

FEKETE-SZEGÖ INEQUALITIES FOR CLASSES OF BI-UNIVALENT FUNCTIONS DEFINED BY SUBORDINATION

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ABSTRACT. In the present investigation, we obtain the Fekete-Szegő inequalities for the classes $S_{\Sigma}(\lambda, \phi)$ and $B(\lambda, \mu, \phi)$ of bi-univalent functions defined by subordination.

1. INTRODUCTION AND DEFINITIONS

Let A denote the class of analytic functions in the unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\},$$

that have the form:

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Further, by S we shall denote the class of all functions in A which are univalent in U .

The Koebe one-quarter theorem [4] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus, every such univalent function has an inverse f^{-1} which satisfies:

$$f^{-1}(f(z)) = z \quad (z \in U),$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots$$

A function $f(z) \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U .

If the functions f and g are analytic in U , then f is said to be subordinate to g , written as $f(z) \prec g(z)$, if there exists a Schwarz function w such that $f(z) = g(w(z))$.

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Let Σ denote the class of bi-univalent functions defined in the unit disk U . For a brief history and interesting examples in the class Σ refer to [11].

Lewin in [7] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [3] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Netanyahu in [9] showed that $\max |a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$.

Brannan and Taha in [2] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $S^*(\alpha)$ and $K(\alpha)$ of starlike and convex function of order α ($0 < \alpha \leq 1$), respectively, (see [9]). Thus, following Brannan and Taha [2], a function $f(z) \in A$ is in the class $S_\Sigma^*(\alpha)$ of strongly bi-starlike functions of order α ($0 < \alpha \leq 1$) if each of the following conditions is satisfied:

$$f \in \Sigma, \quad \left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, z \in U)$$

and

$$\left| \arg \left(\frac{wg'(w)}{g(w)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, w \in U),$$

where g is the extension of f^{-1} to U . Similarly, a function $f(z) \in A$ is in the class $K_\Sigma(\alpha)$ of strongly bi-convex functions of order α ($0 < \alpha \leq 1$) if each of the following conditions is satisfied:

$$f \in \Sigma, \quad \left| \arg \left(\frac{z^2 f''(z) + zf'(z)}{zf'(z)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, z \in U)$$

and

$$\left| \arg \left(\frac{w^2 g''(w) + wg'(w)}{wg'(w)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, w \in U),$$

where g is the extension of f^{-1} to U . The classes $S_\Sigma^*(\alpha)$ and $K_\Sigma(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α , corresponding to the classes $S^*(\alpha)$ and $K(\alpha)$, were introduced analogously. For each of the classes $S_\Sigma^*(\alpha)$ and $K_\Sigma(\alpha)$, they found non-sharp estimates on the initial coefficients. Recently, many authors investigated bounds for various subclasses of bi-univalent functions [5], [11], [12]. The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ for $n = 3, 4, \dots$ is presumably still an open problem.

In this paper we obtain the Fekete Szegő inequalities for the classes $S_\Sigma(\lambda, \phi)$ and $B(\lambda, \mu, \phi)$. These inequalities will result in bounds of the third coefficient which are, in some cases, better than the ones obtained in [1], [6] and [8].

In order to derive our main results, we require the following lemma.

Lemma 1.1. (Pommerenke [10]) *If $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots$ is an analytic function in U with positive real part, then*

$$|p_n| \leq 2 \quad (n \in \mathbb{N} = \{1, 2, \dots\})$$

and

$$(1.2) \quad \left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2}.$$

2. FEKETE-SZEGÖ INEQUALITIES FOR THE CLASS $S_{\Sigma}(\lambda, \phi)$

In the following, let ϕ be an analytic function with positive real part in U , with $\phi(0) = 1$ and $\phi'(0) > 0$. Also, let $\phi(U)$ be starlike with respect to 1 and symmetric with respect to the real axis. Thus, ϕ has the Taylor series expansion

