ADV MATH SCI JOURNAL

Advances in Mathematics: Scientific Journal **3** (2014), no.2, 117–125
ISSN 1857-8365 UDC: 517.546:517.982.4

DIFFERENTIAL SANDWICH THEOREMS WITH A NEW GENERALIZED DERIVATIVE OPERATOR

ANESSA OSHAH AND MASLINA DARUS[1]

ABSTRACT. In the present paper, we will derive certain subordinations and superordination results involving a new generalized derivative operator $\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d}$ for certain normalized analytic functions in the open unit disk. We shall establish sandwich type theorems. These results extend many previously known results.

1. INTRODUCTION

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For $n \in \mathbb{N}$ and $a \in \mathbb{C}$. Let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ..., a \in \mathbb{C}\,.$$

And, let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z + \sum_{\kappa=2}^{\infty} a_\kappa z^\kappa. \tag{1.1}$$

If f and g are analytic functions in U, we say that f subordinate to g, and write $f(z) \prec g(z)(z \in U)$. If there exists the Schwarz function $w(z)$, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, then $f(z) = g(w(z))(z \in U)$. In particular if g is univalent in U then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $p, h \in \mathcal{H}$ and $\psi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. If p and $\psi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if p satisfies the second order differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \quad (z \in U), \tag{1.2}$$

then p is called a solution of the differential supordination (1.2). (If f subordinate to g then g superordinate to f. An analytic function q is called a subordinant of the differential subordination, or more simply a subordinant if $q \prec p$ for all p satisfying (1.2). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.2) is said to be

---

[1] *corresponding author*

2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Analytic function, derivative operator, differential subordination, differential super ordination.

the best subordinant. The best subordinant is unique up to a rotation of U (see[9]). Recently, Miller and Mocanu [10] obtained conditions on h, q and ψ for which the following implication holds:

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \Rightarrow q \prec p, \quad (z \in U).$$

Ali et al. [2], have used the results of Bulboaca [3] to obtain sufficient conditions for normalized analytic functions $f \in \mathcal{A}$ to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)\,,$$

where q_1 and q_2 are given by univalent functions in U with $q_1(0) = 1$, and $q_2(0) = 1$. Also, Tuneski [15] obtained a sufficient condition for starlikeness of $f \in \mathcal{A}$ in terms of the quantity

$$\frac{f''(z)f(z)}{(f'(z))^2}.$$

Very recently, Shanmugam et al. [13] obtained sufficient conditions for the normalized analytic function $f \in \mathcal{A}$ to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z),$$

and

$$q_1(z) \prec \frac{z^2f'(z)}{(f(z))^2} \prec q_2(z).$$

The Hadamard product of function f and g denoted by $f(z) * g(z)$ is defined by

$$f(z) * g(z) = z + \sum_{\kappa=2}^{\infty} a_\kappa b_\kappa z^\kappa,$$

where

$$g(z) = z + \sum_{\kappa=2}^{\infty} b_\kappa z^\kappa.$$

Now, we define a new generalized derivative operator. However, first we give the following:

$$\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}(z) = z + \sum_{\kappa=2}^{\infty} \Lambda_d^{m,\kappa}(\lambda_1, \lambda_2, \ell) z^\kappa, \tag{1.3}$$

where

$$\Lambda_d^{m,\kappa}(\lambda_1, \lambda_2, \ell) = \left[\frac{\ell(1 + (\lambda_1 + \lambda_2)(\kappa - 1)) + d}{\ell(1 + \lambda_2(\kappa - 1)) + d}\right]^m, \tag{1.4}$$

$m, d \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, $\lambda_2 \geq \lambda_1 \geq 0, \ell \geq 0$, and $\ell + d > 0$.

