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SOME STARLIKENESS CONDITIONS CONCERNED WITH THE SECOND COEFFICIENT

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Abstract. Let $\mathcal{A}$ be the class of analytic functions $f(z) = z + a_2z^2 + \cdots$ in the open unit disk $\mathbb{U}$. Some starlikeness conditions for $f(z) \in \mathcal{A}$ missing the second coefficient a_2 were given by V. Singh (see [4]). By considering starlikeness of order α for $f(z) \in \mathcal{A}$ with $a_2 \neq 0$, some starlikeness conditions concerned with the second coefficient a_2 are discussed.

1. Introduction

Let $\mathcal{H}$ denote the class of functions $f(z)$ which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For a positive integer n, let $\mathcal{A}_n$ be the class of functions $f(z) \in \mathcal{H}$ of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$$

with $\mathcal{A}_1 = \mathcal{A}$. The subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in $\mathbb{U}$ is denoted by $\mathcal{S}$. In 1972, Ozaki and Nunokawa [3] proved a univalence criterion for $f(z) \in \mathcal{A}$ as follows.

Lemma 1.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\left| \frac{z^2 f'(z)}{\big(f(z)\big)^2} - 1 \right| < 1 \qquad (z \in \mathbb{U}),$$

then $f(z)$ is univalent in $\mathbb{U}$, which means that $f(z) \in \mathcal{S}$.

Moreover, let $\mathcal{T}_n(\mu)$ denote the class of functions $f(z) \in \mathcal{A}_n$ which satisfy the inequality

$$\left| \frac{z^2 f'(z)}{\big(f(z)\big)^2} - 1 \right| < \mu \qquad (z \in \mathbb{U})$$

for some real number μ with $0 < \mu \leq 1$ and $\mathcal{T}_n(1) = \mathcal{T}_n$. The assertion in Lemma 1.1 gives us that $\mathcal{T}_n(\mu) \subset \mathcal{T}_n \subset \mathcal{S}$.

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A function $f(z) \in \mathcal{A}$ is said to be starlike of order α in $\mathbb{U}$ if it satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}) \tag{1.1}$$

for some real number α with $0 \leq \alpha < 1$. This class is denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{S}^*(0) = \mathcal{S}^*$. It is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^* \subset \mathcal{S}$.

For a positive integer n, we define by $\mathcal{B}_n$ the class of functions $w(z) \in \mathcal{H}$ of the form

$$w(z) = \sum_{k=n}^{\infty} c_k z^k$$

which satisfy the inequality $|w(z)| < 1 \quad (z \in \mathbb{U})$. The following lemma is well-known as Schwarz's lemma (see [1]).

Lemma 1.2. *If $w(z) \in \mathcal{B}_n$, then*

$$|w(z)| \leq |z|^n \tag{1.2}$$

for each point $z \in \mathbb{U}$. The equality in (1.2) is attended for $w(z) = e^{i\varphi} z^n \quad (\varphi \in \mathbb{R})$.

Applying Lemma 1.2 with $n = 2$, Singh [4] discussed starlikeness for $f(z) \in \mathcal{T}_2(\mu)$.

Lemma 1.3. *If $f(z) \in \mathcal{A}_2$ satisfies*

$$\left|\frac{z^2 f'(z)}{\left(f(z)\right)^2} - 1\right| < \frac{1}{\sqrt{2}} \qquad (z \in \mathbb{U}),$$

then $f(z) \in \mathcal{S}^*$. This means that $\mathcal{T}_2(\mu)$ is a subclass of $\mathcal{S}^*$ for $0 < \mu \leq \dfrac{1}{\sqrt{2}}$.

Furthermore, Kuroki, Hayami, Uyanik and Owa in [2] deduced some sufficient condition for $f(z) \in \mathcal{A}_n$ to be starlike of order α in $\mathbb{U}$.

