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PROXIMAL VORONOÏ REGIONS, CONVEX POLYGONS, & LEADER UNFORM TOPOLOGY

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Dedicated to the Memory of Som Naimpally

ABSTRACT. This article introduces proximal Voronoï regions. A main result in this paper is the proof that proximal Voronoï regions are convex polygons. In addition, it is proved that every collection of proximal Voronoï regions has a Leader uniform topology.

1. INTRODUCTION

Klee-Phelps convexity [8, 12] and related work [11] are viewed here in terms of Voronoï regions [13, 14, 15]. A nonempty set A of a space X is a *convex set*, provided $\alpha A + (1-\alpha)A \subset A$ for each $\alpha \in [0,1]$ [1, §1.1, p. 4]. A *simple convex set* is a closed half plane (all points on or on one side of a line in R^2).

Lemma 1.1. [6, §2.1, p. 9] *The intersection of convex sets is convex.*

Proof. Let $A, B \subset \mathbb{R}^2$ be convex sets and let $K = A \cap B$. For every pair points $x, y \in K$, the line segment $\overline{xy}$ connecting x and y belongs to K, since this property holds for all points in A and B. Hence, K is convex. □

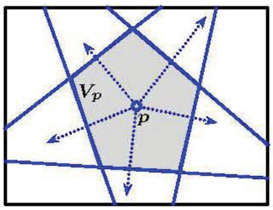

FIGURE 1. V_p = Intersection of closed half-planes

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Let $S \subset \mathbb{R}^2$ be a finite set of n points called sites, $p \in S$. The set S is called the *generating set* [7]. Let H_{pq} be the closed half plane of points at least as close to p as to $q \in S \setminus \{p\}$, defined by

$$H_{pq} = \left\{ x \in R^2 : \|x - p\| \underset{q \in S}{\leq} \|x - q\| \right\}.$$

A *convex polygon* is the intersection of finitely many half-planes [5, §I.1, p. 2]. See, for example, Fig. 1.

Remark 1.1. The Voronoï region V_p depicted as the intersection of finitely many closed half planes in Fig. 1 is a variation of the representation of a Voronoï region in the monograph by H. Edelsbrunner [6, §2.1, p. 10], where each half plane is defined by its outward directed normal vector. The rays from p and perpendicular to the sides of V_p are comparable to the lines leading from the center of the convex polygon in G.L. Dirichlet's drawing [3, §3, p. 216].

2. Preliminaries

Let $S \subset E$, a finite-dimensional normed linear space. Elements of S are called sites to distinguish them from other points in E [6, §2.2, p. 10]. Let $p \in S$. A *Voronoï region* of $p \in S$ (denoted V_p) is defined by

$$V_p = \left\{ x \in E : \|x - p\| \underset{\forall q \in S}{\leq} \|x - q\| \right\}.$$

Remark 2.1. A Voronoï region of a site $p \in S$ contains every point in the plane that is closer to p than to any other site in S [7, §1.1, p. 99]. Let V_p, V_q be Voronoï polygons. If $V_p \cap V_q$ is a line, ray or line segment, then it is called a *Voronoï edge*. If the intersection of three or more Voronoï regions is a point, that point is called a *Voronoï vertex.*

Lemma 2.1. *A Voronoï region of a point is the intersection of closed half planes and each region is a convex polygon.*

Proof. From the definition of a closed half-plane

$$H_{pq} = \left\{ x \in R^2 : \|x - p\| \underset{q \in S}{\leq} \|x - q\| \right\},$$

V_p is the intersection of closed half-planes H_{pq}, for all $q \in S - \{p\}$ [5], forming a polygon. From Lemma 1.1, V_p is a convex. □

A Voronoi diagram of S (denoted by $\mathbb{V}$) is the set of Voronoi regions, one for each site $p \in S$, defined by

$$\mathbb{V} = \bigcup_{p \in S} V_p.$$

Example 2.1. Centroids as Sites in an Image Tessellation.
Let E be a segmentation of a digital image and let $S \subset E$ be a set of sites, where each site is the centroid of a segment in E. In a centroidal approach to the Voronoï tessellation of E, a Voronoï region V_p is defined by the intersection of closed half

plains determined by centroid $p \in S$. *The centroidal approach to Voronoi tessellation was introduced by Q. Du, V. Faber, M. Gunzburger* [4].

3. Main Results

Let V_p, V_z be Voronoï regions of $p, z \in S$, a set of Voronoï sites in a finite-dimensional normed linear Space E that is topological, cl$A$ the closure of a nonempty set A in E. V_p, V_z are *proximal* (denoted by $V_p\ \delta\ V_z$), provided $\mathbb{P} = \mathrm{cl}V_p \cap \mathrm{cl}V_z \neq \emptyset$ [2]. The set $\mathbb{P}$ is called a *proximal Voronoï region.*

Theorem 3.1. *Proximal Voronoï regions are convex polygons.*

Proof. Let $\mathbb{P}$ be a proximal Voronoï region. By definition, $\mathbb{P}$ is the nonempty intersection of convex sets. From Lemma 1.1, $\mathbb{P}$ is convex. Consequently, $\mathbb{P}$ is the intersection of finitely many closed half planes. Hence, from Lemma 2.1, $\mathbb{P}$ is a Voronoï region of a point and is a convex polygon. □

Corollary 3.1. *The intersection of proximal Voronoï regions is either a Voronoï edge or Voronoï point.*

Any two Voronoï regions intersect at least a vertex and at most along their boundaries. Together, the set of Voronoï regions $\mathbb{V}$ cover the entire plane [5, §2.2, p. 10]. For a set of sites $S \subset E$, a Voronoï diagram $\mathbb{D}$ of S is the set of Voronoi regions, one for each site in S.

