ADV MATH SCI JOURNAL

Advances in Mathematics: Scientific Journal **3** (2014), no.2, 83–96
ISSN 1857-8365 UDC: 517.988.5

THE RHODIUS SPECTRA OF SOME NONLINEAR SUPERPOSITION OPERATORS IN THE SPACES OF SEQUENCES

SANELA HALILOVIĆ[1] AND RAMIZ VUGDALIĆ

ABSTRACT. In this paper we study the superposition operator F in l_p spaces of sequences, generated by the function $f(s,u) = a(s) + \varphi(u)$, in case that $\varphi(u) = u^n$ or $\varphi(u) = \sqrt[n]{u}$. We find out the Rhodius spectrum of this operator F. We also give a few examples and compare this spectrum with the point spectrum.

1. INTRODUCTION AND PRELIMINARIES

It is well-known a notion of a spectrum for bounded linear operators and its useful properties. Many various attempts have been made to define and study spectra also for nonlinear operators. In the beginning, the term spectrum was used for nonlinear operators just in the sense of point spectrum. Later, from the last sixties it became clear that a more complete description requires other spectral sets. This led to a number of definitions of nonlinear spectra which are all different. In this paper we are dealing with the Rhodius spectra for some nonlinear superposition operators. This spectrum is becoming reduced to the familiar spectrum in case of linear operator and always contains the eigenvalues of an operator which keeps zero fixed. The Rhodius spectrum is introduced in [4]. The example which shows that the Rhodius spectrum may be empty is due to Georg and Martelli in [5]. Some informations about Rhodius spectrum and other spectra for nonlinear operators and their comparisons and applications can, for example, also be found in [2], [6],[7],[8] and [9]. First of all, let us introduce some definitions and facts for nonlinear superposition operators in space of sequences l_p.

Let $f = f(s,u)$ be a function defined on $\Omega \times \mathbb{R}$ (or $\Omega \times \mathbb{C}$) with the values in $\mathbb{R}$ (or respectively $\mathbb{C}$). Given a function $x = x(s)$, by applying f, we get another function $y = y(s)$ on Ω by:

$$y(s) = f(s, x(s)).$$

In this way, the function f generates an operator F :

$$Fx(s) = f(s, x(s)), \tag{1.1}$$

[1] *corresponding author*

2010 *Mathematics Subject Classification.* 47H30, 47J10.

Key words and phrases. Superposition operator, spectrum of nonlinear operator, spaces of sequences.

which is usually called superposition operator (or composition operator or Nemytskij operator), ([3]). We are going to observe the operator of superposition, defined in the spaces of sequences l_p $(1 \leq p \leq \infty)$, so we have $\Omega = \mathbb{N}$ (i.e. $s \in \mathbb{N}$).

Theorem 1.1. ([1]) *Let $1 \leq p, q < \infty$. Then the following properties are equivalent:*

- *the operator F acts from l_p to l_q;*
- *there are functions $a(s) \in l_q$ and constants $\delta > 0, n \in \mathbb{N}, b \geq 0$, for which $|f(s,u)| \leq a(s) + b|u|^{\frac{p}{q}}$ $(s \geq n, |u| < \delta)$;*
- *for any $\varepsilon > 0$ there exists a function $a_\varepsilon \in l_q$ and constants $\delta_\varepsilon > 0, n_\varepsilon \in \mathbb{N}, b_\varepsilon \geq 0$, for which $\|a_\varepsilon(s)\|_q < \varepsilon$ and*

$$|f(s,u)| \leq a_\varepsilon(s) + b_\varepsilon |u|^{\frac{p}{q}} \ \ (s \geq n_\varepsilon, \ |u| \leq \delta_\varepsilon).$$

Theorem 1.2. ([1]) *Let $1 \leq p, q < \infty$ and let the superposition operator* (1.1)*, generated by the function $f(s,u)$, acts from l_p to l_q. Then this operator is continuous if and only if each of the functions is continuous for every $s \in \mathbb{N}$.*

Definition 1.1. *The set of all eigenvalues of the operator F*

$$\sigma_p(F) = \{\lambda \in \mathbb{K} : Fx = \lambda x \ \ for\ some\ x \neq 0\},$$

($\mathbb{K}$ is the field of real or complex numbers), is called the point spectrum of F.

For the class $\mathfrak{C}(X)$ of all continuous operators F on a Banach space X over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), Rhodius introduced in 1984. the following definition of its spectrum ([4]).

Definition 1.2. ([2]) *For the continuous operator $F : X \to X$ the set*

$$\rho_R(F) = \left\{\lambda \in \mathbb{K} : \lambda I - F\, is\, bijective\, and\, (\lambda I - F)^{-1} \in \mathfrak{C}(X)\right\}$$

is called Rhodius resolvent set, and the set

$$\sigma_R(F) = \mathbb{K} \backslash \rho_R(F)$$

is called Rhodius spectrum.

Remark 1.1. *A point $\lambda \in \mathbb{K}$ belongs to $\rho_R(F)$ if and only if $\lambda I - F$ is a homeomorphism on X.*

If $F0 = 0$, then the problem of investigating injectivity of nonlinear operator $\lambda I - F$, could sometimes be replaced by finding if the equation $(\lambda I - F)x = 0$ has any nontrivial solution. If there are some nontrivial solutions of the equation $(\lambda I - F)x = 0$, then operator $\lambda I - F$ is not injective. Generally this is not true if $F0 \neq 0$. More precisely, if F is nonlinear operator with $F0 = 0$, then $\sigma_p(F) \subseteq \Sigma_i \subseteq \sigma_R(F)$. ($\Sigma_i = \{\lambda \in \mathbb{K} : \lambda I - F\, is\, not\, injective\,\}$).

2. Main Results

Lemma 2.1. *Let the superposition operator F be generated by the function $f(s,u) = a(s) + u^n$, where n is an even number and $a(s)$ is a sequence from the space l_p $(1 \leq p \leq \infty)$. Then the Rhodius spectrum of F is $\sigma_R(F) = \mathbb{R}$ or $(\sigma_R(F) = \mathbb{C})$.*

Proof. Since $a(s) \in l_p$, we can see that operator F can act from l_∞ to l_∞ or according to the Theorem 1.1. F can act from l_p to l_p (or from l_1 to l_p). Denote $a = (a_1, a_2, \ldots) \in l_p$. For $x = (x_1, x_2, \ldots)$ we have

$$Fx = F(x_1, x_2, \ldots) = (a_1 + x_1^n, a_2 + x_2^n, \ldots).$$

Find out if $\lambda I - F$ is an injective operator, for any real λ. Suppose that

$$(\lambda I - F)x = (\lambda I - F)y,$$

for some $x, y \in l_p$. Then

$$(\lambda x_1 - a_1 - x_1^n, \lambda x_2 - a_2 - x_2^n, \ldots) = (\lambda y_1 - a_1 - y_1^n, \lambda y_2 - a_2 - y_2^n, \ldots) \tag{2.1}$$

For $\lambda = 0$ we get

$$\begin{aligned} (-a_1 - x_1^n, -a_2 - x_2^n, \ldots) = (-a_1 - y_1^n, -a_2 - y_2^n, \ldots) &\implies \\ (\forall i \in \mathbb{N}) - a_i - x_i^n = -a_i - y_i^n &\implies \\ (\forall i \in \mathbb{N})\, x_i^n = y_i^n. & \end{aligned}$$