Definition 1.1. *For $f \in \mathcal{A}$, the operator $\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d}$ is defined by*
$\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d} : \mathcal{A} \to \mathcal{A}$

$$\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z) = \mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}(z) * \mathcal{R}^n f(z)\ ,\ (z \in U),$$

where $m, n, d \in \mathbb{N}_0 = \{0, 1, 2, ...\}, \lambda_2 \geq \lambda_1 \geq 0, \ell \geq 0,$ *,and* $\ell + d > 0$*, and* $\mathcal{R}^n f(z)$ *denotes the Ruscheweyh derivative operator* [11] *given by*

$$\mathcal{R}^n f(z) = z + \sum_{\kappa=2}^{\infty} C(n,\kappa) a_\kappa z^\kappa, \qquad (z \in U)\ ,$$

where $C(n,\kappa) = (n+1)_{\kappa-1}/(1)_{\kappa-1}$.

If f given by (1.1), then we easily find from the equality (1.3) that

$$\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z) = z + \sum_{\kappa=2}^{\infty} \Lambda_d^{m,\kappa}(\lambda_1,\lambda_2,\ell) C(n,\kappa) a_\kappa z^\kappa,$$

where $m, n, d \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, $\lambda_2 \geq \lambda_1 \geq 0, \ell \geq 0$, $\ell + d > 0$, and $C(n,\kappa) = (n+1)_{\kappa-1}/(1)_{\kappa-1}$ and $\Lambda_d^{m,\kappa}(\lambda_1,\lambda_2,\ell)$ defined in (1.4).

Note that, x_κ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(x)_\kappa = \begin{cases} 1, & \kappa = 0, x \in \mathbb{C}\backslash\{0\}; \\ x(x+1)(x+2)....(x+\kappa-1), & \kappa \in \mathbb{N} = \{1,2,3,...\}. \end{cases}$$

In particular

$$\mathcal{D}^{0,0}_{\lambda_1,\lambda_2,\ell,d} f(z) = \mathcal{D}^{0,m}_{0,0,1,0} f(z) = f(z), \mathcal{D}^{1,0}_{\lambda_1,\lambda_2,\ell,d} f(z) = zf'(z).$$

It can be easily shown that

$$\big[\ell(1+\lambda_2(\kappa-1)) + d\big]\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,\ell,d} f(z) = \big[\ell(1+\lambda_2(\kappa-1)-\lambda_1) + d\big]\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z) + \ell\lambda_1 z\big(\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z)\big)', \tag{1.5}$$

$$z\big[\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z)\big]' = (n+1)\mathcal{D}^{n+1,m}_{\lambda_1,\lambda_2,\ell,d} f(z) - n\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z). \tag{1.6}$$

Remark 1.1. *For special cases we have the following :*

- $m = 0$ *, we get Ruscheweyh derivative operator* [11],
- $n = \lambda_2 = d = 0, \lambda_1 = \ell = 1$ *, we get Sălăgean derivative operator* [12],
- $n = \lambda_2 = d = 0, \ell = 1,$*we get derivative operator given by Al-Oboudi* [1],
- $\lambda_2 = d = 0, \ell = 1,$ *we get derivative operator given by Darus and Al-Shaqsi* [6],
- $\lambda_2 = 0, \lambda_1 = \ell = d = 1,$ *we get derivative operator given by Uralegaddi and Somanatha* [16],
- $n = \lambda_2 = 0, \lambda_1 = \ell = 1,$ *we get derivative operator given by Cho and Srivastava* [4],
- $\ell = 1, d = 0,$ *we get derivative operator given by Eljamal and Darus* [8],
- $\ell = 1,$ *we get derivative operator given by El-Yagubi and Darus* [7],
- $n = \lambda_2 = 0, \ell = 1,$ *we get derivative operator given by Catas* [5],
- $\lambda_1 = 1, n = \lambda_2 = 0,$ *we get derivative operator given by Swamy* [14].

The main object of the present paper is to find sufficient conditions for certain normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,\ell,d} f(z)}{\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$.