$$(2.1) \quad \phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0).$$

Definition 2.1. A function $f \in \Sigma$ is said to be in $S_{\Sigma}(\lambda, \phi)$, $0 < \alpha \leq 1$ and $0 \leq \lambda \leq 1$, if the following subordinations hold

$$\frac{zf'(z) + (2\lambda^2 - \lambda)z^2f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)zf'(z) + (2\lambda^2 - 3\lambda + 1)f(z)} \prec \phi(z)$$

and

$$\frac{wg'(w) + (2\lambda^2 - \lambda)w^2g''(w)}{4(\lambda - \lambda^2)w + (2\lambda^2 - \lambda)wg'(w) + (2\lambda^2 - 3\lambda + 1)g(w)} \prec \phi(w),$$

where $g(w) = f^{-1}(w)$.

Theorem 2.1. Let f given by (1.1) be in the class $S_{\Sigma}(\lambda, \phi)$ and $\mu \in \mathbb{R}$. Then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1}{2(1+2\lambda^2)} \text{ for } |\mu - 1| \leq \frac{1}{2(2\lambda^2+1)}, \\ \left| 1 + 2\lambda + 15\lambda^2 - 28\lambda^3 + 12\lambda^4 + (1 + 3\lambda - 2\lambda^2)^2 \frac{(B_1 - B_2)}{B_1^2} \right| \\ \frac{B_1^3 |\mu - 1|}{|(12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1)B_1^2 + (1 + 3\lambda - 2\lambda^2)^2 (B_1 - B_2)|} \text{ for } |\mu - 1| \geq \\ \frac{1}{2(2\lambda^2+1)} \cdot \left| 1 + 2\lambda + 15\lambda^2 - 28\lambda^3 + 12\lambda^4 + (1 + 3\lambda - 2\lambda^2)^2 \frac{(B_1 - B_2)}{B_1^2} \right|. \end{cases}$$

Proof. Let $f \in S_{\Sigma}(\lambda, \phi)$ and g be the analytic extension of f^{-1} to U . Then, there exist two functions u and v , analytic in U with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(w)| < 1$, $z, w \in U$, such that

$$(2.2) \quad \frac{zf'(z) + (2\lambda^2 - \lambda)z^2f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)zf'(z) + (2\lambda^2 - 3\lambda + 1)f(z)} = \phi(u(z)) \quad (z \in U)$$

and

$$(2.3) \quad \frac{wg'(w) + (2\lambda^2 - \lambda)w^2g''(w)}{4(\lambda - \lambda^2)w + (2\lambda^2 - \lambda)wg'(w) + (2\lambda^2 - 3\lambda + 1)g(w)} = \phi(v(w)) \quad (w \in U).$$

Next, let define functions p and q by

$$(2.4) \quad p(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + p_1 z + p_2 z^2 + \cdots$$

and

$$(2.5) \quad q(w) = \frac{1 + v(w)}{1 - v(w)} = 1 + q_1 w + q_2 w^2 + \cdots.$$

Clearly, $\operatorname{Re} p(z) > 0$ and $\operatorname{Re} q(w) > 0$. From (2.4) and (2.5) one can derive

$$(2.6) \quad u(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2}p_1 z + \frac{1}{2}\left(p_2 - \frac{1}{2}p_1^2\right)z^2 + \cdots$$

and

$$(2.7) \quad v(w) = \frac{q(w) - 1}{q(w) + 1} = \frac{1}{2}q_1 w + \frac{1}{2}\left(q_2 - \frac{1}{2}q_1^2\right)w^2 + \cdots.$$