Lemma 1.4. *If $f(z) \in \mathcal{A}_n$ with $n \neq 1$ satisfies*

$$\left|\frac{z^2 f'(z)}{\left(f(z)\right)^2} - 1\right| < \frac{(n-1)(1-\alpha)}{\sqrt{(n-1+\alpha)^2 + (1-\alpha)^2}} \qquad (z \in \mathbb{U})$$

for some real number α with $0 \leq \alpha < 1$, then $f(z) \in \mathcal{S}^*(\alpha)$.

In view of Lemma 1.3, Singh [4] discussed some starlikeness condition for $f(z) \in \mathcal{A}$ missing the second coefficient a_2. In the present paper, we consider starlikeness of order α for $f(z) \in \mathcal{A}$ with $a_2 \neq 0$.

2. Main result

By using a certain method of the proof of Lemma 1.4 which was discussed by Kuroki, Hayami, Uyanik and Owa in [2], we deduce some sufficient condition for $f(z) \in \mathcal{A}$ to be starlike of order α in $\mathbb{U}$.

Theorem 2.1. *If* $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k \in \mathcal{A}$ *satisfies*

$$\left| \frac{z^2 f'(z)}{\big(f(z)\big)^2} - 1 \right| < \frac{(1-\alpha)\sqrt{2(1+\alpha^2) - |a_2|^2} - (1-\alpha+2\alpha^2)|a_2|}{2(1+\alpha^2)} \qquad (z \in \mathbb{U})$$

for some real number α *with* $0 \leq \alpha < 1$*, then* $f(z) \in \mathcal{S}^*(\alpha)$*.*

Proof. Suppose that $f(z) \in \mathcal{T}_1(\mu)$. We define the function $w(z)$ by

$$w(z) = \frac{1}{\mu}\left(\frac{z^2 f'(z)}{\big(f(z)\big)^2} - 1 \right) \qquad (z \in \mathbb{U}). \tag{2.1}$$

Noting that

$$\frac{z^2 f'(z)}{\big(f(z)\big)^2} - 1 = \big(a_3 - a_2{}^2\big) z^2 + 2\big(a_4 - 2a_2 a_3 + a_2{}^3\big) z^3 + \cdots ,$$

we have $w(z) \in \mathcal{B}_2$. Dividing the equality (2.1) by z^2 and integrating from 0 to z, we obtain that

$$-\frac{1}{f(z)} + \frac{1}{z} - a_2 = \mu \int_0^z \frac{w(\zeta)}{\zeta^2}\, d\zeta = \frac{\mu}{z} \int_0^1 \frac{w(tz)}{t^2}\, dt \qquad (\zeta = tz),$$

which implies that

$$\frac{z}{f(z)} = 1 - a_2 z - \mu \int_0^1 \frac{w(tz)}{t^2}\, dt \qquad (z \in \mathbb{U}).$$

Thus, we find

$$\frac{z f'(z)}{f(z)} = \frac{1 + \mu\, w(z)}{1 - a_2 z - \mu\, W(z)} \qquad (z \in \mathbb{U}),$$

where

$$W(z) = \int_0^1 \frac{w(tz)}{t^2}\, dt \qquad (z \in \mathbb{U}).$$

In addition, the assertion in Lemma 1.2 with $n = 2$ gives us that

$$|w(z)| \leq |z|^2 \tag{2.2}$$

and

$$\big|W(z)\big| = \left| \int_0^1 \frac{w(tz)}{t^2}\, dt \right| \leq |z|^2 \tag{2.3}$$

for $z \in \mathbb{U}$. Since

$$\left. \frac{z f'(z)}{f(z)} \right|_{z=0} = 1,$$

the inequality (1.1) is equivalent to the condition

$$\frac{z f'(z)}{f(z)} = \frac{1 + \mu\, w(z)}{1 - a_2 z - \mu\, W(z)} \neq \alpha + ik \qquad (z \in \mathbb{U}), \tag{2.4}$$

where $0 \leq \alpha < 1$ and k is any real number. A simple calculation yields that the condition (2.4) is equivalent to

$$F(z) \neq \frac{2(1-\alpha)}{\mu} \qquad (z \in \mathbb{U}), \tag{2.5}$$

where

$$F(z)=\left\{(1-2\alpha)\left(W(z)+\frac{a_2}{\mu}z\right)-w(z)\right\}+\frac{1-\alpha+ik}{1-\alpha-ik}\left(W(z)+\frac{a_2}{\mu}z+w(z)\right).$$