For our study, we may need the following definitions and lemmas:

Definition 1.2. [10] *Denote by Q the set of functions f that are analytic and injective on $\overline{U}\setminus \mathbf{E}(f)$, where*

$$\mathbf{E}(f)=\{\eta\in\partial U: \lim_{z\to\eta} f(z)=\infty\},$$

and are such that $f'(\eta)\neq 0, \eta\in\partial U\setminus\mathbf{E}(f)$.

Lemma 1.1. [9] *Let q be univalent function in U and let θ and ϕ be analytic functions in a domain D containing $q(U)$, with $\phi(w)\neq 0$ when $w\in q(U)$. Set*

$$Q(z)=zq'(z)\phi[q(z)] \quad , \quad h(z)=\theta[q(z)]+Q(z).$$

Suppose that
(i) Q(z) is starlike univalent in U,
(ii) $\Re\left\{\frac{zh'(z)}{Q(z)}\right\}>0$.
If p is analytic in U, with $p(0)=q(0), p(U)\subset D$, and

$$\theta[p(z)]+zp'(z)\phi[p(z)]\prec\theta[q(z)]+zq'(z)\phi[q(z)], \tag{1.7}$$

then $p\prec q$ and q is the best dominate of (1.7).

Lemma 1.2. [3] *Let q be convex univalent in the unit disc U and ϑ and φ be analytic in a domain D containing $q(U)$. Suppose that*
(i) $\Re\{\frac{\vartheta'[q(z)])}{\varphi[q(z)]}\}>0$,
(ii) $zq'(z)\varphi(q(z))$ is starlike univalent in U.
If $p(z)\in\mathcal{H}[q(0),1]\cap Q$, with $p(U)\subseteq D$, and $\vartheta[p(z)]+zp'(z)\varphi[p(z)]$ is univalent in U, and

$$\vartheta[q(z)]+zq'(z)\varphi[q(z)]\prec\vartheta[p(z)]+zp'(z)\varphi[p(z)]$$

then

$$q(z)\prec p(z), \quad (z\in U)$$

and q(z) is the best subordinant.

2. Main Results

Theorem 2.1. *Let $m,n,d\in\mathbb{N}_0=\{0,1,2,...\}, \lambda_2\geq\lambda_1>0, \ell>0, \chi\in\mathbb{C}$ and q be convex univalent in U with $q(0)=1$. Further, for $\alpha,\delta\in\mathbb{C}$, and $\delta\neq 0$ we assume that*

$$\Re\left\{\frac{\alpha}{\delta}+1+\frac{zq''(z)}{q'(z)}\right\}>0. \tag{2.1}$$

Let

$$\Upsilon_{\kappa}^{m,n}(\lambda_1,\lambda_2,\ell,d,\alpha,\delta;z) \tag{2.2}$$
$$= \left[\alpha-\delta(n+2)+\frac{\delta\big[2\ell(1+\lambda_2(\kappa-1))+2d\big]}{\ell\lambda_1}\right]\frac{\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}$$
$$+\frac{\delta\ell\lambda_1(n+1)(n+2)}{\ell(1+\lambda_2(\kappa-1))+d}\frac{\mathcal{D}^{n+2,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}$$
$$-\frac{\delta\ell\lambda_1(n+1)^2}{\ell(1+\lambda_2(\kappa-1))+d}\frac{\mathcal{D}^{n+1,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}$$
$$-\frac{\delta[\ell(1+\lambda_2(\kappa-1))+d]}{\ell\lambda_1}\left(\frac{\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}\right)^2$$
$$-\delta\left(1-\frac{\ell\lambda_1(n+1)}{\ell(1+\lambda_2(\kappa-1))+d}\right)\left(\frac{\ell(1+\lambda_2(\kappa-1))+d}{\ell\lambda_1}-1\right)+\chi$$

If $f\in\mathcal{A}$ *satisfies*

$$\Upsilon_{\kappa}^{m,n}(\lambda_1,\lambda_2,\ell,d,\alpha,\delta;z)\prec\delta zq'(z)+\alpha q(z)+\chi, \tag{2.3}$$

then $\frac{\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}\prec q(z)$ *and* q *is the best dominant.*