Combining (2.1), (2.2), (2.3), (2.6) and (2.7),

$$(2.8) \quad \frac{zf'(z) + (2\lambda^2 - \lambda)z^2 f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)zf'(z) + (2\lambda^2 - 3\lambda + 1)f(z)} = 1 + \frac{1}{2}B_1 p_1 z + \left(\frac{1}{4}B_2 p_1^2 + \frac{1}{2}B_1 \left(p_2 - \frac{1}{2}p_1^2\right)\right) z^2 + \dots$$

and

$$(2.9) \quad \frac{wg'(w) + (2\lambda^2 - \lambda)w^2 g''(w)}{4(\lambda - \lambda^2)w + (2\lambda^2 - \lambda)wg'(w) + (2\lambda^2 - 3\lambda + 1)g(w)} = 1 + \frac{1}{2}B_1 q_1 w + \left(\frac{1}{4}B_2 q_1^2 + \frac{1}{2}B_1 \left(q_2 - \frac{1}{2}q_1^2\right)\right) w^2 + \dots$$

From (2.8) and (2.9), we deduce

$$(2.10) \quad (1 + 3\lambda - 2\lambda^2) a_2 = \frac{1}{2}B_1 p_1,$$

$$(2.11) \quad (12\lambda^4 - 28\lambda^3 + 11\lambda^2 + 2\lambda - 1) a_2^2 + (4\lambda^2 + 2) a_3 = \frac{1}{4}B_2 p_1^2 + \frac{1}{2}B_1 \left(p_2 - \frac{1}{2}p_1^2\right),$$

and

$$(2.12) \quad -(1 + 3\lambda - 2\lambda^2) a_2 = \frac{1}{2}B_1 q_1,$$

$$(2.13) \quad (12\lambda^4 - 28\lambda^3 + 19\lambda^2 + 2\lambda + 3) a_2^2 - (4\lambda^2 + 2) a_3 = \frac{1}{4}B_2 q_1^2 + \frac{1}{2}B_1 \left(q_2 - \frac{1}{2}q_1^2\right).$$

From (2.10) and (2.12) we obtain

$$(2.14) \quad p_1 = -q_1.$$

Subtracting (2.11) from (2.13) and applying (2.14) we have

$$(2.15) \quad a_3 = a_2^2 + \frac{1}{8(1 + 2\lambda^2)} B_1 (p_2 - q_2).$$

By adding (2.11) to (2.13), we get

$$2(12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1) a_2^2 = \frac{1}{2}B_1 (p_2 + q_2) - \frac{1}{4}(B_1 - B_2) (p_1^2 + q_1^2).$$

Combining this with (2.10) and (2.12) leads to

$$(2.16) \quad a_2^2 = \frac{B_1^3 (p_2 + q_2)}{4 \left[(12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1) B_1^2 + (1 + 3\lambda - 2\lambda^2)^2 (B_1 - B_2) \right]}.$$

From (2.15) and (2.16) it follows that

$$a_3 - \mu a_2^2 = B_1 \left[\left(h(\mu) + \frac{1}{8(1 + 2\lambda^2)} \right) p_2 + \left(h(\mu) - \frac{1}{8(1 + 2\lambda^2)} \right) q_2 \right],$$

where

$$h(\mu) = \frac{B_1^2 (1 - \mu)}{4 \left[(12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1) B_1^2 + (1 + 3\lambda - 2\lambda^2)^2 (B_1 - B_2) \right]}.$$

Then, in view of (1.2) and (2.1), we conclude that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1}{2(1+2\lambda^2)} & \text{for } 0 \leq |h(\mu)| \leq \frac{1}{8(1+2\lambda^2)} \\ 4B_1 |h(\mu)| & \text{for } |h(\mu)| \geq \frac{1}{8(1+2\lambda^2)}. \end{cases}$$

