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A NOTE ON WEAK ODD EDGE-COLORINGS OF GRAPHS

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ABSTRACT. An edge-coloring of a graph G is said to be a weak-odd edge-coloring if each non-isolated vertex of G uses at least one color odd number of times on its incident edges. The weak-odd chromatic index $\chi'_{\mathrm{wo}}(G)$ is the minimum number of colors needed for a weak-odd edge-coloring of G. In this paper, we prove that any graph without trivial nonempty components admits a weak-odd edge-coloring, and characterize connected graphs according to the value of their weak-odd chromatic index.

1. INTRODUCTION

Throughout the article we mainly follow terminology and notation used in [2]. A graph G is always regarded as being finite (i.e. having finite nonempty set of vertices $V(G)$, and finite set of edges $E(G)$) with loops and multiple edges allowed. A loopless graph without multiple edges is referred to as a *simple* graph. The parameters $n(G) = |V(G)|$ and $m(G) = |E(G)|$ are called *order* and *size* of G, respectively. Whenever $n(G) = 1$ we say G is *trivial*, and whenever $m(G) = 0$ we say G is *empty*. For $X \subseteq V(G) \cup E(G)$, the subgraph of G obtained by removing the vertices and edges of X is denoted by $G - X$. If $X = \{x\}$ is a singleton, we write $G - x$ rather than $G - \{x\}$. Given a cut vertex v of G, let $V_1, \ldots, V_k$ be the vertex sets of the components of $G - v$, and $H_i = G[V_i \cup \{v\}]$, for $i = 1, \ldots, k$. Each such H_i is called a *v-lobe* of G.

We refer to each vertex v of even (resp. odd) degree $d_G(v)$ as an *even* (resp. *odd*) vertex of G. In particular, a vertex of degree equal to 0 (resp. 1) is an *isolated* (resp. *pendant*) vertex. A graph is called *even* (resp. *odd*) whenever all its vertices are even (resp. odd).

Given a (not necessarily proper) edge-coloring φ of a graph G and a color c, we denote the fiber $\varphi^{-1}(c)$ by E_c, and the spanning subgraph of G with edge set E_c by G_c. For a vertex $v \in V(G)$, we say that c *appears at* v if $d_{G_c}(v) > 0$. Moreover, we say that c is *odd at* v whenever $d_{G_c}(v)$ is odd. The edge-coloring φ is *weak-odd at* v whenever at least one color is odd at v. And if this holds for every non-isolated vertex v of G, then we speak of a *weak-odd edge-coloring* of G. Thus, we say that φ is a weak-odd edge-coloring of G if each non-isolated vertex appears as an odd vertex in at least one of the subgraphs induced by the different color classes.

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A weak-odd edge-coloring of G which uses at most k colors is referred to as a *weak-odd k-edge-coloring*, and we then say that G is *weak-odd k-edge-colorable.* Whenever G admits a weak-odd edge-coloring, the *weak-odd chromatic index* $\chi'_{\mathrm{wo}}(G)$ is defined to be the least integer k for which G is weak-odd k-edge-colorable.

Since each loop at a vertex v colored with c contributes 2 to the count of appearances of c at v, it is obvious that a necessary and sufficient condition for the existence of a weak-odd edge-coloring of G is the absence of vertices incident only to loops. Apart from this, the presence of loops does not influence the existence nor changes the value of the index $\chi'_{\mathrm{wo}}(G)$.

A similar notion of *odd edge-coloring* of a graph G was introduced by Pyber in his survey on graph coverings [7] as an edge decomposition of G into (edge disjoint) odd subgraphs. Equivalently, it is an edge-coloring of G such that at each non-isolated vertex every appearing color is odd. In his work, Pyber considered simple graphs and proved the following result.

Theorem 1.1 (Pyber, 1991). *Every simple graph admits an odd edge-coloring with at most* 4 *colors.*

In [6], the authors considered the same notion for loopless graphs and proved an analogous result.

Theorem 1.2 (Lužar et al., 2013). *Every loopless graph admits an odd edge-coloring with at most* 6 *colors.*

Furthermore, a characterization of the loopless graphs needing the maximum 6 colors is given in [6].

In the next section, we consider the related notion of weak-odd edge-coloring of graphs in more detail. We provide a tight upper bound for the weak-odd chromatic index and characterize graphs G according to the value of $\chi'_{\mathrm{wo}}(G)$.

2. Weak-odd edge-coloring

We have already determined which graphs G are weak-odd edge-colorable. It is a simple matter to characterize those having $\chi'_{\mathrm{wo}}(G) \leq 1$. Namely, $\chi'_{\mathrm{wo}}(G) = 0$ holds exclusively for the empty graphs G, while $\chi'_{\mathrm{wo}}(G) = 1$ if and only if G is nonempty and the subgraph induced by the non-isolated vertices is odd.