Proof. Define the function p(z) by

$$p(z)=\frac{\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)},\quad z\in U.$$

Then the function p(z) is analytic in U and $p(0)$=1. Therefore, by using (1.5),(1.6) we have

$$\Upsilon_{\kappa}^{m,n}(\lambda_1,\lambda_2,\ell,d,\alpha,\delta;z) \tag{2.4}$$
$$= \left[\alpha-\delta(n+2)+\frac{\delta\big[2\ell(1+\lambda_2(\kappa-1))+2d\big]}{\ell\lambda_1}\right]\frac{\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}$$
$$+\frac{\delta\ell\lambda_1(n+1)(n+2)}{\ell(1+\lambda_2(\kappa-1))+d}\frac{\mathcal{D}^{n+2,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}$$
$$-\frac{\delta\ell\lambda_1(n+1)^2}{\ell(1+\lambda_2(\kappa-1))+d}\frac{\mathcal{D}^{n+1,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}$$
$$-\frac{\delta[\ell(1+\lambda_2(\kappa-1))+d]}{\ell\lambda_1}\left(\frac{\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}\right)^2$$
$$-\delta\left(1-\frac{\ell\lambda_1(n+1)}{\ell(1+\lambda_2(\kappa-1))+d}\right)\left(\frac{\ell(1+\lambda_2(\kappa-1))+d}{\ell\lambda_1}-1\right)+\chi$$
$$=\delta zp'(z)+\alpha p(z)+\chi.$$

By using (2.4) in (2.3) we get

$$\delta zp'(z)+\alpha p(z)+\chi\prec\delta zq'(z)+\alpha q(z)+\chi.$$

By sitting

$$\theta(w)=\alpha w+\chi\quad\text{and}\quad\phi(w)=\delta,$$

it can be easily observed that θ, ϕ are analytic function in $\mathbb{C} \setminus \{0\}$, and that $\phi(w) \neq 0$. Also we see that

$$Q(z) = zq'(z)\phi\big[q(z)\big] = z\delta q'(z),$$

and

$$h(z) = Q(z) + \theta\big[q(z)\big] = z\delta q'(z) + \alpha q(z) + \chi.$$

We can calculate

$$\Re\left\{\frac{zQ'(z)}{Q(z)}\right\} = \Re\left\{1 + \frac{zq''(z)}{q(z)}\right\}.$$

We have q is convex, hence $\Re\left\{\frac{zQ'(z)}{Q(z)}\right\} > 0$, then Q is starlike univalent in U , and

$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\} = \Re\left\{\frac{\alpha}{\delta} + 1 + \frac{zq''(z)}{q(z)}\right\} > 0.$$

Since Q is starlike, and $\Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0,\ z \in U.$ and then, by using Lemma 1.1 we deduce that the subordination (2.3) implies $p(z) \prec q(z)$, and the function q is the best dominant of (2.3). □

Remark 2.1. *Many authors have studied sandwich Theorem, one can refer to* [6] *and* [5] *to fined similar results were obtained earlier.*

For the choices $q(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$ and $q(z) = (\frac{1+z}{1-z})^{\gamma}, 0 < \gamma \leq 1$ in Theorem 2.1, we get the following results.

Corollary 2.1. *Let $m, n, d \in \mathbb{N}_0 = \{0, 1, 2, ...\}, \lambda_2 \geq \lambda_1 > 0, \ell > 0$. Assume that* (2.3) *holds. If $f \in \mathcal{A}$ then,*

$$\Upsilon_{\kappa}^{m,n}(\lambda_1, \lambda_2, \ell, d, \alpha, \delta; z) \prec \frac{\delta(A-B)z}{(1+Bz)^2} + \alpha\left(\frac{1+Az}{1+Bz}\right) + \chi,$$

where $\Upsilon_{\kappa}^{m,n}(\lambda_1, \lambda_2, \ell, d, \alpha, \delta; z)$ is as defined by (2.2)*, then*

$$\frac{\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,\ell,d} f(z)}{\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z)} \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

In particular

$$\Upsilon_{\kappa}^{m,n}(\lambda_1, \lambda_2, \ell, d, \alpha, \delta; z) \prec \frac{2\delta z}{(1+z)^2} + \alpha\left(\frac{1+z}{1-z}\right) + \chi,$$

implies

$$\frac{\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,\ell,d} f(z)}{\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z)} \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is the best dominant.