□

Taking $\mu = 1$ or $\mu = 0$ we get

Corollary 2.1. *If $f \in S_\Sigma(\lambda, \phi)$ then*

$$(2.17) \quad |a_3 - a_2^2| \leq \frac{B_1}{2(1+2\lambda^2)}.$$

Corollary 2.2. *If $f \in S_\Sigma(\lambda, \phi)$ then*

$$(2.18) \quad |a_3| \leq \begin{cases} \frac{B_1}{2(1+2\lambda^2)} \text{ for } \frac{B_1-B_2}{B_1^2} \in \\ \left(-\infty, -\frac{12\lambda^4-28\lambda^3+19\lambda^2+2\lambda+3}{(1+3\lambda-2\lambda^2)^2}\right] \cup \left[-\frac{12\lambda^4-28\lambda^3+11\lambda^2+2\lambda-1}{(1+3\lambda-2\lambda^2)^2}, \infty\right) \\ \frac{B_1^3}{[(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1)B_1^2+(1+3\lambda-2\lambda^2)^2(B_1-B_2)]} \text{ for } \frac{B_1-B_2}{B_1^2} \in \\ \left[-\frac{12\lambda^4-28\lambda^3+19\lambda^2+2\lambda+3}{(1+3\lambda-2\lambda^2)^2}, -\frac{12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1}{(1+3\lambda-2\lambda^2)^2}\right) \cup \\ \left(-\frac{12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1}{(1+3\lambda-2\lambda^2)^2}, -\frac{12\lambda^4-28\lambda^3+11\lambda^2+2\lambda-1}{(1+3\lambda-2\lambda^2)^2}\right]. \end{cases}$$

Corollary 2.3. *If*

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \dots \quad (0 < \alpha \leq 1),$$

then inequalities (2.17) and (2.18) become

$$|a_3 - a_2^2| \leq \frac{\alpha}{1+2\lambda^2}$$

and

$$|a_3| \leq \begin{cases} \frac{\alpha}{1+2\lambda^2} \text{ for } \\ \alpha \in \left(0, \frac{(1+3\lambda-2\lambda^2)^2}{-20\lambda^4+44\lambda^3-17\lambda^2+2\lambda+3}\right] \cup \left[\frac{(1+3\lambda-2\lambda^2)^2}{-20\lambda^4+44\lambda^3-33\lambda^2+2\lambda-5}, 1\right] \\ \frac{4\alpha^2}{(20\lambda^4-44\lambda^3+25\lambda^2-2\lambda+1)\alpha+(1+3\lambda-2\lambda^2)^2} \text{ for } \\ \alpha \in \left[\frac{(1+3\lambda-2\lambda^2)^2}{-20\lambda^4+44\lambda^3-17\lambda^2+2\lambda+3}, \frac{(1+3\lambda-2\lambda^2)^2}{-20\lambda^4+44\lambda^3-25\lambda^2+2\lambda-1}\right) \\ \cup \left(\frac{(1+3\lambda-2\lambda^2)^2}{-20\lambda^4+44\lambda^3-25\lambda^2+2\lambda-1}, \frac{(1+3\lambda-2\lambda^2)^2}{-20\lambda^4+44\lambda^3-33\lambda^2+2\lambda-5}\right] \end{cases}$$

Corollary 2.4. *If*

$$\phi(z) = \frac{1+(1-2\alpha)z}{1-z} = 1 + 2(1-\alpha)z + 2(1-\alpha)z^2 + \dots \quad (0 \leq \alpha < 1),$$

then inequalities (2.17) and (2.18) become

$$|a_3 - a_2^2| \leq \frac{1-\alpha}{1+2\lambda^2}$$

and

$$|a_3| \leq \frac{2(1-\alpha)}{12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1}.$$

Remark 2.1. *Corollary 2.3 and Corollary 2.4 provide an improvement of the estimate of $|a_3|$ obtained by Magesh and Yamini [8].*

