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SOME PROPERTIES OF CERTAIN NON-ANALYTIC FUNCTIONS

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ABSTRACT. Let $\mathcal{N}$ be the class of functions $f(z,\overline{z})$ which are non-analytic in the open unit disk $\mathbb{U}$. Many classes of analytic functions $f(z)$ in $\mathbb{U}$ are studied by mathematicians in the world. But, we have only few papers for non-analytic functions $f(z,\overline{z})$ in $\mathbb{U}$. The purpose of the present paper is to discuss some properties of non-analytic functions $f(z,\overline{z})$ in $\mathbb{U}$ with some examples.

1. INTRODUCTION

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n ,$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} \,|\, |z| < 1\}$. If $f(z) \in \mathcal{A}$ maps the unit circle on to the starlike curve with respect to the origin, then $f(z)$ is said to be starlike with respect to the origin in $\mathbb{U}$. Also, if $f(z) \in \mathcal{A}$ maps the unit circle on to the convex curve, then $f(z)$ is said to be convex in $\mathbb{U}$. A function $f(z) \in \mathcal{A}$ is said to be starlike of order α if it satisfies:

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U}),$$

for some real α $(0 \leqq \alpha < 1)$. Also, a function $f(z) \in \mathcal{A}$ is said to be convex of order α if it satisfies

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U}),$$

for some real α $(0 \leqq \alpha < 1)$. It is well known that $f(z)$ is convex of order α in $\mathbb{U}$ if and only if $zf'(z)$ is starlike of order α in $\mathbb{U}$ (see Robertson [3], Komatu [2], Goodman [1]). We see that:

$$f(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \quad (z \in \mathbb{U})$$

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is starlike of order α in $\mathbb{U}$ and that:

$$f(z)=\begin{cases}\dfrac{1-(1-z)^{2\alpha-1}}{2\alpha-1} & \left(\alpha\neq\dfrac{1}{2};z\in\mathbb{U}\right)\\ \\ -\log(1-z) & \left(\alpha=\dfrac{1}{2};z\in\mathbb{U}\right)\end{cases}$$

is convex of order α in $\mathbb{U}$.

Let $\mathcal{N}$ denote the class of functions:

$$f(z,\overline{z})=\sum_{n=1}^{\infty} f_n(z,\overline{z}),$$

defined in $\mathbb{U}$, where $z=x+iy$ and $\overline{z}=x-iy$. If $f(z,\overline{z})\in\mathcal{N}$ maps $\mathbb{U}$ onto the starlike domain, then we say that $f(z,\overline{z})$ is non-analytic and starlike in $\mathbb{U}$. Further, we say that $f(z,\overline{z})$ is non-analytic and convex in $\mathbb{U}$ if it maps $\mathbb{U}$ onto the convex domain. If we consider a function $f(z,\overline{z})$ given by:

$$f(z,\overline{z})=\frac{\overline{z}}{(1-\overline{z})^2}=\overline{z}+\sum_{n=2}^{\infty} n\overline{z}^n,$$

then $f(z,\overline{z})$ is non-analytic and starlike in $\mathbb{U}$, and we see that a function:

$$f(z,\overline{z})=\frac{\overline{z}}{1-\overline{z}}=\overline{z}+\sum_{n=2}^{\infty}\overline{z}^n,$$

is non-analytic and convex in $\mathbb{U}$. But if we consider

$$f(z,\overline{z})=\frac{z}{(1-\overline{z})^2}=z+|z|^2\sum_{n=2}^{\infty} n\overline{z}^{n-2}, \tag{1.1}$$

then $f(z,\overline{z})$ is non-analytic, but it is not starlike in $\mathbb{U}$. Also, if we take

$$f(z,\overline{z})=\frac{z}{1-\overline{z}}=z+|z|^2\sum_{n=2}^{\infty}\overline{z}^{n-2}, \tag{1.2}$$

then $f(z,\overline{z})$ is non-analytic, but it is not convex in $\mathbb{U}$.

