paths.** Let P_n be a path on an even number of vertices. A minimum vertex cover, S consists of alternate vertices of P_n starting with the first vertex of the path. Since $|S| = \frac{n}{2}$, $t(H(S)) = 1$, and $t(P_n) = 1$, we have:

$$tc(P_n)_s(even) = \frac{|S|t(H(S))}{t(P_n)} = \frac{n}{2}\,,$$

so that

$$Asyp(P_n)(even) = \infty\,.$$

For a path on an odd number of vertices, the number of vertices in $H(S)$, where S consists of alternative vertices of P_n starting with the second vertex of the path will be $\frac{n-1}{2}$, so that:

$$tc(P_n)_s(odd) = \frac{|S|t(H(S))}{t(P_n)} = \frac{n-1}{2}\,,$$

such that:

$$Asyp(P_n) = \infty\,.$$

2.5. **Wheel graph.** The wheel graph, W_n, on an odd number of vertices n, has a cycle of even length with each vertex joined to a center. The number of spanning trees of this wheel is $t(W_n) = \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2$ and the minimum vertex cover, S will involve alternate vertices of the even cycle and the center vertex.

Thus, $t'(H(S)) = 1$ and:

$$\begin{aligned} tc(W_n)_s(odd) &= \frac{|S|t(H(S))}{t(W_n)} \\ &= \frac{n}{\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2} \\ &\approx \frac{n}{2(\frac{3}{2})^n}(\text{ for n large})\,, \end{aligned}$$

so that:

$$Asyp(W_n)_s(odd) = 0$$

2.6. **Ladder graph.** The ladder graph, $L_{n/2,n/2}$ on an even number of vertices, has $t(L_{n/2,n/2}) = \dfrac{(2+\sqrt{3})^{\frac{n}{2}} - (2-\sqrt{3})^{\frac{n}{2}}}{\sqrt{3}}$ and $t(H(S)) = 1$, where S is taken as follows; let P and P' be the two paths, each having $n/2$ vertices, of the ladder, with edges between matched vertices of the two paths. By taking S as the set of alternating vertices on P and P', where the first vertex of P is selected and the second vertex P' is selected so that S will have $n/2$ vertices. Then we have:

$$\begin{aligned} tc(L_{n/2,n/2})_s &= \frac{|S|t(H(S))}{t(L_{n/2,n/2})} \\ &= \frac{n\sqrt{3}}{2(2+\sqrt{3})^n - 2(2-\sqrt{3})^n} \,. \end{aligned}$$

Since $(2+\sqrt{3})^n$ dominates $(2-\sqrt{3})^n$ for large n, we have:

$$f(n) = \frac{n\sqrt{3}}{2(2+\sqrt{3})^n - 2(2-\sqrt{3})^n} \approx \frac{n\sqrt{3}}{2(2+\sqrt{3})^n} \,,$$

so that:

$$Asyp(L_{n/2,n/2})_s = 0 \,.$$

2.7. **Star graph of rays of length 1.** Let $S_{n,1}$ be the star graph on n vertices with $n-1$ rays of length 1. Then its center is its minimum covering set so that:

$$tc(S_{n-1})_s = \frac{|S|t(H(S))}{t(S_{n,1})} \,,$$

hence:

$$Asyp(S_{n,1})_s = 1 \,.$$

2.8. **Star graph with k rays of length 2.** Let $S_{n,k(2)}$ be the star graph on n vertices with k rays of length 2, so that $n = 2k+1$(odd). The set S of vertices, a distance 1 from the center form the minimum covering of this graph so that $|S| = \dfrac{n-1}{2}$ and $t(H(S)) = 1$, so that:

$$tc(S_{n,k(2)})_s = \frac{|S|t(H(S))}{t(S_{n,k(2)})} = \frac{n-1}{2} \,,$$

and

$$Asyp(S_{n,k(2)})_s = \infty \,.$$

2.9. **Sun graph.** Take a cycle on $n/2$ vertices and attached an end vertex to each vertex of the cycle to form the sun graph, SN_n on n vertices. Since $t(SN_n) = n$ and S consists of all the vertices of the cycle so that $t(H(S)) = n$. Hence:

$$\begin{aligned} tc(SN_n)_s &= \frac{|S|t(H(S))}{t(S_{n,k(2)})} \\ &= \frac{n.n}{n} = n \,, \end{aligned}$$

so that:

$$Asyp(SN_n)_s = \infty \,.$$

Theorem 2.1. *The tree-cover ratios and tree-cover asymptotes of the following graphs are:*

$$tc(K_n)_s = \Big(\frac{n-1}{n}\Big)^{n-2} \text{ and } Asyp(K_n)_s = e^{-1}\,;$$
$$tc(C_n)_s = \frac{1}{2} \text{ and } Asyp(C_n)_s = \frac{1}{2}\,;$$
$$tc(K_{n/2,n/2})_s = \Big(\frac{2}{n}\Big)^{n-3} \text{ and } Asyp(K_{n/2,n/2})_s = 0\,;$$
$$tc(P_n)_s(even) = \frac{n}{2};\ T(P_n)_s(odd) = \frac{n-1}{2} \text{ and } Asyp(K_n)_s = \infty\,;$$
$$tc(W_n)_s = \frac{n}{(\frac{3+\sqrt{5}}{2})^n + (\frac{3-\sqrt{5}}{2})^n - 2} \text{ and } Asyp(W_n)_s = 0\,;$$
$$tc(L_{n/2,n/2})_s = \frac{n\sqrt{3}}{(2+\sqrt{3})^n - (2-\sqrt{3})^n} \text{ and } Asyp(L_{n/2,n/2})_s = 0\,;$$
$$tc(S_{n,1})_s = 1 \text{ and } Asyp(S_{n,1})_s = 1\,;$$
$$tc(S_{n,k(2)})_s = n-1 \text{ and } Asyp(S_{n,k(2)})_s = \infty\,;$$
$$tc(SN_n)_s = n \text{ and } Asyp(SN_n)_s = \infty\,.$$

Conjecture 2.1. *The non-zero tree-cover asymptote of classes of graphs takes on the smallest value of e^{-1} for complete graphs, i.e. if $Asyp(\xi)_s \neq 0$, then*

$$Asyp(\xi)_s \in [\frac{1}{e}, \infty]\,.$$

The following theorem may help in proving the conjecture 2.1:

Theorem 2.2. *Suppose ξ is a class of graphs for which:*

$$t(G)_s = \frac{|S|t(H(S))}{t(G)} = \frac{(n-k)(n-k)^{n-p-1}}{n^{n-p}}$$

for which $G \in \xi$, $p \in \mathbb{N}$, $k > 1$, and $p > 2$.

That is, the number of spanning trees of G is n^{n-p} and G has a (minimum) vertex cover S with $n-k$ vertices and spanning trees $(n-k)^{n-p-1}$. Then, $Asyp(\xi) = \frac{1}{e^{\frac{1}{k}}} > \frac{1}{e}$.

Proof. Let $q = \left(\frac{n-k}{n}\right)^{n-p}$. Then,

$$q = \Big(\frac{n-k}{n}\Big)^{n-p} \Longrightarrow \ln q = \frac{\ln(1-\frac{k}{n})}{\frac{1}{n-p}}\,.$$

As $n \to \infty$,

$$\ln p = \frac{-(n-p)^2}{(1-\frac{k}{n})(kn^2)} \approx \frac{-n^2}{kn^2} = -\frac{1}{k}$$

$$\Longrightarrow q \to e^{-\frac{1}{k}} \quad \text{as} \quad n \to \infty\,.$$

Since, $k > 1$, $\Rightarrow \frac{1}{k} < 1$; thus, $e^{\frac{1}{k}} < e \Rightarrow \frac{1}{e^{\frac{1}{k}}} > \frac{1}{e}$, proving the theorem. □

3. Tree-cover area of classes of graphs

We introduce another dimension by integrating this tree-cover ratio.