3. FEKETE-SZEGÖ INEQUALITIES FOR THE FUNCTION CLASS $B(\lambda, \mu, \phi)$

Definition 3.1. *A function $f \in \Sigma$ is said to be in $B(\lambda, \mu, \phi)$, $\lambda \geq 0$, $0 \leq \mu \leq \lambda \leq 1$, if the following subordinations hold*

$$\frac{\lambda \mu z^3 f'''(z) + (2\lambda \mu + \lambda - \mu) z^2 f''(z) + z f'(z)}{\lambda \mu z^2 f''(z) + (\lambda - \mu) z f'(z) + (1 - \lambda + \mu) f(z)} \prec \phi(z)$$

and

$$\frac{\lambda \mu w^3 g'''(w) + (2\lambda \mu + \lambda - \mu) w^2 g''(w) + w g'(w)}{\lambda \mu w^2 g''(w) + (\lambda - \mu) w g'(w) + (1 - \lambda + \mu) g(w)} \prec \phi(w),$$

where $g(w) = f^{-1}(w)$ (see [1]).

Theorem 3.1. *Let f given by (1.1) be in the class $B(\lambda, \mu, \phi)$ and $\delta \in \mathbb{R}$. Then*

$$|a_3 - \delta a_2^2| \leq \begin{cases} \frac{B_1}{2(1+2\lambda-2\mu+6\lambda\mu)} \text{ for } |\delta - 1| \leq \frac{1}{2(1+2\lambda-2\mu+6\lambda\mu)}. \\ \left| 2 + 4\lambda - 4\mu + 12\lambda\mu - (1 + \lambda - \mu + 2\lambda\mu)^2 \frac{(B_1^2 - B_1 + B_2)}{B_1^2} \right| \\ \frac{B_1^3 |\delta - 1|}{|(2+4\lambda-4\mu+12\lambda\mu)B_1^2 - (1+\lambda-\mu+2\lambda\mu)^2(B_1^2 - B_1 + B_2)|} \text{ for } \\ |\delta - 1| \geq \frac{1}{2(1+2\lambda-2\mu+6\lambda\mu)}. \\ \left| 2 + 4\lambda - 4\mu + 12\lambda\mu - (1 + \lambda - \mu + 2\lambda\mu)^2 \frac{(B_1^2 - B_1 + B_2)}{B_1^2} \right| \end{cases}.$$

Proof. Let $f \in B(\lambda, \mu, \phi)$ and g be the analytic extension of f^{-1} to U . Then, there exist two functions u and v , analytic in U with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(w)| < 1$ ($z, w \in U$) such that

$$(3.1) \quad \frac{\lambda \mu z^3 f'''(z) + (2\lambda \mu + \lambda - \mu) z^2 f''(z) + z f'(z)}{\lambda \mu z^2 f''(z) + (\lambda - \mu) z f'(z) + (1 - \lambda + \mu) f(z)} = \phi(u(z)) \quad (z \in U)$$

and

$$(3.2) \quad \frac{\lambda \mu w^3 g'''(w) + (2\lambda \mu + \lambda - \mu) w^2 g''(w) + w g'(w)}{\lambda \mu w^2 g''(w) + (\lambda - \mu) w g'(w) + (1 - \lambda + \mu) g(w)} = \phi(v(w)) \quad (w \in U).$$

Combining (2.1), (3.1), (3.2), (2.6) and (2.7), we get

$$(3.3) \quad \begin{aligned} & \frac{\lambda \mu z^3 f'''(z) + (2\lambda \mu + \lambda - \mu) z^2 f''(z) + z f'(z)}{\lambda \mu z^2 f''(z) + (\lambda - \mu) z f'(z) + (1 - \lambda + \mu) f(z)} \\ &= 1 + \frac{1}{2} B_1 p_1 z + \left(\frac{1}{4} B_2 p_1^2 + \frac{1}{2} B_1 (p_2 - \frac{1}{2} p_1^2) \right) z^2 + \dots \end{aligned}$$

and

$$(3.4) \quad \begin{aligned} & \frac{\lambda \mu w^3 g'''(w) + (2\lambda \mu + \lambda - \mu) w^2 g''(w) + w g'(w)}{\lambda \mu w^2 g''(w) + (\lambda - \mu) w g'(w) + (1 - \lambda + \mu) g(w)} \\ &= 1 + \frac{1}{2} B_1 q_1 w + \left(\frac{1}{4} B_2 q_1^2 + \frac{1}{2} B_1 (q_2 - \frac{1}{2} q_1^2) \right) w^2 + \dots \end{aligned}$$