2. Jacobian of non-analytic functions

For $z=x+iy$ and $f(z,\overline{z})=u+iv$, Jacobian $J(x,y)$ of $f(z,\overline{z})$ is defined by:

$$J(x,y)=\frac{\partial(u,v)}{\partial(x,y)}=|f_z|^2-|f_{\overline{z}}|^2 \quad (z\in\mathbb{U}).$$

If $J(x,y)>0$, then $f(z,\overline{z})$ is said to be a sense preserving mapping, and if $J(x,y)<0$, then we say that $f(z,\overline{z})$ is sense reversing mapping.

We first consider the following property of $f(z,\overline{z})$.

Theorem 2.1. *Let a function $f(z,\overline{z}) \in \mathcal{N}$ is defined by*

$$f(z,\overline{z}) = z + a_n \overline{z}^k z^{n-k},$$

with some integers n and k which satisfy $2 \leqq 2k \leqq n$. If $|a_n| \leqq \frac{1}{n}$ or $|a_n| \geqq \frac{1}{n-2k}$, then $f(z,\overline{z})$ is sense preserving mapping in $\mathbb{U}$.

Proof. Let us consider the Jacobian $J(x,y)$ of $f(z,\overline{z})$ with $a_n = |a_n|e^{i\phi}$ and $z = re^{i\theta}$. Then we have that:

$$\begin{aligned} J(x,y) &= |f_z|^2 - |f_{\overline{z}}|^2 \\ &= \left|1 + (n-k)a_n \overline{z}^k z^{n-k-1}\right|^2 - \left|k a_n \overline{z}^{k-1} z^{n-k}\right|^2 \\ &= 1 + n(n-k)|a_n|^2 r^{2(n-1)} + 2(n-k)|a_n| r^{n-1} \cos((n-2k-1)\theta + \phi) \\ &\geqq \left(1 - (n-2k)|a_n| r^{n-1}\right)\left(1 - n|a_n| r^{n-1}\right) \\ &> (1 - (n-2k)|a_n|)(1 - n|a_n|) \geqq 0 \end{aligned}$$

for $|a_n| \leqq \frac{1}{n}$ or $|a_n| \geqq \frac{1}{n-2k}$. This shows that $f(z,\overline{z})$ is sense preserving mapping in $\mathbb{U}$. □

Theorem 2.2. *Let*

$$f(z,\overline{z}) = \overline{z} + a_n \overline{z}^{n-k} z^k,$$

with some integers n and k which satisfy $2 \leqq 2k \leqq n$. If $|a_n| \leqq \frac{1}{n}$ or $|a_n| \geqq \frac{1}{n-2k}$, then $f(z,\overline{z})$ is sense reversing mapping in $\mathbb{U}$.

Proof. Letting $a_n = |a_n|e^{i\phi}$ and $z = re^{i\theta}$, we see that

$$\begin{aligned} J(x,y) &= |f_z|^2 - |f_{\overline{z}}|^2 \\ &= \left|k a_n \overline{z}^{n-k} z^{k-1}\right|^2 - \left|1 + (n-k)a_n \overline{z}^{n-k-1} z^k\right|^2 \\ &= -\left(1 + n(n-2k)|a_n|^2 r^{2(n-1)} + 2(n-k)|a_n r^{n-1} \cos((2k+1-n)\theta + \phi)\right) \\ &< -(1 - (n-2k)|a_n|)(1 - n|a_n|) \leqq 0 \end{aligned}$$

for $|a_n| \leqq \frac{1}{n}$ or $|a_n| \geqq \frac{1}{n-2k}$. □

Next, we consider some coefficient problem for sense preserving of $f(z,\overline{z})$.

Theorem 2.3. *If a function $f(z,\overline{z})$ given by*

$$f(z,\overline{z}) = z + \sum_{n=k+1}^{\infty} a_n \overline{z}^k z^{n-k} satisfies:$$

$$\sum_{n=k+1}^{\infty} n|a_n| \leqq 1, \tag{2.1}$$

then $f(z,\overline{z})$ is sense preserving mapping in $\mathbb{U}$.