Number n is an even number, so it doesn't have to follow $x_i = y_i, (\forall i \in \mathbb{N})$. This is not injective (nor bijective) mapping so $0 \in \sigma_R(F)$. If $\lambda \neq 0$ then from equality (2.1) we get $(\forall i \in \mathbb{N})$:

$$\begin{aligned} &\lambda x_i - a_i - x_i^n = \lambda y_i - a_i - y_i^n \\ &\lambda x_i - x_i^n = \lambda y_i - y_i^n \iff \lambda(x_i - y_i) = x_i^n - y_i^n \\ &\lambda(x_i - y_i) = (x_i - y_i)\left(x_i^{n-1} + x_i^{n-2} y_i + \ldots + x_i y_i^{n-2} + y_i^{n-1}\right) \implies \\ &(x_i = y_i) \vee \left(x_i^{n-1} + x_i^{n-2} y_i + \ldots + x_i y_i^{n-2} + y_i^{n-1} - \lambda = 0\right). \end{aligned}$$

Hence we get the equation of an odd degree $n-1$, there are always at least one real (nontrivial) solution and $\lambda I - F$ is not injective mapping. We proved that $\lambda I - F$ is not bijective mapping for any real λ, so the Rhodius spectrum of this superposition operator F is : $\sigma_R(F) = \mathbb{R}$. In case that sequences were defined in $\mathbb{C}$ we would get the Rhodius spectrum $\sigma_R(F) = \mathbb{C}$. □

Lemma 2.2. *Let the superposition operator F be generated by the function $f(s,u) = a(s) + u^n$, where n is an even number and $a(s)$ is a sequence from the space l_p $(1 \leq p \leq \infty)$. If $a(s) = 0$, $\forall s \in \mathbb{N}$, then the point spectrum is $\sigma_p(F) = \mathbb{R} \setminus \{0\}$; if $a(s) \leq 0$, $\forall s \in \mathbb{N}$ and $(\exists s \in \mathbb{N}) a(s) < 0$, then $\sigma_p(F) = \mathbb{R}$. If $\sup_{s \in \mathbb{N}} a(s) > 0$ and $n = 2$ then the point spectrum of F is*

$$\sigma_p(F) = \left(-\infty, -2\sqrt{\sup_{s \in \mathbb{N}} a(s)}\right] \cup \left[2\sqrt{\sup_{s \in \mathbb{N}} a(s)}, \infty\right). \tag{2.2}$$

Proof. Find out has the equation $(\lambda I - F)x = 0$ any nontirivial solution. For $\lambda = 0$ we have

$$-Fx = \left(-a_1 - x_1^n, -a_2 - x_2^n, \ldots\right) = (0, 0, \ldots) \implies$$

$$-a_i - x_i^n = 0, \; \forall i \in \mathbb{N}. \tag{2.3}$$

If $a_i = 0$, $\forall i \in \mathbb{N}$, then from (2.3) it follows that there is only a trivial solution $x_i = 0, \forall i \in \mathbb{N}$ (i.e. $x = (0, 0, \ldots)$) and $0 \notin \sigma_p(F)$. If $a_i \leq 0$, $\forall i \in \mathbb{N}$, and $(\exists i \in \mathbb{N}) a_i < 0$, then from (2.3) it follows that $x_i^n = -a_i \geq 0$, $\forall i \in \mathbb{N}$ so there are nontrivial solutions with $x_i = \pm \sqrt[n]{-a_i}$ ($x = (\sqrt[n]{-a_1}, \sqrt[n]{-a_2}, \ldots)$) and it means that $0 \in \sigma_p(F)$. If $\sup\limits_{s \in \mathbb{N}} a(s) > 0$ then there exists a positive number a_i and for such i equation (2.3) has no real solutions. It follows $0 \notin \sigma_p(F)$. Let's see if the equation $(\lambda I - F)x = 0$, for $\lambda \neq 0$, has some solution away from zero

$$(\lambda I - F)(x_1, x_2, \ldots) = \left(\lambda x_1 - a_1 - x_1^n, \lambda x_2 - a_2 - x_2^n, \ldots\right) = (0, 0, \ldots) \implies$$

$$x_i^n - \lambda x_i + a_i = 0, \; \forall i \in \mathbb{N} \tag{2.4}$$

If $a_i = 0$, $\forall i \in \mathbb{N}$, then from (2.4) it follows that

$$x_i \left(x_i^{n-1} - \lambda\right) = 0, \forall i \in \mathbb{N} \implies (x_i = 0) \vee \left(x_i = \sqrt[n-1]{\lambda}\right).$$

There are nontrivial solutions (for example $x = (\sqrt[n-1]{\lambda}, 0, \ldots)$) so $\lambda \in \sigma_p(F)$. If $a_i \leq 0$, $\forall i \in \mathbb{N}$, and $(\exists i \in \mathbb{N}) a_i < 0$, then from (2.4) it follows:

$$x_i^n = \lambda x_i - a_i, \; \forall i \in \mathbb{N} \tag{2.5}$$

It is clear that for $a_i = 0$ there are two solutions $x_{i,1} = 0$ and $x_{i,2} = \sqrt[n-1]{\lambda}$. When $a_i < 0$ we can see that graph of the function $y = \lambda x - a_i$ intersects the graph of the function $y = x^n$. It means that there is a (nontrivial) solution of the equation (2.5). Hence $\lambda \in \sigma_p(F)$. Since $a \in l_p$, sequence a is bounded so $\sup\limits_{s \in \mathbb{N}} a(s) < \infty$. If $\sup\limits_{s \in \mathbb{N}} a(s) > 0$ and $n = 2$ the equations (2.4) become

$$x_i^2 - \lambda x_i + a_i = 0, \; \forall i \in \mathbb{N}. \tag{2.6}$$

For any $i \in \mathbb{N}$ the equation (2.6) has real solutions if its discriminant is nonnegative, i.e. $\lambda^2 - 4a_i \geq 0$. So equations (2.6) has real solutions if $\lambda^2 \geq 4a_i, \forall i \in \mathbb{N}$, which gives us the condition $\lambda^2 \geq 4\sup\limits_{s \in \mathbb{N}} a(s) > 0$. Now we get that for

$$\lambda \in \left(-\infty, -2 \cdot \sqrt{\sup_{s \in \mathbb{N}} a(s)}\right] \cup \left[2 \cdot \sqrt{\sup_{s \in \mathbb{N}} a(s)}, \infty\right) \equiv J$$

the equation $(\lambda I - F)x = 0$ has (nontrivial) solutions, so the point spectrum is

$$\sigma_p(F) = J.$$

$\square$

Example 2.1. *Let the function* $f(s, u) = \frac{1}{s(s+1)} + u^2$ *generate a superposition operator* $F : l_1 \to l_1$.

$$F(x_1, x_2, \ldots) = \left(\frac{1}{1 \cdot 2} + x_1^2, \frac{1}{2 \cdot 3} + x_2^2, \ldots\right).$$