Corollary 2.2. *Let $m, n, d \in \mathbb{N}_0 = \{0, 1, 2, ...\}, \lambda_2 \geq \lambda_1 > 0, \ell > 0, 0 < \gamma \leq 1$. Assume that* (2.3) *holds. If $f \in \mathcal{A}$ then,*

$$\Upsilon_{\kappa}^{m,n}(\lambda_1, \lambda_2, \ell, d, \alpha, \delta; z) \prec \frac{2\delta\gamma z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\gamma-1} + \alpha\left(\frac{1+z}{1-z}\right)^{\gamma} + \chi,$$

implies

$$\frac{\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\gamma},$$

and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best dominant.*

Theorem 2.2. *Let q be convex univalent in U with* $q(0) = 1$ *, for* $\alpha, \delta \in \mathbb{C}$ *, and* $\delta \neq 0$ *let us assume that*

$$\Re\left\{\frac{\alpha}{\delta} q'(z)\right\} > 0. \tag{2.5}$$

If $f \in \mathcal{A}$,

$$\frac{\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)} \in \mathcal{H}[q(0), 1] \cap Q,$$

and $\Upsilon^{m,n}_{\kappa}(\lambda_1, \lambda_2, \ell, d, \alpha, \delta; z)$ *given by* (2.2) *is univalent in U and satisfies the following superordination*

$$\delta z q'(z) + \alpha q(z) + \chi \prec \Upsilon^{m,n}_{\kappa}(\lambda_1, \lambda_2, \ell, d, \alpha, \delta; z), \tag{2.6}$$

then

$$q(z) \prec \frac{\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)},$$

and q is the best subordinant.

Proof. Taking

$$\theta(w) = \alpha w + \chi \quad \text{and} \quad \phi(w) = \delta,$$

it is easily observed that θ, ϕ are analytic function in $\mathbb{C} \setminus \{0\}$, and that $\phi(w) \neq 0$. Since q is a convex (univalent) function it follows that

$$\Re\left\{\frac{\vartheta'[q(z)])}{\varphi[q(z)]}\right\} = \Re\left\{\frac{\alpha}{\delta} q'(z)\right\} > 0.$$

Thus the assertion (2.6) of Theorem 2.2 follows by an application of Lemma 1.2.

□

Combining Theorem 2.1 and Theorem 2.2, we get the following sandwich theorem for the linear operator $\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)$.

Theorem 2.3. *Let* q_i *be convex univalent in U with* $q_i(0) = 1, (i = 1, 2)$. *Suppose that* $q_1(z)$ *satisfies* (2.5) *and* $q_2(z)$ *satisfies* (2.1).
Let $f \in \mathcal{A}$, $\frac{\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)} \in \mathcal{H}[q(0), 1] \cap Q$, *and* $\Upsilon^{m,n}_{\kappa}(\lambda_1, \lambda_2, \ell, d, \alpha, \delta; z)$ *is univalent in U, where* $\Upsilon^{m,n}_{\kappa}(\lambda_1, \lambda_2, \ell, d, \alpha, \delta; z)$ *is defined in* (2.2), *and*

$$\delta z q_1'(z) + \alpha q_1(z) + \chi \prec \Upsilon^{m,n}_{\kappa}(\lambda_1, \lambda_2, \ell, d, \alpha, \delta; z) \prec \delta z q_2'(z) + \alpha q_2(z) + \chi,$$

then

$$q_1(z) \prec \frac{\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)} \prec q_2(z),$$