Proof. We know that it is enough to show that $\left|\frac{f_z}{f_{\overline{z}}}\right| > 1$ by means of the definition for $J(x,y)$. Indeed, we have that

$$\left|\frac{f_z}{f_{\overline{z}}}\right| = \frac{\left|1+\sum_{n=k+1}^{\infty}(n-k)a_n\overline{z}^k z^{n-k-1}\right|}{\left|\sum_{n=k+1}^{\infty}ka_n\overline{z}^k z^{n-k-1}\right|}$$

$$> \frac{1-\sum_{n=k+1}^{\infty}(n-k)|a_n|}{\sum_{n=k+1}^{\infty}k|a_n|}.$$

Therefore, we see that $|f_z| > |f_{\overline{z}}|$ if $f(z,\overline{z})$ satisfies (2.1). This completes the proof of the theorem. □

3. Convexity of some non-analytic functions

In this section, we consider some problems for convexity of $f(z,\overline{z})$.

Theorem 3.1. *A function*

$$f(z,\overline{z}) = z + a_2\overline{z}z + a_3\overline{z}^2 z\,,$$

is non-analytic and convex in $\mathbb{U}$*, where* a_2 *and* a_3 *are real numbers.*

Proof. Writing $z = er^{i\theta}$, we have that:

$$f(z,\overline{z}) = z + a_2|z|^2 + a_3|z|^2\overline{z}$$

$$= re^{i\theta} + a_2r^2 + a_3r^3e^{-i\theta}$$

$$= u + iv\,,$$

where

$$u = a_2r^2 + r(1+a_3r^2)\cos\theta$$

and

$$v = r(1-a_3r^2)\sin\theta\,.$$

It follows that

$$\frac{(u-a_2r^2)^2}{r^2(1+a_3r^2)^2} + \frac{v^2}{r^2(1-a_3r^2)^2} = 1 \quad (0<r<1)\,.$$

Therefor, $f(z,\overline{z})$ maps $|z| \leqq r < 1$ onto the elliptic domain containing the origin which is convex.

Further, letting $r \to 1$, we have that:

$$\frac{(u-a_2)^2}{(1+a_3)^2} + \frac{v^2}{(1-a_3)^2} = 1\,.$$

□

We give an example for Theorem 3.1.

Example 3.1. *If we consider a function*

$$f(z,\overline{z}) = z + \frac{1}{2}\overline{z}z + \frac{1}{3}\overline{z}^2 z\,,$$

then we have that:

$$\frac{\left(u - \frac{1}{2}r^2\right)^2}{r^2\left(1 + \frac{1}{3}r^2\right)^2} + \frac{v^2}{r^2\left(1 - \frac{1}{3}r^2\right)^2} = 1 \quad (0 < r < 1)\,. \tag{3.1}$$

Letting $r \to 1$, the elliptic equation (3.1) becomes:

$$\frac{\left(u - \frac{1}{2}\right)^2}{\left(\frac{4}{3}\right)^2} + \frac{v^2}{\left(\frac{2}{3}\right)^2} = 1\,.$$

This gives us that $f(z,\overline{z})$ is non-analytic, sense preserving and convex in $\mathbb{U}$.

Next, we derive:

Theorem 3.2. *A function*

$$f(z,\overline{z}) = \overline{z} + a_2\overline{z}z + a_3\overline{z}z^2\,,$$

is non-analytic and convex in $\mathbb{U}$, where a_2 and a_3 are real numbers.