Corollary 3.2. *A Voronoï diagram* $\mathbb{D}$ *equals* $\mathbb{V}$.

The partition of a plane E with a finite set of n sites into n Voronoï polygons is known as a Dirichlet tessellation, named after G.L. Dirichlet [16] (see [3]). A *cover* (covering) of a space X is a collection $\mathcal{U}$ of subsets of X whose union contains X (*i.e.*, $\mathcal{U} \supseteq X$) [17, §15], [10, §7.1].

Corollary 3.3. *A Dirichlet tessellation* $\mathbb{D}$ *of the Euclidean plane* E *is a covering of* E.

Recall that the Euclidean space $E = R^2$ is a metric space. The topology in a metric space results from determining which points are close to each set in the space. A point $x \in E$ is close to $A \subset E$, provided the Hausdorff distance $d(x, A) = inf\,\{||x - a|| : a \in A\} = 0$. Let X, Y be a pair of metric spaces, $f : X \longrightarrow Y$ is a function such that for each $x \in X$, there is a unique $f(x) \in Y$. A continuous function preserves the closeness (proximity) between points and sets, *i.e.*, $f(x)$ is close to $f(B)$ whenever x is close to B. In a proximity space, one set A is near another set B, provided $A\ \delta\ B$, *i.e.*, the closure of A has at least one element in common with the closure of B. The set A is close to the set B, provided the Čech distance $D(A, B) = inf\,\{||a - b|| : a \in A, b \in B\} = 0$. In that case, we write $A\ \delta\ B$ (A and B are proximal). A *uniformly continuous mapping* is a function that preserves proximity between sets, *i.e.*, $f(A)\ \delta\ f(B)$ whenever $A\ \delta\ B$. A *Leader uniform topology* is determined by finding those points that are close to each given set in E.

Theorem 3.2. *Let S be a set of two or more sites, $p \in S, V_p \in \mathbb{D}$ in the Euclidean space R^2. Then*
1° V_p is near at least one other Voronoï region in $\mathbb{D}$.
2° Let p, y be sites in S. $\{y\}\ \delta\ \{p\} \Rightarrow \{y\}\ \delta\ V_p$.
3° V_p is close to Voronoï region V_y if and only if $d(x, V_y) = 0$ for at least one $x \in V_p$.
4° A mapping $f : V_p \longrightarrow V_y$ is uniformly continuous, provided $f(V_p)\ \delta\ f(V_y)$ whenever $V_p\ \delta\ V_y$.

Proof.
1°: Assume S contains at least 2 sites. Let $p \in S, y \in S \setminus \{y\}$ such that V_p, V_y have at least one closed half plane in common. Then $V_p\ \delta\ V_y$.
2°: If $\{y\}\ \delta\ \{p\}$, then $\|y - p\| = 0$, since $y \in \{y\} \cap \{p\}$. Consequently, $\{y\} \cap \mathrm{cl}(V_p) \neq \emptyset$. Hence, $\{y\}\ \delta\ \mathrm{cl}(V_p)$.
3°: $V_p\ \delta\ V_y \Leftrightarrow$ exists $x \in \mathrm{cl}(V_p)\ \cap\ \mathrm{cl}(V_y) \Leftrightarrow\ d(x, V_y) = 0$.
4°: Let $f(V_p)\ \delta\ f(V_y)$ whenever $V_p\ \delta\ V_y$. Then, by definition, $f : V_p \longrightarrow V_y$ is uniformly continuous. ☐

Theorem 3.3. *Every collection of proximal Voronoï regions has a Leader uniform topology (application of* [9]*).*

Proof. Assume $\mathbb{D}$ has more than one Voronoï region. For each $V_p \in \mathbb{D}$, find all $V_y \in \mathbb{D}$ that are close to V_p. For each V_p, this procedure determines a family of Voronoï regions that are near V_p. Let τ be a collection of families of proximal Voronoï regions. Let $A, B \in \tau$. $A \cap B \in \tau$, since either $A \cap B = \emptyset$ or, from Theorem 3.2.1°, there is at least one Voronoï region $V_p \in A \cap B$, *i.e.*, $V_p\ \delta\ A$ and $V_p\ \delta\ B$. Hence, $A \cap B \in \tau$. Similarly, $A \cup B \in \tau$, since $V_p\ \delta\ A$ or $V_p\ \delta\ B$ for each $V_p \in A \cup B$. Also, $\mathbb{D}, \emptyset$ are in τ. Then, τ is a Leader uniform topology in $\mathbb{D}$. ☐

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