We notice that the condition (2.5) holds true for

$$\sup_{z\in\mathbb{U},\, w(z)\in\mathcal{B}_2,\, k\in\mathbb{R}}|F(z)|\leq\frac{2(1-\alpha)}{\mu}. \tag{2.6}$$

It follows from (2.2) and (2.3) that

$$\sup_{z\in\mathbb{U},\, w(z)\in\mathcal{B}_2,\, k\in\mathbb{R}}|F(z)|$$

$$\leq\sup_{z\in\mathbb{U},\, w(z)\in\mathcal{B}_2}\left\{\left|(1-2\alpha)\left(W(z)+\frac{a_2}{\mu}z\right)-w(z)\right|+\left|W(z)+\frac{a_2}{\mu}z+w(z)\right|\right\}$$

$$\leq\sup_{z\in\mathbb{U},\, w(z)\in\mathcal{B}_2}\sqrt{2\left(\left|(1-2\alpha)\left(W(z)+\frac{a_2}{\mu}z\right)-w(z)\right|^2+\left|W(z)+\frac{a_2}{\mu}z+w(z)\right|^2\right)}$$

$$\leq 2\sup_{z\in\mathbb{U},\, w(z)\in\mathcal{B}_2}\sqrt{(1-\alpha)^2\left(|W(z)|+\frac{|a_2|}{\mu}|z|\right)^2+\left\{|w(z)|+\alpha\left(|W(z)|+\frac{|a_2|}{\mu}|z|\right)\right\}^2}$$

$$\leq 2\sqrt{(1-\alpha)^2\left(1+\frac{|a_2|}{\mu}\right)^2+\left\{1+\alpha\left(1+\frac{|a_2|}{\mu}\right)\right\}^2}.$$

We note that

$$\sqrt{(1-\alpha)^2\left(1+\frac{|a_2|}{\mu}\right)^2+\left\{1+\alpha\left(1+\frac{|a_2|}{\mu}\right)\right\}^2}\leq\frac{1-\alpha}{\mu},$$

if

$$0<\mu\leq\frac{(1-\alpha)\sqrt{2(1+\alpha^2)-|a_2|^2}-(1-\alpha+2\alpha^2)|a_2|}{2(1+\alpha^2)} \tag{2.7}$$

with $0\leq\alpha<1$. From this fact, we see that $F(z)$ satisfies the inequality (2.6) for some real number μ with the condition (2.7). Therefore, we find that $f(z)\in\mathcal{T}_1(\mu)$ satisfies the inequality (1.1) for some real number μ with the condition (2.7). This completes the proof of Theorem 2.1. $\square$

Remark 2.1. *From Theorem 2.1, we find that the class $\mathcal{T}_1(\mu)$ is a subclass of $\mathcal{S}^*(\alpha)$ for some real number μ with the condition* (2.7).

Remark 2.2. *If we take $a_2=0$ in Theorem 2.1, then we obtain the assertion of Lemma 1.4 with $n=2$.*

3. Some special cases

In this section, we discuss some special cases of Theorem 2.1 and their examples.

Letting $\alpha = 0$ in Theorem 2.1, we obtain

Corollary 3.1. *If $f(z) = z + a_2 z^2 + \cdots \in \mathcal{A}$ satisfies*

$$\left| \frac{z^2 f'(z)}{\big(f(z)\big)^2} - 1 \right| < \frac{\sqrt{2 - |a_2|^2} - |a_2|}{2} \qquad (z \in \mathbb{U}), \tag{3.1}$$

then $f(z) \in \mathcal{S}^*$.