Definition 3.1. *If ξ is a class of graphs and $tc(G)_s = \frac{|S|t(H(S))}{t(G)} = f(n)$ for each $G \in \xi$, where n is the order of G, and m is the size, then the* **tree-cover area** *of ξ is defined as:*

$$Ar(\xi_n) = \frac{2m}{n} \int f(n)dn.$$

for min p defined, given that $Ar(\xi_p) = 0$.

Here, $\frac{2m}{n}$ represents the *average degree* of the graph G, and the integral part as the *tree-cover height*, $h(G)$ of the graph.

3.1. Examples of tree-cover areas of classes of graphs.

Example 3.1. *Cycle, C_n.*

If C_n is a cycle on an even number of vertices, then $tc(C_n) = \frac{1}{2}$, so that the tree-cover height of cycles, $h(C_n)$, is $\int \frac{1}{2}$. Hence, the tree-cover area of cycles, $Ar(C_n)$ is given as:

$$Ar(C_n) = \frac{2n}{n} \int \frac{1}{2}dn = 2(\frac{n}{2} + c);$$

$$f(3) = 0 \Longrightarrow c = -\frac{3}{2},$$

hence:

$$Ar(C_n) = n - 3.$$

Example 3.2. *Paths, P_n.*

If P_n is a path on an even number of vertices, then $tc(P_n) = \frac{n}{2}$, so that

$$Ar(P_n) = \frac{2n-2}{n} \int \frac{n}{2}dn = \frac{n-1}{n}(\frac{n^2}{2} + c),$$

$$f(2) = 0 \Longrightarrow c = -2,$$

hence,

$$Ar(P_n) = \frac{n-1}{n}\left(\frac{n^2}{2} - 2\right) ; (n \text{ even}).$$

If P_n is a path on an odd number of vertices, then:

$$Ar(P_n) = \frac{n-1}{n}\left(\frac{n^2}{2} - n + c\right) ; (n \text{ odd}),$$

but,

$$Ar(P_2) = 0 \Longrightarrow c = 0,$$

hence:

$$Ar(P_n) = \frac{n-1}{n}\left(\frac{n^2}{2} - n\right) ; (n \text{ odd}).$$

$n=2$ *to:*	$h(K_n)$	$Ar(K_n)$
2	*0.0000*	*0.0000*
3	*0.7932*	*1.5864*
4	*1.4004*	*4.2012*
5	*1.9352*	*7.7408*
6	*2.4373*	*12.1860*

TABLE 1. Tree-cover area of complete graphs

Example 3.3. *Star graphs with rays of length 1,* $S_{n,1}$.

$$Ar(S_{n,1}) = \frac{2n-2}{n}\int dn = \frac{2n-2}{n}(n+c)\,;$$
$$Ar(S_{1,1}) = 0 \Longrightarrow c = -2\,;$$
$$Ar(S_{n,1}) = \frac{2(n-1)(n-2)}{n}\,.$$

Example 3.4. *Star graphs with* k *rays of length 2,* $S_{1,k(2)}$.

$$Ar(S_{n,k(2)}) = \frac{2n-2}{n}\int (\frac{n}{2}-\frac{1}{2})dn = \frac{2n-2}{n}(\frac{n^2}{4}-\frac{n}{2}+c)\,;$$
$$Ar(S_{1,k(2)}) = 0 \Longrightarrow c = -\frac{3}{4}\,;$$
$$Ar(S_{1,k(2)}) = \frac{2n-2}{n}(\frac{n^2}{4}-\frac{n}{2}-\frac{3}{4})\,.$$

Example 3.5. *Sun graph,* SN_n .

$$Ar(SN_n) = 2\int n dn = 2(\frac{n^2}{2}+c)\,;$$
$$Ar(SN_6) = 0 \Longrightarrow c = -18\,;$$
$$Ar(SN_n) = n^2 - 36\,.$$

For the following examples, we approximate the tree-cover heights and areas in tables 1 and 2.