Proof. With the same method of the proof of Theorem 3.1, we obtain that:

$$\frac{(u - a_2 r^2)^2}{r^2(1 + a_3 r^2)^2} + \frac{v^2}{r^2(1 - a_3 r^2)^2} = 1 \quad (0 < r < 1)\,,$$

and

$$\frac{(u - a_2)^2}{(1 + a_3)^2} + \frac{v^2}{(1 - a_3)^2} = 1\,.$$

This shows that $f(z,\overline{z})$ is non-analytic and convex in $\mathbb{U}$. □

Example 3.2. *If $f(z,\overline{z})$ is given by:*

$$f(z,\overline{z}) = \overline{z} + \frac{1}{2}\overline{z}z + \frac{1}{3}\overline{z}z^2\,,$$

then we have that:

$$\frac{\left(u - \frac{1}{2}r^2\right)^2}{r^2\left(1 + \frac{1}{3}r^2\right)^2} + \frac{v^2}{r^2\left(1 - \frac{1}{3}r^2\right)^2} = 1 \quad (0 < r < 1)\,,$$

and

$$\frac{\left(u - \frac{1}{2}\right)^2}{\left(\frac{4}{3}\right)^2} + \frac{v^2}{\left(\frac{2}{3}\right)^2} = 1\,.$$

This shows us that $f(z,\overline{z})$ is non-analytic, sense reserving and convex in $\mathbb{U}$.

4. Appendix

We know that the function $f(z,\overline{z})$ given by (1.2) is not starlike in $\mathbb{U}$. But we see that the function $f(z,\overline{z})$ is starlike in $|z| < \dfrac{1}{2}$ and convex in $|z| < 0.44$ with the help of Mathematica as follows.

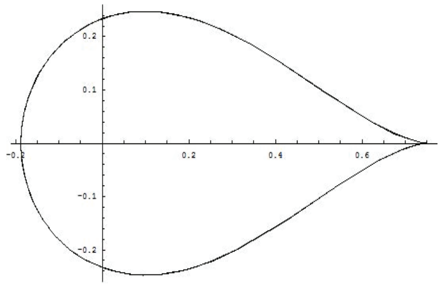

Figure 1. The image for $|z| < \frac{1}{2}$ of $f(z,\overline{z})$

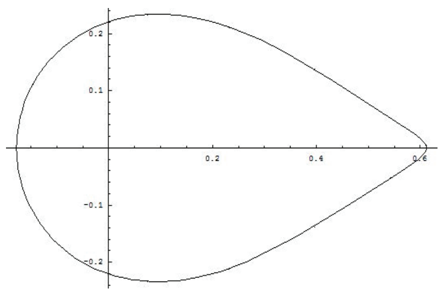

Figure 2. The image for $|z| < 0.44$ of $f(z,\overline{z})$

We also see that the function $f(z,\overline{z})$ defined by (1.1) is not convex in $\mathbb{U}$. But, we say that $f(z,\overline{z})$ is starlike in $|z| < \dfrac{1}{3}$ and convex in $|z| < 0.3$ with the help of Mathematica as follows.

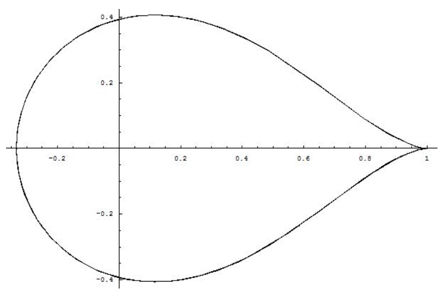

Figure 3. The image for $|z| < \frac{1}{3}$ of $f(z,\overline{z})$

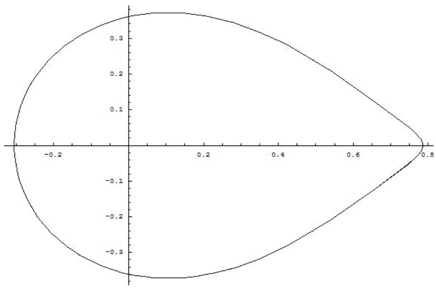

FIGURE 4. The image for $|z| < 0.3$ of $f(z, \overline{z})$

We check some radius of non-analytic functions $f(z, \overline{z})$ to be starlike or convex. But we have to say that we don't have any idea to find the sharp radius for these problems. Thus, we need to do discuss these problems future.

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