From (3.3) and (3.4), we deduce

$$(3.5) \quad (1 + \lambda - \mu + 2\lambda\mu) a_2 = \frac{1}{2} B_1 p_1,$$

$$(3.6) \quad 2(1 + 2\lambda - 2\mu + 6\lambda\mu) a_3 - (1 + \lambda - \mu + 2\lambda\mu)^2 a_2^2 = \frac{1}{4} B_2 p_1^2 + \frac{1}{2} B_1 \left(p_2 - \frac{1}{2} p_1^2 \right),$$

and

$$(3.7) \quad -(1 + \lambda - \mu + 2\lambda\mu) a_2 = \frac{1}{2} B_1 q_1,$$

$$(3.8)$$

$$2(1 + 2\lambda - 2\mu + 6\lambda\mu) (2a_2^2 - a_3) - (1 + \lambda - \mu + 2\lambda\mu)^2 a_2^2 = \frac{1}{4} B_2 q_1^2 + \frac{1}{2} B_1 \left(q_2 - \frac{1}{2} q_1^2 \right).$$

From (3.5) and (3.7) we obtain

$$(3.9) \quad p_1 = -q_1.$$

Subtracting (3.6) from (3.8) and applying (3.9) we have

$$(3.10) \quad a_3 = a_2^2 + \frac{1}{8(1 + 2\lambda - 2\mu + 6\lambda\mu)} B_1 (p_2 - q_2).$$

By adding (3.6) to (3.8), we get

$$4(1 + 2\lambda - 2\mu + 6\lambda\mu) a_2^2 - 2(1 + \lambda - \mu + 2\lambda\mu)^2 a_2^2 = \frac{1}{2} B_1 (p_2 + q_2) - \frac{1}{4} (B_1 - B_2) (p_1^2 + q_1^2).$$

Combining this with (3.5) and (3.7) leads to

$$(3.11) \quad a_2^2 = \frac{B_1^3 (p_2 + q_2)}{4 \left[(2 + 4\lambda - 4\mu + 12\lambda\mu) B_1^2 - (1 + \lambda - \mu + 2\lambda\mu)^2 (B_1^2 - B_1 + B_2) \right]}.$$

From (3.10) and (3.11) it follows that

$$a_3 - \delta a_2^2 = B_1 \left[\left(h(\delta) + \frac{1}{8(1 + 2\lambda - 2\mu + 6\lambda\mu)} \right) p_2 + \left(h(\delta) - \frac{1}{8(1 + 2\lambda - 2\mu + 6\lambda\mu)} \right) q_2 \right],$$

where

$$h(\delta) = \frac{B_1^2 (1 - \delta)}{4 \left[(2 + 4\lambda - 4\mu + 12\lambda\mu) B_1^2 - (1 + \lambda - \mu + 2\lambda\mu)^2 (B_1^2 - B_1 + B_2) \right]}.$$

Then, in view of (1.2) and (2.1), we conclude that

$$|a_3| \leq \begin{cases} \frac{B_1}{2 + 4\lambda - 4\mu + 12\lambda\mu} & \text{for } 0 \leq |h(\delta)| \leq \frac{1}{8(1 + 2\lambda - 2\mu + 6\lambda\mu)} \\ 4B_1 |h(\delta)| & \text{for } |h(\delta)| \geq \frac{1}{8(1 + 2\lambda - 2\mu + 6\lambda\mu)} \end{cases}.$$

□

Taking $\delta = 1$ or $\delta = 0$ we get

Corollary 3.1. *If $f \in \mathcal{B}(\lambda, \mu, \phi)$ then*

$$(3.12) \quad |a_3 - a_2^2| \leq \frac{B_1}{2 + 4\lambda - 4\mu + 12\lambda\mu}.$$

