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UNLIMITED MOMENTS OF SWITCHING FOR DIFFERENTIAL EQUATIONS WITH VARIABLE STRUCTURE AND IMPULSES

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ABSTRACT. Systems of homogeneous differential equations with variable structure (i.e. variable right hand side) and impulsive effects are the main object of study in this paper. The switching moments (in which the structure changes and the effects are realized) are determined by the switching functions which are defined in the phase space of the system. These switching moments are specific to the solution of each initial problem. They coincide with the moments at which the trajectory of initial problem cancels successively each one of the switching functions. Sufficient conditions for unlimited moments of switching are found for the indicated systems.

1. INTRODUCTION

The applications of differential equations with variable structure (without impulsive effects) are mostly in the control theory: [4], [8], [11], [12], [13] and [19].

The impulsive equations (with fixed structure) are mainly used in the description and study of dynamic processes, subjected to the discrete external influences: [1], [2], [3], [10], [14], [15], [17], [18], [20], [21] and [22]. The differential equations with variable structure and impulses are introduced in [16]. Some properties of their solutions are studied in [5], [7] and [9].

The solutions of such class differential equations are continuous piecewise functions. The moments of discontinuity of solutions coincide with the moments when the trajectory successively cancels any one of the switching functions. At these moments, the right hand side of the system is changing and the impulsive effect takes place. These moments are called switching. If the switching moments have a compression point, then the solution is not continuable at the right hand side of this point. Therefore, in this case, a number of important properties of the solutions can not be studied. Moreover, they cannot be defined properly. The above mentioned qualities include: stability, monotony, oscillation, periodicity equivalence, etc. The sufficient conditions which ensure that the switching points do not have a compression point have been found in the paper.

Finally, we point out the articles [5], [6] and [7] where the discontinuous dynamic processes from practice are investigated using the above-described class of equations. This fact confirms the need to research deeper the systems differential equations with variable structure and impulses.

2. THE PRELIMINARY RESULTS AND NOTES

Further, we denote the Euclidean norm and scalar product in R^n by $\|.\|$ and $\langle .,.\rangle$, respectively. For the points $x = (x^1, x^2, ..., x^n)$ and $y = (y^1, y^2, ..., y^n)$ in R^n, we have:

$$\langle x, y \rangle = x^1 y^1 + x^2 y^2 + ... + x^n y^n \,,$$

$$\|x\| = \left(\langle x, y \rangle\right)^{\frac{1}{2}} = \left(\left(x^1\right)^2 + \left(x^2\right)^2 + ... + \left(x^n\right)^2\right)^{\frac{1}{2}}.$$

The Euclidean distance between non-empty sets $X, Y \subset R^n$ is:

$$\rho(X, Y) = inf\big\{\|x - y\|; x \in X, \ y \in Y\big\}\,.$$

The main object of investigation in this paper is the following initial problem for systems differential equations with variable structure and impulses:

$$(2.1) \qquad \frac{dx}{dt} = f_i(x), \quad \varphi_i\big(x(t)\big) \neq 0, \ i.e. \ t_{i-1} < t < t_i;$$

$$(2.2) \qquad x(t_i + 0) = J_i\big(x(t_i)\big), \quad \varphi_i\big(x(t_i)\big) = 0, \ i = 1, 2, ...;$$

$$(2.3) \qquad x(0) = x_0,$$

where:

- phase space G of system considered (2.1), (2.2) is a non-empty domain in R^n;
- the functions $f_i : G \to R^n$;
- the functions $\varphi_i : G \to R$;
- the functions $J_i : G \to G$;
- an initial point $x_0 \in G$.

The sets $\Phi_i = \{x \in G; \ \varphi_i(x) = 0\}$, $i = 1, 2, ...$, are named switching. The solution of problem (2.1), (2.2), (2.3) is denoted by $x(t; x_0)$. It is fulfilled:

1.1. For $0 = t_0 \leq t < t_1$, the solution of the studied problem coincides with the solution $x_1(t; x_0)$ of problem

$$\frac{dx}{dt} = f_1(x), \quad x(t_0) = x_0 = x_0^+;$$

1.2. For $t_0 \leq t < t_1$, it is satisfied $\varphi_1\big(x(t; x_0)\big) = \varphi_1\big(x_1(t; x_0^+)\big) > 0$;

1.3. For $t = t_1$, we have $\varphi_1\big(x(t_1; x_0)\big) = \varphi_1\big(x_1(t_1; x_0^+)\big) = 0$. We denote $x_1 = x(t_1; x_0)$;