It is not hard to find out for which $\lambda \in \mathbb{R}$, the equation $(\lambda I - F)\,x = 0$ has nontrivial solutions, so we get ***the point spectrum of*** *F:*

$$\sigma_p(F) = \left(-\infty, -\sqrt{2}\right] \cup \left[\sqrt{2}, +\infty\right).$$

We can see that $\sup\limits_{s\in\mathbb{N}} \frac{1}{s(s+1)} = \frac{1}{2}$ and $2\sqrt{\sup\limits_{s\in\mathbb{N}} a(s)} = 2 \cdot \sqrt{\frac{1}{2}} = \sqrt{2}$, so we really get the point spectrum (2.2) *from Lemma 2.2. The operator $\lambda I - F$ is not injective for any $\lambda \in \mathbb{R}$, so* ***the Rhodius spectrum of*** *F* ***is:***

$$\sigma_R(F) = \mathbb{R}.$$

Lemma 2.3. *Let the superposition operator F be generated by the function $f(s,u) = a(s) + \sqrt[n]{|u|}$, where n is an even number and $a(s)$ is a sequence from the space l_p $(1 \le p \le \infty)$. Then the Rhodius spectrum of F is $\sigma_R(F) = \mathbb{R}$ (or $\sigma_R(F) = \mathbb{C}$).*

Proof. Denote $a = (a_1, a_2, \ldots) \in l_p$. For $x = (x_1, x_2, \ldots)$ we have

$$Fx = F(x_1, x_2, \ldots) = \left(a_1 + \sqrt[n]{|x_1|}, a_2 + \sqrt[n]{|x_2|}, \ldots\right).$$

Consider the operator $(\lambda I - F)(x_1, x_2, \ldots) = \left(\lambda x_1 - a_1 - \sqrt[n]{|x_1|}, \lambda x_2 - a_2 - \sqrt[n]{|x_2|}, \ldots\right)$. For $\lambda = 0$, it becomes $-F(x_1, x_2, \ldots) = \left(-a_1 - \sqrt[n]{|x_1|}, -a_2 - \sqrt[n]{|x_2|}, \ldots\right)$. Suppose that

$$-Fx = -Fy,$$

for some $x, y \in l_p$. Then

$$\left(-a_1 - \sqrt[n]{|x_1|}, -a_2 - \sqrt[n]{|x_2|}, \ldots\right) = \left(-a_1 - \sqrt[n]{|y_1|}, -a_2 - \sqrt[n]{|y_2|}, \ldots\right) \implies$$
$$-a_i - \sqrt[n]{|x_i|} = -a_i - \sqrt[n]{|y_i|},\ \forall i \in \mathbb{N} \implies$$
$$|x_i| = |y_i|,\ i \in \mathbb{N}.$$

This is not injective mapping since it doesn't have to follow $x = y$. Hence,

$$0 \in \sigma_R(F). \tag{2.7}$$

Now let $\lambda \neq 0$ and consider the equation $(\lambda I - F)\,x = (\lambda I - F)\,y$ for $x, y \in l_p$. We are interested in does it have to follow: $x = y$.

$$\left(\lambda x_1 - a_1 - \sqrt[n]{|x_1|}, \lambda x_2 - a_2 - \sqrt[n]{|x_2|}, \ldots\right) = \left(\lambda y_1 - a_1 - \sqrt[n]{|y_1|}, \lambda y_2 - a_2 - \sqrt[n]{|y_2|}, \ldots\right)$$
$$\lambda x_i - a_i - \sqrt[n]{|x_i|} = \lambda y_i - a_i - \sqrt[n]{|y_i|},\ \forall i \in \mathbb{N} \implies$$
$$\lambda(x_i - y_i) = \sqrt[n]{|x_i|} - \sqrt[n]{|y_i|},\ \forall i \in \mathbb{N}.$$

There are following possibilities: a) $x_i = y_i$; b) $(x_i \ge 0, y_i \ge 0)$ then

$$\left[(\sqrt[n]{x_i})^{n-1} + (\sqrt[n]{x_i})^{n-2}\sqrt[n]{y_i} + \ldots + (\sqrt[n]{y_i})^{n-1}\right] - \frac{1}{\lambda} = 0;$$

c) $(x_i < 0, y_i < 0)$ then

$$\left[(\sqrt[n]{x_i})^{n-1} + (\sqrt[n]{x_i})^{n-2}\sqrt[n]{y_i} + \ldots + (\sqrt[n]{y_i})^{n-1}\right] + \frac{1}{\lambda} = 0;$$

d) $x_i \cdot y_i < 0$. then

$$\left(\sqrt[n]{|x_i|}\right)^n + \left(\sqrt[n]{|y_i|}\right)^n = \pm\frac{1}{\lambda}\left(\sqrt[n]{|x_i|} - \sqrt[n]{|y_i|}\right).$$

The equation in d) isn't always solvable in $\mathbb{R}$ and anyway then it holds $x \neq y$. The equations b) and c) have at least one real solution x_i for fixed y_i (if we denote $p = \sqrt[n]{|x_i|}$, $q = \sqrt[n]{|y_i|}$ they become an odd-degree polynomial equations) and we have that $x_i \neq y_i$. So from $(\lambda I - F)\, x = (\lambda I - F)\, y$ it doesn't have to follow that $x = y$. We have proved that operator $\lambda I - F$ is not injective for any $\lambda \neq 0$ and that's why

$$\mathbb{R}\backslash\{0\} \subseteq \sigma_R(F) \tag{2.8}$$

From (2.7) and (2.8) it follows that the Rhodius spectrum of this operator F is $\sigma_R(F) = \mathbb{R}$. In case that sequences were defined in $\mathbb{C}$ we would get the Rhodius spectrum $\sigma_R(F) = \mathbb{C}$. □

Lemma 2.4. *Let the superposition operator F be generated by the function $f(s,u) = a(s) + \sqrt[n]{|u|}$, where n is an even number and $a(s)$ is a sequence from the space l_p $(1 \leq p \leq \infty)$. If $a(s) = 0$, $\forall s \in \mathbb{N}$, then the point spectrum is $\sigma_p(F) = \mathbb{R}\backslash\{0\}$; if $a(s) \leq 0$, $\forall s \in \mathbb{N}$ and $(\exists s \in \mathbb{N})a(s) < 0$, then $\sigma_p(F) = \mathbb{R}$; if $(\exists s \in \mathbb{N})a(s) > 0$, then $\sigma_p(F) = \mathbb{R}\backslash\{0\}$.*