Example 3.6. *Complete graph,* K_n .

$$Ar(K_n) = (n-1)\int \left(\frac{n-1}{n}\right)^{n-2} dn\,.$$

Using trapezoid rule, we get the tree-cover area for cases $n = 3, 4, 5, 6$*, where the height is found by starting from* $n = 2$ *in table 1.*

Example 3.7. *Complete split-bipartite graph,* $K_{n/2,n/2}$.

$$Ar(K_{n/2,n/2}) = \frac{n}{2}\int \left(\frac{2}{n}\right)^{n-3} dn\,.$$

By using the Trapezium rule to obtain the tree-cover height and area for $n = 2$ *to* $n = 2, 4, 6, 8$ *in table 2.*

n=2 to:	$h(K_n)$	$Ar(K_n)$
2	*0.0000*	*0.0000*
4	*1.8323*	*3.6646*
6	*2.2260*	*6.6780*
8	*2.4700*	*9.8800*

TABLE 2. Tree-cover area of complete split-bipartite graphs

$n = 6$	$h(\xi)$	$Ar(\xi)$
C_n	1.5000	3.0000
P_n	16.0000	13.3333
$S_{n,1}$	4.0000	6.6667
SN_n	0.0000	0.0000
K_n	2.4372	12.1860
$K_{n/2,n/2}$	2.2.2260	6.6780

TABLE 3. Tree-cover heights and areas for classes of graphs

3.2. **Which class of graph has the largest tree-cover area?** The star graphs with k rays of length 2 are not defined for n even, but for $n = 5$, the tree-cover area of the complete graph is larger than the tree-cover area of the star graph. Comparing all tree-cover areas up to $n = 6$, we see that the complete graph has the largest area for regular graphs.

4. CONCLUSION: KNOWN AND NEW RESULTS

4.1. **Combining spanning trees and vertex coverings.** In this paper, we combined the concepts of spanning trees, $t(G)$, and vertex cover, S of a graph, G, to introduce a new concept of a tree-cover ratio of G, where $H(S)$ is the subgraph of G induced by S:

$$\frac{|S|t(H(S))}{t(G)}.$$

This ratio was motivated by the fact that the general tree-cover ratio for complete graphs, as a function of order n of such graphs is, $\left(1 - \frac{1}{n}\right)^{n-2}$, and has an asymptotic convergence of e^{-1}, which is identical to the probability of choosing the best applicant selected in the secretary problem. This resulted in the consideration of the asymptotic convergence of the tree-cover ratio of the classes of graphs. We introduced the integration of the tree-cover ratio which allowed for the idea of tree-cover areas of classes of graphs.

We propose that the tree-cover asymptote of complete graphs is the smallest amongst all such possible positive tree-cover asymptotes of classes of graphs, and that the tree-cover area of complete graphs dominates all other tree-cover areas of classes of regular graphs.

Future research may involve considering the tree-cover ratio of the complement of classes of graphs discussed here. We could have considered the reciprocal of the tree-cover ratio, i.e.

$$[tc(G)_s]^{-1} = \frac{t(G)}{|S|t(H(S))}\,.$$

For example, the reciprocal of the tree-cover ratio of complete graphs would have the asymptotic convergence of e, while paths on an even number of vertices would have a reciprocal tree-cover ratio asymptote of 0,(which is the same as the tree-cover asymptote of complete split-bipartite graph), and reciprocal tree-cover area of

$$\frac{2n-2}{n}\int \frac{2}{n}dn = \frac{2(n-1)}{n}(2\ln n + c)\,.$$

4.2. **Known and new results: ratios, asymptotes and areas.** For the complete graph on n vertices, the following are known results:

(1) The vertex expansion ratio which has asymptote 1(see [8]).
(2) The Hall ratio which converges to infinity (see [5]).
(3) The eigen-ratio which converges to -1 (see [10]).
(4) The central radius ratio which has asymptote 1 (see [1]).

For complete graphs on n vertices, the following are the new results presented in this paper.

Tree cover ratio, $tc(K_n)$ is $\left(\frac{n-1}{n}\right)^{n-2}$, which converges to e^{-1}, and the tree-cover area, $Ar(K_n) = (n-1)\int\left(\frac{n-1}{n}\right)^{n-2}$.

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