1.4. The equality $x(t_1 + 0; x_0) = J_1\big(x(t_1; x_0)\big) = J_1(x_1) = x_1^+$ is valid;

2.1. For $t_1 < t < t_2$, the solution of problem (2.1), (2.2), (2.3) coincides with the solution $x_2(t; x_1^+)$ of problem

$$\frac{dx}{dt} = f_2(x), \quad x(t_1) = x_0 = x_1^+;$$

2.2. For $t_1 < t < t_2$, it is fulfilled $\varphi_2\big(x(t; x_0)\big) = \varphi_2\big(x_2(t; x_1^+)\big) > 0$;

2.3. For $t = t_2$, we have $\varphi_2\big(x(t_2; x_0)\big) = \varphi_2\big(x_2(t_2; x_1^+)\big) = 0$;

2.4. For $t = t_2$, the impulsive equality $x(t_2 + 0; x_0) = J_2\big(x(t_2; x_0)\big) = J_2(x_2) = x_2^+$ is valid, etc.

Time constants t_i, $i = 1, 2, ...$, are named switching moments. We introduce the notations $x_i = x(t_i; x_0)$ and $x_i^+ = J_i(x_i)$, $i = 1, 2, ...$

In the arbitrary interval of continuity (t_{i-1}, t_i), the solution of problem (2.1), (2.2), (2.3) starts at the point $x_{i-1}^+ \in G$. In this interval, the right hand side of the system coincides with the function $f_i(x)$. At the moment t_i, the solution of considered system reaches the switching set Φ_i. At the same moment t_i, the solution is subjected to the impulsive effect. It means that the function $x(t; x_0)$ has a finite jump, i.e. first type discontinuity at the point t_i. Furthermore, just then, the right hand side of the system is changed. The right hand side of system coincides with function $f_{i+1}(x)$ in the next interval of continuity of the solution.

We introduce the following conditions:

H1. The functions $f_i \in C[G, R^n]$, $i = 1, 2, ...$

H2. There exist constants $C_{f_i} > 0$ such that

$$(\forall x \in G) \Rightarrow \|f_i(x)\| \le C_{f_i}, \ i = 1, 2, ...$$

H3. For any point $x_0 \in G$ and for each $i = 1, 2, ...$, the solution $x_i(t; x_0)$ of the initial problem

$$\frac{dx}{dt} = f_i(x), \quad x(0) = x_0$$

exists and is unique for $t \ge 0$.

H4. There exist constants $C_{Lip\varphi_i} > 0$ such that

$$(\forall x', x'' \in G) \Rightarrow |\varphi_i(x') - \varphi_i(x'')| \le C_{Lip\varphi_i}\|x' - x''\|, \ i = 1, 2, ...$$

H5. There exist constants $C_{J_i} > 0$ such that

$$(\forall x \in \Phi_i) \Rightarrow \big|\varphi_{i+1}\big(J_i(x)\big)\big| \ge C_{J_i}, \ i = 1, 2, ...$$

3. Main results

Theorem 3.1. *Let the conditions H1-H5 be satisfied. If the series*

$$\sum_{i=1}^{\infty} \frac{C_{J_i}}{C_{f_{i+1}} C_{Lip\varphi_{i+1}}}$$

is divergent, then the switching moments for system (2.1), (2.2) have no compression point.

Proof. If the switching moments are finite number, then the statement of the theorem is trivial. Let the switching moments $t_1, t_2, ...$ be infinitely many. We will evaluate below the difference $t_{i+1} - t_i$ for each $i = 1, 2, ...$ By condition H2, we have:

$$(3.1) \qquad \big\|x(t_{i+1}; x_0) - x(t_i + 0; x_0)\big\| = \big\|x_{i+1}(t_{i+1}; x_i^+) - x_{i+1}(t_i; x_i^+)\big\|$$

$$\le \int_{t_i}^{t_{i+1}} \Big\|f_{i+1}\big(x_{i+1}(\tau; x_i^+)\big)\Big\| d\tau$$

$$\le C_{f_{i+1}}(t_{i+1} - t_i).$$

From the assessment above, using the conditions H4 and H5 successively, we obtain:

$$(3.2) \qquad t_{i+1} - t_i \geq \frac{1}{C_{f_{i+1}}} \left\| x(t_{i+1}; x_0) - x(t_i + 0; x_0) \right\|$$

$$\geq \frac{1}{C_{f_{i+1}} C_{Lip\varphi_{i+1}}} \left| \varphi_{i+1}\big(x(t_{i+1}; x_0) \big) - \varphi_{i+1}\big(x(t_i + 0; x_0) \big) \right|$$

$$\geq \frac{1}{C_{f_{i+1}} C_{Lip\varphi_{i+1}}} \left| \varphi_{i+1}\big(x(t_i + 0; x_0) \big) \right|$$

$$= \frac{1}{C_{f_{i+1}} C_{Lip\varphi_{i+1}}} \left| \varphi_{i+1}\Big(J_i\big(x(t_i; x_0) \big) \Big) \right|$$