Proof. Find out has the equation $(\lambda I - F)x = 0$ any nontirivial solution. For $\lambda = 0$ we have

$$-Fx = \left(-a_1 - \sqrt[n]{|x_1|}, -a_2 - \sqrt[n]{|x_2|}, \ldots\right) = (0,0,\ldots) \implies$$

$$a_i + \sqrt[n]{|x_i|} = 0,\ \forall i \in \mathbb{N}. \tag{2.9}$$

If $a_i = 0$, $\forall i \in \mathbb{N}$, then from (2.9) it follows that there is only a trivial solution $x_i = 0, \forall i \in \mathbb{N}$ (i.e. $x = (0,0,\ldots)$) and $0 \notin \sigma_p(F)$. If $a_i \leq 0$, $\forall i \in \mathbb{N}$, and $(\exists i \in \mathbb{N})a_i < 0$, then from (2.9) it follows that $\sqrt[n]{|x_i|} = -a_i \geq 0$, $\forall i \in \mathbb{N}$ so there are nontrivial solutions with $x_{i_{1,2}} = \pm(-a_i)^n$ and it means that $0 \in \sigma_p(F)$. If $\exists a_i > 0$ then equation (2.9) has no solutions. Let's see if the equation $(\lambda I - F)x = 0$, for $\lambda \neq 0$, has any solution away from zero

$$(\lambda I - F)(x_1, x_2, \ldots) = \left(\lambda x_1 - a_1 - \sqrt[n]{|x_1|}, \lambda x_2 - a_2 - \sqrt[n]{|x_2|}, \ldots\right) = (0,0,\ldots)$$

$$\sqrt[n]{|x_i|} - \lambda x_i + a_i = 0,\ \forall i \in \mathbb{N} \tag{2.10}$$

If $a_i = 0$, $\forall i \in \mathbb{N}$, then from (2.10) it follows that

$$\begin{cases} (x_i = 0) \vee \left(x_i = \lambda^{-\frac{n}{n-1}}\right), & for\ \lambda > 0 \\ (x_i = 0) \vee \left(x_i = -\lambda^{-\frac{n}{n-1}}\right), & for\ \lambda < 0. \end{cases} \quad (\forall i \in \mathbb{N}).$$

There are nontrivial solutions so $\lambda \in \sigma_p(F)$. Denote now:

$$y(x_i) = \sqrt[n]{|x_i|} - \lambda x_i + a_i, \tag{2.11}$$

If $a_i < 0$ we can see that $y(0) = a_i < 0$ and $y\left(\frac{a_i}{\lambda}\right) = \sqrt[n]{\left|\frac{a_i}{\lambda}\right|} > 0$. Since the function y is continuous, from the Intermediate ValueTheorem, it follows that there exists $x_i \neq 0$ such that $y(x_i) = 0$ ($x_i \in \left(0, \frac{a_i}{\lambda}\right)$, if $\lambda < 0$ and $x_i \in \left(\frac{a_i}{\lambda}, 0\right)$, if $\lambda > 0$). It means that there is a (nontrivial) solution of the equations (2.11) and the point spectrum of F (in the case $a(s) \leq 0$, $\forall s \in \mathbb{N}$ and $(\exists s \in \mathbb{N}) a(s) < 0$) is $\sigma_p(F) = \mathbb{R}$. If $a_i > 0$ and $n = 2$ then equation (2.10) becomes:

$$\sqrt{|x_i|} = \lambda x_i - a_i. \tag{2.12}$$

It is not hard to see that this equation is solvable in $\mathbb{R}$ for every $\lambda \neq 0$ (if $\lambda > 0$ then $x_i > 0$; if $\lambda < 0$ then $x_i < 0$). Denote this solution with $x_i = p \in \mathbb{R}$, and consider the equation $\sqrt[n]{|x_i|} = \lambda x_i - a_i$, where n is an even number and $n > 2$. Since p is a solution of the equation (2.12) we have $\sqrt{|p|} = \lambda p - a_i$. From (2.11) we get $y(0) = a_i > 0$ and

$$y(p) = \sqrt[n]{|p|} - \lambda p + a_i = \sqrt[n]{|p|} - (\lambda p - a_i) = \sqrt[n]{|p|} - \sqrt{|p|}.$$

If $|p| > 1$ then $y(p) = \sqrt[n]{|p|} - \sqrt{|p|} < 0$ and from the Intermediate Value Theorem it follows that there is some x_i such that $y(x_i) = \sqrt[n]{|x_i|} - \lambda x_i + a_i = 0$ ($x_i \in (0, p)$ if $p > 1$ or $x_i \in (p, 0)$ if $p < -1$). If $0 < p < 1$ then $\lambda > 0$ and since the function $g(x_i) = \sqrt{|x_i|} - \lambda x_i + a_i$ is continuous (and has the only one solution p) and $g(0) > 0$ then it has to be

$$g(1) = 1 - \lambda + a_i < 0.$$

Now we have that $y(1) = \sqrt[n]{|1|} - \lambda + a_i = 1 - \lambda + a_i < 0$ and since $y(0) = a_i > 0$ and y is a continuous function, it follows that $(\exists x_i \in (0,1))\ y(x_i) = 0$. If $-1 < p < 0$ then $\lambda < 0$ and since the function g is continuous (and has the only one solution p) and $g(0) > 0$, it has to be $g(-1) = \sqrt{|-1|} + \lambda + a_i = 1 + \lambda + a_i < 0$. Now we have that $y(-1) = \sqrt[n]{|-1|} + \lambda + a_i = 1 + \lambda + a_i < 0$ and since $y(0) = a_i > 0$ and y is a continuous function, it follows that $(\exists x_i \in (-1, 0))\ y(x_i) = 0$. So we proved that if $\exists a_i > 0$ then the point spectrum of F is $\sigma_p(F) = \mathbb{R} \setminus \{0\}$. $\square$

Lemma 2.5. *Let the superposition operator F be generated by the function $f(s,u) = a(s) + u^n$, where n is an odd number, $n \geq 3$ and $a(s)$ is a sequence from the space l_p $(1 \leq p \leq \infty)$. Then the Rhodius spectrum of F is $\sigma_R(F) = (0, \infty)$ (or $\sigma_R(F) = \mathbb{C}$).*

Proof. Let the superposition operator $F : l_p \to l_p$, be generated by the function $f(s,u) = a(s) + u^n$, (n is an odd number, $a \in l_p$)

$$Fx = F(x_1, x_2, \ldots) = (a_1 + x_1^n, a_2 + x_2^n, \ldots).$$

Consider the operator

$$(\lambda I - F)(x_1, x_2, \ldots) = (\lambda x_1 - a_1 - x_1^n, \lambda x_2 - a_2 - x_2^n, \ldots).$$

For $\lambda = 0$, the operator $-F$ is injective, because from
$-Fx = -Fy \iff (-a_1 - x_1^n, -a_2 - x_2^n, \ldots) = (-a_1 - y_1^n, -a_2 - y_2^n, \ldots)$, we get

$$-a_i - x_i^n = -a_i - y_i^n, \forall i \in \mathbb{N} \implies x_i^n = y_i^n, \forall i \in \mathbb{N} \implies x = y.$$

The operator $-F$ is surjective because for arbitrary $y \in l_q$ there are some $x \in l_p$ such that $-Fx = y$. Really:

$$-Fx = (-a_1 - x_1^n, -a_2 - x_2^n, \ldots) = (y_1, y_2, \ldots) \iff$$
$$x = (\sqrt[n]{-a_1 - y_1}, \sqrt[n]{-a_2 - y_2}, \ldots).$$