and q_1 *and* q_2 *respectively, the best subordinant and the best dominant .*

Corollary 2.3. *Let* $f \in \mathcal{A}$, $\frac{\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)} \in \mathcal{H}[q(0),1] \cap Q$, *and*

$$\frac{\delta(A_1-B_1)z}{(1+B_1z)^2} + \alpha\left(\frac{1+A_1z}{1+B_1z}\right) + \chi \prec \Upsilon^{m,n}_{\kappa}(\lambda_1,\lambda_2,\ell,d,\alpha,\delta;z) \prec \frac{\delta(A_2-B_2)z}{(1+B_2z)^2} + \alpha\left(\frac{1+A_2z}{1+B_2z}\right) + \chi,$$

then $\frac{1+A_1z}{1+B_1z}$ *and* $\frac{1+A_2z}{1+B_2z}$ *respectively, the best subordinant and the best dominant.*

Acknowledgment

The work here is supported by FRGSTOPDOWN/2013/ST06/UKM/01/1.

References

[1] F.M. Al-Oboudi: *On univalent functions defined by derivative operator*, International Journal of Mathematics and Mathematical Sciences, **27** (2004), 1429-1436.
[2] R. M. Ali, V. Ravichandran, K. G. Subramanian: *Differential sandwich theorems for certain analytic functions*, Far East J. Math. Sci., **15**(1) (2004), 87–94.
[3] T. Bulboaca: *Classes of first order differential superordinations*, Demonstratio Math., **35**(2) (2002), 287–292.
[4] N.E. Cho, H.M. Srivastava: *Argument estimates of certain analytic functions defined by a class of multiplier transformations*, Math. Comput. Modelling, **37**(1-2) (2003), 39–49.
[5] A. Catas: *On a certain differential sandwich theorem associated with a new generalized derivative Operator*, General Mathematics, *17*(4) (2009), 83–95.
[6] M. Darus, K. Al-Shaqsi: *Differential sandwich theorems with generalize derivative operator*, International Journal of Computing and Mathematical Sciences, **22** (2008), 75–78.
[7] E. El-Yaguby, M.Darus: *Subclasses of analytic functions defined by new generalized derivative operator*, Journal of Quality Measurement and Analysis (JQMA), **9**(1) (2013), 47–56.
[8] E. A. Eljamal, M.Darus: *Subordination results defined by a new differential operator*, Acta Universitatis Apulensis, **27** (2011), 121-126.
[9] S. S. Miller, P.T. Mocanu: *Differential Subordinations Theory and Applications*, Pure and Applied Mathematics, No. 225, Marcel Dekker, New York, 2000.
[10] S. S. Miller, P. T. Mocanu: *Subordinates of differential superordinations*, Complex Var. Elliptic Equ., **48**(10) (2003), 815–826.
[11] S. Ruscheweyh: *A new criteria for univalent function*, Proc. Amer. Math. Soc., **49** (1975), 109 – 115.
[12] G.S. Sălăgean: *Subclasses of univalent functions*, in Proceedings of the Complex Analysis 5th Romanian Finnish Seminar Part 1, **1013** (1983), 362–372.
[13] T. N. Shanmugam, V. Ravichandran, S. Sivasubramanian: *Differential sandwich theorems for some subclasses of analytic functions*, Aust. J. Math. Anal. Appl., **3** (1) (2006) Art. 8, 11 pp.
[14] S.R. Swamy: *Inclusion properties of certain subclasses of analytic functions*, International Mathematical Forum, **7**(36) (2012) 1751–1760.
[15] N. Tuneski: *On certain sufficient conditions for starlikeness*, Int. J. Math. Math. Sci., **23**(8) (2000), 521–527.
[16] B.A. Uralegaddi, C. Somanatha: *Certain classes of univalent functions*, In Current Topics in Analytic Function Theory, River Edge, NJ:World Scientific, (1992), 371–374.

School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600 Selangor, Malaysia
E-mail address: anessa.oshah@yahoo.com

School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600 Selangor, Malaysia
E-mail address: maslina@ukm.edu.my