$$\geq \frac{C_{J_i}}{C_{f_{i+1}} C_{Lip\varphi_{i+1}}} \, .$$

It is fulfilled:

$$(3.3) \qquad \lim_{i \to \infty} t_i = \lim_{i \to \infty} \big((t_i - t_{i-1}) + (t_{i-1} - t_{i-2}) + \ldots + (t_2 - t_1) + (t_1 - t_0) \big) + t_0$$

$$\geq \lim_{i \to \infty} \left(\frac{C_{J_1}}{C_{f_2} C_{Lip\varphi_2}} + \frac{C_{J_2}}{C_{f_3} C_{Lip\varphi_3}} + \ldots + \frac{C_{J_{i-1}}}{C_{f_i} C_{Lip\varphi_i}} \right) + t_0$$

$$= \sum_{i=1}^{\infty} \frac{C_{J_i}}{C_{f_{i+1}} C_{Lip\varphi_{i+1}}} + t_0 = \infty \, .$$

Therefore, the switching moments do not have a compression point. The theorem is proved. $\square$

In the next theorem, we change condition H4 by the following:

H6. There exist constants $C_{\varphi_i} > 0$ such that

$$(\forall x \in G) \Rightarrow \left| \varphi_i(x) \right| \leq C_{\varphi_i} \rho(x, \Phi_i), \ i = 1, 2, \ldots .$$

Theorem 3.2. *Let the conditions H1, H2, H3, H5 and H6 be fulfilled. If the series*

$$\sum_{i=1}^{\infty} \frac{C_{J_i}}{C_{f_i} C_{\varphi_i}}$$

is divergent, then the switching moments for system (2.1), (2.2) have no compression point.

Proof. We will consider the case of innumerable switching moments. From (3.1), similar to (3.2), we obtain:

$$t_{i+1} - t_i \geq \frac{1}{C_{f_{i+1}}} \left\| x(t_{i+1}; x_0) - x(t_i + 0; x_0) \right\|$$

$$\geq \frac{1}{C_{f_{i+1}}} \rho\big(x(t_i + 0; x_0), \Phi_{i+1} \big)$$

$$\geq \frac{1}{C_{f_{i+1}} C_{\varphi_{i+1}}} \left| \varphi_{i+1}\big(x(t_i + 0; x_0) \big) \right|$$

$$= \frac{1}{C_{f_{i+1}} C_{\varphi_{i+1}}} \left| \varphi_{i+1}\big(\big(J_i(x(t_i; x_0)) \big) \big) \right|$$

$$\geq \frac{C_{J_{i+1}}}{C_{f_{i+1}} C_{\varphi_{i+1}}} \, .$$

From the inequality above, analogously to (3.3), we obtain $\lim_{i \to \infty} t_i = \infty$. The theorem is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 3.1. *It should be noted that condition H4 is substantially different from condition H6. For example, it follows by H4 that the functions φ_i, $i = 1, 2, ...$, are continuous. On the other hand, it can be shown that condition H6 is satisfied by discontinuous functions.*

The following conditions will be used in the next theorem.

H7. There exist constants C_{D_i} and domains D_i, $\Phi_i \subset D_i \subset G$, such that

$$(\forall x \in \partial D_i \cap G) \Rightarrow \left|\varphi_i(x)\right| \geq C_{D_i}, \ i = 1, 2, ...$$

H8. It is satisfied

$$(\forall x \in \Phi_i) \Rightarrow J_i(x) \in G \backslash D_{i+1}, \ i = 1, 2, ...$$

Theorem 3.3. *Let the conditions H1, H2, H3, H6, H7 and H8 be fulfilled. If the series*

$$\sum_{i=1}^{\infty} \frac{C_{D_i}}{C_{f_i} C_{\varphi_i}}$$

is divergent, then the switching moments for system (2.1), (2.2) do not posses a compression point.

Proof. Once again, we prove the case of innumerable switching moments for system (2.1), (2.2). The theorem is proven by showing that $\lim_{i \to \infty} t_i = \infty$. We consider the point $x_i^+ = x(t_i + 0; x_0) = J_i\big(x(t_i; x_0)\big) = J_i(x_i)$. As $x_i \in \Phi_i$ and under condition H8, we have $x_i^+ \in G \backslash D_{i+1}$. We go into the solution $x_{i+1}(t; x_i^+)$ of initial value problem

$$\frac{dx}{dt} = f_{i+1}(x), \ x(t_i) = x_i^+ \, .$$

The following inclusions are satisfied:

(3.4) $\qquad\qquad x_{i+1}(t_i; x_i^+) = x_i^+ \in G \backslash D_{i+1},$

$\qquad\qquad\quad\ x_{i+1}(t_{i+1}; x_i^+) = x(t_{i+1}; x_0) = x_{i+1} \in \Phi_{i+1} \subset D_{i+1} \, .$