Let now $\lambda \neq 0$:

$$(\lambda I - F)(x_1, x_2, \ldots) = (\lambda I - F)(y_1, y_2, \ldots)$$
$$(\lambda x_1 - a_1 - x_1^n, \lambda x_2 - a_2 - x_2^n, \ldots) = (\lambda y_1 - a_1 - y_1^n, \lambda y_2 - a_2 - y_2^n, \ldots) \tag{2.13}$$

$$\lambda x_i - a_i - x_i^n = \lambda y_i - a_i - y_i^n, \forall i \in \mathbb{N}$$
$$x_i^n - \lambda x_i = y_i^n - \lambda y_i, \forall i \in \mathbb{N}. \tag{2.14}$$

$$\Longleftrightarrow (x_i^n - y_i^n) = \lambda (x_i - y_i), \forall i \in \mathbb{N}. \tag{2.15}$$

From (2.15) we get $(x_i = y_i)$ or

$$x_i^{n-1} + x_i^{n-2} y_i + \ldots + x_i \; y_i^{n-2} + y_i^{n-1} - \lambda = 0. \tag{2.16}$$

If $\lambda < 0$ then for $(x_i \geq 0 \wedge y_i \geq 0)$, or $(x_i \leq 0 \wedge y_i \leq 0)$, we have that

$$x_i^{n-1} + x_i^{n-2} y_i + \ldots + x_i \; y_i^{n-1} + y_i^{n-1} \geq 0 \quad \text{and}$$
$$x_i^{n-1} + x_i^{n-2} y_i + \ldots + x_i \; y_i^{n-1} + y_i^{n-1} - \lambda > 0.$$

If $\lambda < 0$ and $(x_i \geq 0 \wedge y_i \leq 0)$ then: a) for $x_i \geq -y_i$ we have

$$x_i^{n-1} \geq x_i^{n-2}(-y_i)$$
$$x_i^{n-3} y_i^2 \geq x_i^{n-4}(-y_i)^3$$
$$\vdots$$
$$x_i^2 y_i^{n-3} \geq x_i (-y_i)^{n-2}.$$

From these inequalities by summarizing we get

$$x_i^{n-1} + x_i^{n-3} y_i^2 + \ldots + x_i^2 y_i^{n-3} \geq -x_i^{n-2} y_i - x_i^{n-4} y_i^3 - \ldots - x_i y_i^{n-2} \implies$$
$$x_i^{n-1} + x_i^{n-2} y_i + x_i^{n-3} y_i^2 + x_i^{n-4} y_i^3 + \ldots x_i^2 y_i^{n-3} + x_i y_i^{n-2} \geq 0 \; .$$

By adding two members $y_i^{n-1} \geq 0$ and $-\lambda > 0$, to the left side, we get

$$x_i^{n-1} + x_i^{n-2} y_i + \ldots + x_i \; y_i^{n-2} + y_i^{n-1} - \lambda > 0 \; .$$

b) For $x_i \leq -y_i$, we have

$$y_i^{n-1} \geq x_i (-y_i)^{n-2}$$
$$x_i^2 y_i^{n-3} \geq x_i^3 (-y_i)^{n-4}$$
$$\vdots$$
$$x_i^{n-3} y_i^2 \geq x_i^{n-2}(-y_i).$$

From these inequalities by summerizing we get

$$y_i^{n-1} + x_i^2 y_i^{n-3} + \ldots + x_i^{n-3} y_i^2 \geq -x_i y_i^{n-2} - x_i^3 y^{n-4} - \ldots - x_i^{n-2} y_i \implies$$
$$y_i^{n-1} + x_i y_i^{n-2} + x_i^2 y_i^{n-3} + x_i^3 y^{n-4} + \ldots + x_i^{n-3} y_i^2 + x_i^{n-2} y_i \geq 0 \; .$$

By adding two members $x_i^{n-1} \geq 0$ and $-\lambda > 0$, to the left side, we get

$$x_i^{n-1} + x_i^{n-2} y_i + \ldots + x_i \; y_i^{n-2} + y_i^{n-1} - \lambda > 0.$$

If $\lambda < 0$ and $(x_i \leq 0 \wedge y_i \geq 0)$ we can analogously get the same inequality. So any way, from (2.13) it follows that $x = y$ and $\lambda I - F$ is an injective operator (for $\lambda < 0$). We can see from the equations (2.14) that this operator $\lambda I - F$ is injective if the operator $\lambda I - G$ (where G is operator generated by the function $g(s,u) = u^n$, (n is odd number)) is injective. Let's find out if the equation $(\lambda I - G)x = 0$ has any nontrivial solutions for $\lambda > 0$.

$$\begin{aligned} (\lambda I - G)(x_1, x_2, \ldots) &= (0, 0, \ldots) \\ (\lambda x_1 - x_1^n, \lambda x_2 - x_2^n, \ldots) &= (0, 0, \ldots) \\ \lambda x_i - x_i^n &= 0, \forall i \in \mathbb{N} \\ x_i\left(\lambda - x_i^{n-1}\right) &= 0, \forall i \in \mathbb{N} \\ (x_i = 0 \vee x_i^{n-1} &= \lambda), \forall i \in \mathbb{N}. \end{aligned} \tag{2.17}$$

If $\lambda < 0$ then there is only trivial solution $x = (0, 0, \ldots)$. If $\lambda > 0$, then it is possible that $x_i = \pm \sqrt[n-1]{\lambda}$ for some $i \in \mathbb{N}$, so the equation (2.17) has nontrivial solutions also, such as $\left(\sqrt[n-1]{\lambda}, 0, 0, \ldots\right)$. This implies (since $G0 = 0$) for $\lambda > 0$, that operator $\lambda I - G$ is not injective and also $\lambda I - F$ is not injective. It means that positive numbers λ do not belong to the Rhodius resolvent set, so they belong to the Rhodius spectrum, i.e. $(0, \infty) \subseteq \sigma_R(F)$. Let us see for $\lambda \neq 0$ and arbitrary $y \in l_q$, whether exists $x \in l_p$ such that $(\lambda I - F)x = y$.

$$\begin{aligned} (\lambda x_1 - a_1 - x_1^n, \lambda x_2 - a_2 - x_2^n, \ldots) &= (y_1, y_2, \ldots) \implies \\ \lambda x_i - a_i - x_i^n &= y_i, \forall i \in \mathbb{N} \\ x_i^n - \lambda x_i + a_i + y_i &= 0, \forall i \in \mathbb{N}. \end{aligned}$$

These odd-degree polynomial equations have at least one real solutions x_i for every $y_i \in \mathbb{R}$. It means that operator $\lambda I - F$ is onto for $\lambda \neq 0$. For $\lambda \leq 0$ operator $\lambda I - F$ is bijective and now we need to research if $(\lambda I - F)^{-1}$ is a continuous operator. For $\lambda = 0$ we have

$$(-F)^{-1}(x_1, x_2, \ldots) = (\sqrt[n]{-a_1 - x_1}, \sqrt[n]{-a_2 - x_2}, \ldots)$$

and this is continuous mapping. It follows from the Theorem 1.2., because $f(i,u) = \sqrt[n]{-a_i - u}$ are continuous functions $\forall i \in \mathbb{N}$. For $\lambda < 0$:

$$\begin{aligned} (\lambda I - F)(x_1, x_2, \ldots) &= (y_1, y_2, \ldots) \\ (\lambda x_1 - a_1 - x_1^n, \lambda x_2 - a_2 - y_2^n, \ldots) &= (y_1, y_2, \ldots). \end{aligned}$$