Example 3.1. *Noting that*

$$\frac{\sqrt{2 - |a_2|^2} - |a_2|}{2} = \frac{1}{2} \qquad \text{when} \quad a_2 = \frac{\sqrt{3} - 1}{2},$$

let us consider the function $f(z)$ given by

$$f(z) = \frac{z}{1 - \frac{\sqrt{3}-1}{2} z - \frac{1}{2} z^2} = z + \frac{\sqrt{3} - 1}{2} z^2 + \frac{3 - \sqrt{3}}{2} z^3 + \cdots \qquad (z \in \mathbb{U}) \tag{3.2}$$

in Corollary 3.1. It follows from (3.2) that

$$\left| \frac{z^2 f(z)}{\big(f(z)\big)^2} - 1 \right| = \left| \frac{1}{2} z^2 \right| < \frac{1}{2} \qquad (z \in \mathbb{U}).$$

Thus, we find that $f(z)$ given by (3.2) satisfies the inequality (3.1) with $a_2 = \dfrac{\sqrt{3} - 1}{2}$.

On the other hand, a simple check gives us that

$$\operatorname{Re}\left(\frac{z f'(z)}{f(z)} \right) = \operatorname{Re}\left(\frac{1 + \frac{1}{2} z^2}{1 - \frac{\sqrt{3}-1}{2} z - \frac{1}{2} z^2} \right)$$

$$> \frac{33 + \big(5\sqrt{3} - 3\big)\sqrt{12\sqrt{3} - 6}}{198} = 0.2765\cdots > 0 \qquad (z \in \mathbb{U}).$$

This leads that $f(z)$ given by (3.2) belongs to the class $\mathcal{S}^*$.

Furthermore, putting $\alpha = \dfrac{1}{2}$ in Theorem 2.1, we have

Corollary 3.2. *If $f(z) = z + a_2 z^2 + \cdots \in \mathcal{A}$ satisfies:*

$$\left| \frac{z^2 f'(z)}{\big(f(z)\big)^2} - 1 \right| < \frac{\sqrt{\frac{5}{2} - |a_2|^2} - 2|a_2|}{5} \qquad (z \in \mathbb{U}), \tag{3.3}$$

then $f(z) \in \mathcal{S}^* \left(\dfrac{1}{2} \right)$.

Example 3.2. *Noting that*

$$\frac{\sqrt{\frac{5}{2} - |a_2|^2} - 2|a_2|}{5} = \frac{1}{10} \qquad \text{when} \quad a_2 = \frac{1}{2},$$

let us consider the function $f(z)$ given by

$$f(z) = \frac{z}{1 - \frac{1}{2}z - \frac{1}{10}z^2} = z + \frac{1}{2}z^2 + \frac{1}{20}z^3 + \cdots \qquad (z \in \mathbb{U}) \tag{3.4}$$

in Corollary 3.2. It is easy to check that

$$\left| \frac{z^2 f(z)}{\left(f(z)\right)^2} - 1 \right| = \left| \frac{1}{10} z^2 \right| < \frac{1}{10} \qquad (z \in \mathbb{U}) .$$

Then, we see that $f(z)$ given by (3.4) satisfies the inequality (3.3) with $a_2 = \frac{1}{2}$.

Moreover, we can observe that

$$\begin{aligned} \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) &= \operatorname{Re}\left(\frac{1 + \frac{1}{10}z^2}{1 - \frac{1}{2}z - \frac{1}{10}z^2}\right) \\ &> \frac{10153 + 792\sqrt{71}}{24534} = 0.6858\cdots > \frac{1}{2} \qquad (z \in \mathbb{U}), \end{aligned}$$

which implies that $f(z)$ given by (3.4) belongs to the class $\mathcal{S}^*\left(\frac{1}{2}\right)$.

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