Corollary 3.2. *If $f \in B(\lambda, \mu, \phi)$ then*

$$(3.13) \quad |a_3| \leq \begin{cases} \frac{B_1}{2+4\lambda-4\mu+12\lambda\mu} & \text{for } \frac{B_1^2-B_1+B_2}{B_1^2} \in (-\infty, 0] \cup \left[\frac{4(1+2\lambda-2\mu+6\lambda\mu)}{(1+\lambda-\mu+2\lambda\mu)^2}, \infty \right) \\ \frac{B_1^3}{|(2+4\lambda-4\mu+12\lambda\mu)B_1^2-(1+\lambda-\mu+2\lambda\mu)^2(B_1^2-B_1+B_2)|} & \text{for } \frac{B_1^2-B_1+B_2}{B_1^2} \in \left[0, \frac{2(1+2\lambda-2\mu+6\lambda\mu)}{(1+\lambda-\mu+2\lambda\mu)^2} \right) \cup \left(\frac{2(1+2\lambda-2\mu+6\lambda\mu)}{(1+\lambda-\mu+2\lambda\mu)^2}, \frac{4(1+2\lambda-2\mu+6\lambda\mu)}{(1+\lambda-\mu+2\lambda\mu)^2} \right] \end{cases}.$$

Corollary 3.3. *If*

$$\phi(z) = \left(\frac{1+z}{1-z} \right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \dots \quad (0 < \alpha \leq 1),$$

then inequalities (3.12) and (3.13) become

$$|a_3 - a_2^2| \leq \frac{\alpha}{1+2\lambda-2\mu+6\lambda\mu}$$

and

$$|a_3| \leq \begin{cases} \frac{\alpha}{1+2\lambda-2\mu+6\lambda\mu} & \text{for } \alpha \in (0, \frac{1}{3}] \cup \left[\frac{(1+\lambda-\mu+2\lambda\mu)^2}{3(1+\lambda-\mu+2\lambda\mu)^2-8(1+2\lambda-2\mu+6\lambda\mu)}, 1 \right) \\ \frac{4\alpha^2}{4(1+2\lambda-2\mu+6\lambda\mu)\alpha-(1+\lambda-\mu+2\lambda\mu)^2(3\alpha-1)} & \text{for } \alpha \in \left[\frac{1}{3}, \frac{(1+\lambda-\mu+2\lambda\mu)^2}{3(1+\lambda-\mu+2\lambda\mu)^2-4(1+2\lambda-2\mu+6\lambda\mu)} \right) \\ \cup \left(\frac{(1+\lambda-\mu+2\lambda\mu)^2}{3(1+\lambda-\mu+2\lambda\mu)^2-4(1+2\lambda-2\mu+6\lambda\mu)}, \frac{(1+\lambda-\mu+2\lambda\mu)^2}{3(1+\lambda-\mu+2\lambda\mu)^2-8(1+2\lambda-2\mu+6\lambda\mu)} \right] \end{cases}.$$

Corollary 3.4. *If*

$$\phi(z) = \frac{1+(1-2\alpha)z}{1-z} = 1 + 2(1-\alpha)z + 2(1-\alpha)z^2 + \dots \quad (0 \leq \alpha < 1),$$

then inequalities (3.12) and (3.13) become

$$|a_3 - a_2^2| \leq \frac{1-\alpha}{1+2\lambda-2\mu+6\lambda\mu}$$

and

$$|a_3| \leq \frac{2(1-\alpha)}{2(1+2\lambda-2\mu+6\lambda\mu) - (1+\lambda-\mu+2\lambda\mu)^2}.$$

Remark 3.1. *Corollary 3.3 and Corollary 3.4 provide an improvement of the estimate of $|a_3|$ obtained previously by Keerthi and Raja [6].*

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