The function $f : \mathbb{R} \to \mathbb{R}$, $f(i,u) = \lambda u - a_i - u^n$ is bijective and decreasing (for $\lambda < 0$) and continuous, $\forall i \in \mathbb{N}$, so there exists its inverse $f^{-1}(i,u)$ (which is also bijective, decreasing and continuous function) $\forall i \in \mathbb{N}$. Now from the Theorem 1.2 follows that operator $(\lambda I - F)^{-1}$, generated by $f^{-1}(i,u)$, is continuous operator. We proved that for $\lambda \leq 0$ the operator $(\lambda I - F)$ is bijective and $(\lambda I - F)^{-1}$ is continuous operator, so the Rhodius resolvent set is $\rho_R(F) = (-\infty, 0]$ and the Rhodius spectrum of F is $\sigma_R(F) = (0, \infty)$. □

Lemma 2.6. *Let the superposition operator F be generated by the function $f(s,u) = a(s) + u^n$, where n is an odd number, $n \geq 3$ and $a(s)$ is a sequence from the space l_p $(1 \leq p \leq \infty)$. Then the point spectrum of F is $\sigma_p(F) = (0, \infty)$ if $a(s) = 0, \forall s \in \mathbb{N}$ and $\sigma_p(F) = \mathbb{R}$ if $(\exists s \in \mathbb{N})$ $a(s) \neq 0$.*

Proof. Find out has the equation $(\lambda I - F)x = 0$ any nontirivial solution. For $\lambda = 0$ we have

$$-F(x_1, x_2, \ldots) = (-a_1 - x_1^n, -a_2 - x_2^n, \ldots) = (0, 0, \ldots)$$
$$\iff x_i^n + a_i = 0, \forall i \in \mathbb{N}$$

If $a_i = 0$ then $x_i = 0$; if $a_i \neq 0$ then $x_i = \sqrt[n]{-a_i} \in \mathbb{R}$. Hence for $a(s) = 0, \forall s \in \mathbb{N}$ it values $0 \notin \sigma_p(F)$ and if $(\exists s \in \mathbb{N})\ a(s) \neq 0$ then $0 \in \sigma_p(F)$. Let now $\lambda \neq 0$ and see if the following equation has only trivial solution:

$$(\lambda I - F)(x_1, x_2, \ldots) = (\lambda x_1 - a_1 - x_1^n, \lambda x_2 - a_2 - x_2^n, \ldots) = (0, 0, \ldots)$$
$$\iff x_i^n - \lambda x_i + a_i = 0, \forall i \in \mathbb{N} \tag{2.18}$$

If $a_i = 0$ then

$$x_i\left(x_i^{n-1} - \lambda\right) = 0 \iff \left(x_i = 0 \lor x_i^{n-1} = \lambda\right),$$

so for $\lambda < 0$ (as $n - 1$ is an even number), there is only a trivial solution $x_i = 0$ and for $\lambda > 0$ there is a nontrivial solution also,

$$x_i = \pm \sqrt[n-1]{\lambda}$$

$$\left(x = \left(\sqrt[n-1]{\lambda}, 0, 0, \ldots\right) \text{ is one of nontrivial solutions of the equation } (\lambda I - F)x = 0\right).$$

If $a_i \neq 0$ then equation (2.18) has at least one real (nontrivial) solution x_i because it is an odd-degree polynomial equation. So if $a(s) = 0, \forall s \in \mathbb{N}$ then $\sigma_p(F) = (0, \infty)$. If $(\exists s \in \mathbb{N})\ a(s) \neq 0$ then the point spectrum of F is $\sigma_p(F) = \mathbb{R}$. □

Lemma 2.7. *Let the superposition operator F be generated by the function $f(s, u) = a(s) + \sqrt[n]{u}$, where n is an odd number, $n \geq 3$ and $a(s)$ is a sequence from the space l_p $(1 \leq p \leq \infty)$. Then the Rhodius spectrum of F is $\sigma_R(F) = (0, \infty)$ (or $\sigma_R(F) = \mathbb{C}$).*

Proof. Let the superposition operator F, be generated by the function $f(s, u) = a(s) + \sqrt[n]{u}$, ($n$ is an odd number, $a \in l_p$)

$$Fx = F(x_1, x_2, \ldots) = (a_1 + \sqrt[n]{x_1}, a_2 + \sqrt[n]{x_2}, \ldots).$$

Consider the operator

$$(\lambda I - F)(x_1, x_2, \ldots) = (\lambda x_1 - a_1 - \sqrt[n]{x_1}, \lambda x_2 - a_2 - \sqrt[n]{x_2}, \ldots).$$

For $\lambda = 0$, from

$$-Fx = -Fy \iff (-a_1 - \sqrt[n]{x_1}, -a_2 - \sqrt[n]{x_2}, \ldots) = (-a_1 - \sqrt[n]{y_1}, -a_2 - \sqrt[n]{y_2}, \ldots)$$

we get

$$-a_i - \sqrt[n]{x_i} = -a_i - \sqrt[n]{y_i}, \forall i \in \mathbb{N} \implies \sqrt[n]{x_i} = \sqrt[n]{y_i}, \forall i \in \mathbb{N} \implies x = y.$$

Hence, the operator $-F$ is injective.

For $\lambda \neq 0$ we have

$$\begin{aligned}(\lambda I - F)\,x &= (\lambda I - F)\,y\\ (\lambda x_1 - a_1 - \sqrt[n]{x_1}, \lambda x_2 - a_2 - \sqrt[n]{x_2}, \ldots) &= (\lambda x_1 - a_1 - \sqrt[n]{y_1}, \lambda x_2 - a_2 - \sqrt[n]{y_2}, \ldots)\\ \lambda x_i - a_i - \sqrt[n]{x_i} &= \lambda y_i - a_i - \sqrt[n]{y_i}, \forall i \in \mathbb{N} \implies\\ \lambda x_i - \sqrt[n]{x_i} &= \lambda y_i - \sqrt[n]{y_i}, \forall i \in \mathbb{N} \qquad (2.19)\\ \lambda\,(x_i - y_i) &= (\sqrt[n]{x_i} - \sqrt[n]{y_i})\,, \forall i \in \mathbb{N}.\end{aligned}$$

Now it follows $x_i = y_i$ or

$$(\sqrt[n]{x_i})^{n-1} + (\sqrt[n]{x_i})^{n-2}\,\sqrt[n]{y_i} + \ldots + (\sqrt[n]{y_i})^{n-1} - \frac{1}{\lambda} = 0. \tag{2.20}$$

Denote $\sqrt[n]{x_i} = k_i$ and $\sqrt[n]{y_i} = l_i$ then (2.20) becomes

$$k_i^{n-1} + k_i^{n-2} l_i + \ldots k_i l_i^{n-2} + l_i^{n-1} - \frac{1}{\lambda} = 0, \tag{2.21}$$

which is equivalent to the equation (2.16). We have already shown in the proof of the Lemma 2.5 that for $\lambda < 0$ the equation (2.21) is not true for any k_i and l_i. Hence it must be $x_i = y_i, \forall n \in \mathbb{N}$ and $\lambda I - F$ is injective for $\lambda < 0$. We can see from the equations (2.20) that this operator $\lambda I - F$ is injective if the operator $\lambda I - G$ (where G is operator generated by the function $g\,(s,u) = \sqrt[n]{u}$, (n is odd number)) is injective. Let's find out if the equation $(\lambda I - G)x = 0$ has any nontrivial solutions for $\lambda > 0$.