Given a graph G, let T be an even-sized subset of $V(G)$. Following [2], a spanning subgraph H of G is called a *T-join* if $d_H(v)$ is odd for every $v \in T$ and even for every $v \in V(G)\backslash T$. For example, if P is an xy-path in G (i.e. a path having endvertices x and y), the spanning subgraph of G with edge set $E(P)$ is an $\{x, y\}$-join. An obvious necessary condition for the existence of a T-join is that T intersects every component of G in an even number of vertices (possibly equal to 0). The following classical result about T-joins claims that this condition is also sufficient (see e.g. [8]).

Lemma 2.1. *Given a connected graph G, for every even-sized subset T of $V(G)$ there exists a T-join in G.*

In particular, for a connected graph G of even order, let T denote the subset of even vertices. Since T is even-sized, a T-join H can be found in G. By setting $K = G - E(H)$ we obtain an *odd factor* of G, i.e. a spanning odd subgraph K. This proves the following result.

Lemma 2.2. *Every connected graph G of even order has an odd factor K.*

While considering edge-colorings, it suffices to restrict attention to connected graphs. As an immediate consequence of the last lemma, we have the next proposition.

Proposition 2.1. *If G is a connected graph of even order, then $\chi'_{\mathrm{wo}}(G) \leq 2$.*

Note that by leaving out the constraint about the order of G, we can do almost as good, as the following proposition demonstrates.

Proposition 2.2. *Given a connected graph G and a vertex $w \in V(G)$, there exists a 2-edge-coloring of G that is weak-odd at each vertex distinct from w.*

Proof. By Proposition 2.1, we may assume that G is of odd order. Consider first the case when w is a non-cut vertex of G. Lemma 2.2 implies that $G - w$ has an odd factor K. Thus, by coloring each member of $E(K)$ with 1 and each member of $E(G)\backslash E(K)$ with 2, we obtain a 2-edge-coloring of G which is weak-odd at each vertex distinct from w.

Consider now the case when w is a cut vertex of G. Denote by $V_1, \dots, V_k$ the vertex sets of the components of $G - w$, and let $H_i = G[V_i \cup \{w\}]$, $i = 1, \dots, k$, be the w-lobes of G. For each i, since w is a non-cut vertex of H_i, there exists an edge-coloring φ_i of H_i with the color set $\{1, 2\}$, which is weak-odd at each vertex distinct from w. Then, the union $\varphi_1 \cup \dots \cup \varphi_k$ satisfies the same for G. □

Corollary 2.1. *If a connected graph G has at least one odd vertex, then $\chi'_{\mathrm{wo}}(G) \leq 2$.*

Proof. Let w be an odd vertex of G. Clearly, any edge-coloring of G is weak-odd at w. By the previous proposition, there exists a 2-edge-coloring φ of G which is weak-odd at every vertex distinct from w. Thus, φ is a weak-odd 2-edge-coloring of G. □

Next, we show that three colors suffice for a weak-odd edge-coloring of any connected nontrivial graph.

Proposition 2.3. *For every connected nontrivial graph G it holds that $\chi'_{\mathrm{wo}}(G) \leq 3$. Moreover, equality is attained if and only if G is a nontrivial even graph of odd order.*

Proof. We may restrict to even connected graphs of odd order at least 3. Let G be such and take v to be any non-cut vertex of G (there are at least two such vertices). By Lemma 2.2, there exists an odd factor K of $G - v$. Select an arbitrary edge e incident to v. We obtain a weak-odd 3-edge-coloring of G by coloring $E(K)$ with 1, the edge e with 2, and each of the remaining non-colored edges with 3.

For the second part of the statement, suppose there exists a nontrivial connected even graph G of odd order that is weak-odd 2-edge-colorable. For such an edge-coloring, each color class induces an odd factor of G. But this is clearly impossible, for it implies that any such odd factor is a graph with odd number of odd vertices. This completes the proof. □

Thus, we are able to characterize the connected graphs according to the value of their weak-odd chromatic index. Recall from the introduction that a connected graph admits weak-odd edge-colorings if and only if its edge set does not consist only of loops.

Theorem 2.1. *Given a connected nontrivial graph G, it holds that*

$$\chi'_{\mathrm{wo}}(G) = \begin{cases} 1 & \textit{if } G \textit{ is odd}, \\ 3 & \textit{if } G \textit{ is even and of odd order}, \\ 2 & \textit{otherwise}. \end{cases}$$

3. Concluding remarks and further work

The notion of weak-odd edge-coloring of graphs naturally gives an analogous notion for digraphs. Namely, a (not necessarily proper) edge-coloring of a digraph D is said to be *weak-odd* whenever for each vertex $v \in V(D)$ at least one color c satisfies the following requirement: if $d^+(v) > 0$ then c appears an odd number of times on the outgoing edges at v; and if $d^-(v) > 0$ then c appears an odd number of times on the ingoing edges at v. We will address this matter in forthcoming works (for example, in [5]).

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