$$\begin{aligned}&(\lambda I - G)\,(x_1, x_2, \ldots) = (0, 0, \ldots)\\ &(\lambda x_1 - \sqrt[n]{x_1}, \lambda x_2 - \sqrt[n]{x_2}, \ldots) = (0, 0, \ldots)\\ &\iff \lambda x_i - \sqrt[n]{x_i} = 0, \forall i \in \mathbb{N} \iff \sqrt[n]{x_i}\left(\lambda \sqrt[n]{x_i^{n-1}} - 1\right) = 0, \forall i \in \mathbb{N}\\ &\qquad\qquad\qquad\qquad \iff \left(x_i = 0 \ \vee \ x_i^{n-1} = \left(\frac{1}{\lambda}\right)^n\right), \forall i \in \mathbb{N}.\end{aligned}$$

If $\lambda < 0$ then there is only trivial solution $x = (0, 0, \ldots)$. If $\lambda > 0$, then it is possible that $x_i = \pm\lambda^{-\frac{n}{n-1}}$, so there are nontrivial solutions also, such as $\left(\lambda^{-\frac{n}{n-1}}, 0, 0, \ldots\right)$. This implies (since $G0 = 0$) for $\lambda > 0$, that operator $\lambda I - G$ is not injective and also $\lambda I - F$ is not injective. It means that positive numbers λ do not belong to the Rhodius resolvent set, so they belong to the Rhodius spectrum, i.e.$(0, \infty) \subseteq \sigma_R\,(F)$. We are interested now is the operator $(\lambda I - F)$ surjective. For $\lambda = 0$ and given $y \in l_q$ we have:

$$\begin{aligned}(-F)\,(x_1, x_2, \ldots) &= (y_1, y_2, \ldots)\\ (-a_1 - \sqrt[n]{x_1}, -a_2 - \sqrt[n]{x_2}, \ldots) &= (y_1, y_2, \ldots) \implies\\ -a_i - \sqrt[n]{x_i} &= y_i, \ \forall i \in \mathbb{N}\\ x_i &= (-a_i - y_i)^n\,, \ \forall i \in \mathbb{N}.\end{aligned}$$

We get that for every y there is $x = \left((-a_1 - y_1,)^n, (-a_2 - y_2,)^n, \ldots\right)$ such that $(-F)\,x = y$. For $\lambda \neq 0$ and given $y \in l_q$ we have:

$$\begin{aligned}(\lambda I - F)\,(x_1, x_2, \ldots) &= (y_1, y_2, \ldots)\\ (\lambda x_1 - a_1 - \sqrt[n]{x_1}, \lambda x_2 - a_2 - \sqrt[n]{x_2}, \ldots) &= (y_1, y_2, \ldots) \implies\\ \lambda x_i - a_i - \sqrt[n]{x_i} &= y_i, \ \forall i \in \mathbb{N}.\end{aligned}$$

Consider this equation (for $i \in \mathbb{N}$) :

$$\begin{aligned} \sqrt[n]{x_i} &= \lambda x_i - a_i - y_i \implies \\ x_i &= (\lambda x_i - (a_i + y_i))^n \\ x_i &= \sum_{k=0}^{n} \binom{n}{k} (\lambda x_i)^{n-k} \cdot (-(a_i + y_i))^k \end{aligned}$$

$$\lambda^n x_i^n - n\lambda^{n-1} (a_i + y_i) x_i^{n-1} + \ldots + \left(\lambda n (a_i + y_i)^{n-1} - 1\right) x_i - (a_i + y_i)^n = 0.$$

This equation has real solution x_i (since this is an odd-degree polynomial equation), so $(\lambda I - F)$ is surjective operator for every $\lambda \in \mathbb{R}$. For $\lambda \leq 0$ operator $\lambda I - F$ is bijective and now we need to research if $(\lambda I - F)^{-1}$ is a continuous operator. For $\lambda = 0$ we have

$$(-F)^{-1} (x_1, x_2, \ldots) = ((-a_1 - x_1)^n, (-a_2 - x_2)^n, \ldots)$$

and this is continuous mapping. It follows from the Theorem 1.2., because $f(i, u) = (-a_i - y_i)^n$ are continuous functions $\forall i \in \mathbb{N}$. For $\lambda < 0$:

$$\begin{aligned} (\lambda I - F)(x_1, x_2, \ldots) &= (y_1, y_2, \ldots) \\ (\lambda x_1 - a_1 - \sqrt[n]{x_1}, \lambda x_2 - a_2 - \sqrt[n]{x_2}, \ldots) &= (y_1, y_2, \ldots). \end{aligned}$$

The function $f : \mathbb{R} \to \mathbb{R}$, $f(i, u) = \lambda x_i - a_i - \sqrt[n]{x_i}$ is bijective and decreasing (for $\lambda < 0$) and continuous, $\forall i \in \mathbb{N}$, so there exists its inverse $f^{-1}(i, u)$ (which is also bijective, decreasing and continuous function) $\forall i \in \mathbb{N}$. Now from the Theorem 1.2. follows that operator $(\lambda I - F)^{-1}$, generated by $f^{-1}(i, u)$, is a continuous operator. We proved that for $\lambda \leq 0$ the operator $(\lambda I - F)$ is bijective and $(\lambda I - F)^{-1}$ is a continuous operator, so the Rhodius resolvent set is $\rho_R(F) = (-\infty, 0]$ and the Rhodius spectrum of F is $\sigma_R(F) = (0, \infty)$. □

Lemma 2.8. *Let the superposition operator F be generated by the function $f(s, u) = a(s) + \sqrt[n]{u}$, where n is an odd number, $n \geq 3$, and $a(s)$ is a sequence from the space l_p $(1 \leq p \leq \infty)$. Then the point spectrum of F is $\sigma_p(F) = (0, \infty)$ if $a(s) = 0, \forall s \in \mathbb{N}$ and $\sigma_p(F) = \mathbb{R}$ if $(\exists s \in \mathbb{N})$ $a(s) \neq 0$.*

Proof. Consider the equation $(\lambda I - F) x = 0$. For $\lambda = 0$ we have

$$-Fx = (-a_1 - \sqrt[n]{x_1}, -a_2 - \sqrt[n]{x_2}, \ldots) = (0, 0, \ldots)$$

$$-a_i - \sqrt[n]{x_i} = 0, \forall i \in \mathbb{N}.$$

In the case that $a_i = 0, \forall i \in \mathbb{N}$, there is only a trivial solution $x = (0, 0, \ldots)$ and $0 \notin \sigma_p(F)$. In the case that $\exists\ i \in \mathbb{N}, a_i \neq 0$, we have $x_i = (-a_i)^n \neq 0$, so there is also nontrivial solution and $0 \in \sigma_p(F)$. For $\lambda \neq 0$ we have

$$\begin{aligned} (\lambda I - F) x = (\lambda x_1 - a_1 - \sqrt[n]{x_1}, \lambda x_2 - a_2 - \sqrt[n]{x_2}, \ldots) &= (0, 0, \ldots) \\ \lambda x_i - a_i - \sqrt[n]{x_i} &= 0, \forall i \in \mathbb{N}. \end{aligned} \tag{2.22}$$

In the case that $a_i = 0, \forall i \in \mathbb{N}$, we have

$$\sqrt[n]{x_i} \left(1 - \lambda \sqrt[n]{x_i^{n-1}}\right) = 0 \implies (x_i = 0) \vee \left(\sqrt[n]{x_i^{n-1}} = \frac{1}{\lambda}\right).$$

If $\lambda < 0$ the only solution is $x_i = 0$ (since $n-1$ is even number); if $\lambda > 0$ there is nontrivial solution also, $x_i = \pm \sqrt[n-1]{\left(\frac{1}{\lambda}\right)^n}$. Hence the point spectrum of F, if $a(s) = 0, \forall s \in \mathbb{N}$, is $\sigma_p(F) = (0, \infty)$. In the case that $(\exists i \in \mathbb{N}) a_i \neq 0$ then from (2.22) we get

$$x_i = (\lambda x_i - a_i)^n \implies$$
$$\lambda^n x_i^n - n\lambda^{n-1} a_i x_i^{n-1} + \ldots + \left(\lambda n a_i^{n-1} - 1\right) x_i - a_i^n = 0.$$

There exists always a real (nontrivial) solution x_i of this equation since it is an odd-degree polynomial equation. Hence the point spectrum of F, if $(\exists s \in \mathbb{N})\ a(s) \neq 0$, is $\sigma_p(F) = \mathbb{R}$. $\square$

Example 2.2. *Consider an operator $F : l_p \to l_q$ generated by the function $f(s,u) = au^2 + bu$, $(a \neq 0)$. We have that $F0 = 0$.*

$$(\lambda I - F)(x_1, x_2, \ldots) = \left(\lambda x_1 - a x_1^2 - b x_1, \lambda x_2 - a x_2^2 - b x_2, \ldots\right)$$
$$(\lambda I - F)x = 0 \iff \left(\lambda x_i - a x_i^2 - b x_i = 0\right), \forall i \in \mathbb{N}$$
$$x_i\,(a x_i + b - \lambda) = 0, \forall i \in \mathbb{N} \iff \left(x_i = 0 \vee x_i = \frac{\lambda - b}{a}\right), \forall i \in \mathbb{N}.$$

If $\lambda = b$ the equation $(\lambda I - F)x = 0$ has only a trivial solution, hence $b \notin \sigma_p(F)$.

*If $\lambda \neq b$ the equation $(\lambda I - F)x = 0$ has beside trivial solution, also nontrivial solutions (such as $x = \left(\frac{\lambda - b}{a}, 0, 0, \ldots\right)$), hence **the point spectrum is** $\sigma_p(F) = \mathbb{R} \setminus \{b\}$. For $\lambda \in \mathbb{R} \setminus \{b\}$ operator $\lambda I - F$ is not injective. Let's check it for $\lambda = b$.*

$$(bI - F)x = (bI - F)y \implies (b x_i - a x_i^2 - b x_i = b y_i - a y_i^2 - b y_i), \forall i \in \mathbb{N}$$
$$\implies -a x_i^2 = -a y_i^2, \forall i \in \mathbb{N} \Rightarrow x_i = y_i, \forall i \in \mathbb{N}.$$

Hence operator $\lambda I - F$ is not injective (for instance $(1,0,0,\ldots) \neq (-1,0,0,\ldots)$ and $(bI - F)(1,0,0,\ldots) = (bI - F)(-1,0,0,\ldots) = (-a,0,0,\ldots)$).

Operator $\lambda I - F$ is not surjective for any real number λ, because in order to solving operator equation $(\lambda I - F)\,x = y$ we come to solving the square equations

$$\left(\lambda x_i - a x_i^2 - b x_i = y_i\right), \forall i \in \mathbb{N},$$

and they do not have always real solutions (for arbitrary $y_i \in \mathbb{R}$).

(In set of complex numbers $\mathbb{C}$, these solutions always exist, i.e. the operator $\lambda I - F$ is surjective for every $\lambda \in \mathbb{C}$).

*Hence, **the Rhodius spectrum is** $\sigma_R(F) = \mathbb{R}$ **(or** $\sigma_R(F) = \mathbb{C}$**).***

We can summerize Lemma 2.1 and Lemma 2.3 in the following Theorem:

Theorem 2.1. *Let the superposition operator F be generated by the function $f(s,u) = a(s) + |u|^{(2k)^p}$, where $k \in \mathbb{N}$, $p \in \{-1, 1\}$ and $a(s)$ is a sequence from the space l_p $(1 \leq p \leq \infty)$. Then the Rhodius spectrum of F is $\sigma_R(F) = \mathbb{R}$ (or $\sigma_R(F) = \mathbb{C}$).*

We can summerize Lemma 2.5 and Lemma 2.7 in the following Theorem:

Theorem 2.2. *Let the superposition operator F be generated by the function $f(s,u) = a(s) + u^{(2k+1)^p}$, where $k \in \mathbb{N}$, $p \in \{-1, 1\}$ and $a(s)$ is a sequence from the space l_p $(1 \leq p \leq \infty)$. Then the Rhodius spectrum of F is $\sigma_R(F) = (0, \infty)$ (or $\sigma_R(F) = \mathbb{C}$).*

References

[1] F. Dedagich, P.P. Zabreiko: *Operator Superpositions in the Spaces l_p*, Sibirskii Matematicheskii Zhurnal, **28**(1) (1987) 86–98.

[2] J. Appell, E.D. Pascale, A.Vignoli: *Nonlinear Spectral Theory*, Walter de Gruyter, Berlin-New York, 2004.

[3] J. Appell, P.P. Zabreiko: *Nonlinear superposition operators*, Cambridge University Press, 1990.

[4] A. Rhodius: *Über numerische Wertebereiche und Spektralwertrabschätzungen*, Acta Sci. Math. **47** (1984), 465-470.

[5] K.Georg, M.Martelli: *On spectral theory for nonlinear operators*, J. Funct.Anal. **24** (1977), 140–147.

[6] S. Halilović: *Nonlinear Spectral Theories and Solvability of Nonlinear Hammerstein Equations*, Math. Balkanica, N.S. **24**(3-4) (2010), 321–327.

[7] E. Petronela: *Apllications of Spectral Methods to Boundary Value Problems for Differential Equations*, ISRN Mathematical Analysis, **2011**, 1–13.

[8] P. Catana: *Different Spectra for Nonlinear operators*, An. St. Univ. Ovidius Constanta, **13**(1) (2005), 5–14.

[9] J. Appell, E. De Pascale, A. Vignoli: *A comparison of different spectra for nonlinear operators*, Nonlinear Anal., Theory Methods Appl. **40**A (2000), 73–90.

Department of Mathematics
University of Tuzla
Bosnia and Herzegovina
E-mail address: sanela.halilovic@untz.ba

Department of Mathematics
University of Tuzla
Bosnia and Herzegovina
E-mail address: ramiz.vugdalic@untz.ba