Using (3.4) and continuity of the curve $x_{i+1}(t; x_i^+)$, it follows that

$$(\exists \, t_{\partial D_{i+1}}, \ t_i < t_{\partial D_{i+1}} < t_{i+1}): \ x_{i+1}(t_{\partial D_{i+1}}; x_i^+) = x(t_{\partial D_{i+1}}; x_0) \in \partial D_{i+1} \cap G \, .$$

Then, according to condition H7, we obtain:

$$\left|\varphi_{i+1}\big(x(t_{\partial D_{i+1}}; x_0)\big)\right| \geq C_{D_{i+1}} \, .$$

On the other hand, we have:

$$\left\|x(t_{i+1}; x_0) - x(t_{\partial D_{i+1}}; x_0)\right\| \leq C_{f_{i+1}}(t_{i+1} - t_{\partial D_{i+1}}) \, ,$$

i.e.,

$$t_{i+1} - t_{\partial D_{i+1}} \geq \frac{1}{C_{f_{i+1}}} \big\| x(t_{i+1}; x_0) - x(t_{\partial D_{i+1}}; x_0) \big\|$$

$$\geq \frac{1}{C_{f_{i+1}}} \rho\big(x(t_{\partial D_{i+1}}; x_0), \Phi_{i+1}\big)$$

$$\geq \frac{1}{C_{f_{i+1}} C_{\varphi_{i+1}}} \big| \varphi_{i+1}\big(x(t_{\partial D_{i+1}}; x_0)\big) \big|$$

$$\geq \frac{C_{D_{i+1}}}{C_{f_{i+1}} C_{\varphi_{i+1}}} \, .$$

From the estimate above, we find

$$t_{i+1} - t_i > t_{i+1} - t_{\partial D_{i+1}} \geq \frac{C_{D_{i+1}}}{C_{f_{i+1}} C_{\varphi_{i+1}}}, \quad i = 1, 2, \dots.$$

and it follows $\lim_{i \to \infty} t_i = \infty$. The theorem is proved.   □

4. APPLICATION

A number of isolated populations grow optimally (in a specific sense: for example, their growth is relatively more intensive), if the amount of their biomass is maintained within certain limits. Typically, these quantitative restrictions are highly dependent upon food stocks, living environment, intra-competition, etc. In order to maintain the biomass population in these optimal limits, it is possible to carry out external, discreet effects which consist of the withdrawal or adding the biomass. One option is the duration of such external impacts to be negligible compared to the total duration of the process of development of the isolated species. In this variant, the maintenance of the optimal range of the population biomass, the influences are performed instantaneously in the form of impulses. Differential equations with variable structure and impulses are one suitable mathematical tool for such processes. It is natural to assume that the impulsive effects, consisting in adding or withdrawal of certain biomass amounts are carry out when the amount of biomass reaches to the fixed in advance limits which restrict the optimum biomass levels.

The impulsive equation of Gompertz is an adequate mathematical model of such processes. The corresponding initial problem has the form:

(4.1) $\dfrac{dm}{dt} = m(r - \gamma \ln m), \ \varphi_i\big(m(t)\big) = m_{maxi} - m(t) > 0, \ t_{i-1} < t < t_i,$

(4.2) $m(t_i + 0) = m(t_i) + I_i\big(m(t_i)\big) = J_i\big(m(t_i)\big), \ \varphi_i\big(m(t_i)\big) = m_{maxi} - m(t_i) = 0,$

(4.3) $m(0) = m_0,$

where:

 - $m = m(t)$ is the amount of biomass at the moment $t \geq 0$, $m(t) \in \left(0, \exp\left(\frac{r}{\gamma}\right)\right)$;
 - the right hand side of the equation does not change at the various intervals of continuity of the solution of problem (4.1), (4.2), (4.3). We have $f_i(m) = f(m) = m(r - \gamma \ln m)$, $f : G \to R$ for $i = 1, 2, \dots$;
 - $r = const > 0$ is the reproductive potential of the species;

- $\gamma = const > 0$ is a coefficient of intra specific competition;
- $m_{maxi} > 0$, $i = 1, 2, ...$, are the upper barrier constants which determine optimal upper limit of the biomass amount. These constants are specific to each interval of the continuity of solution and it is fulfilled $0 < m_{maxi} < \exp\left(\frac{r}{\gamma}\right)$, $i = 1, 2, ...$;
- the switching functions $\varphi_i(m) = m_{maxi} - m$, $\varphi_i : G \to R$, $i = 1, 2, ...$;
- each one of the switching sets consists of one point. We have $\Phi_i = \{m_{maxi}\}$, $i = 1, 2, ...$;
- the functions $I_i : G \to (-m_{maxi}, 0)$, $i = 1, 2, ...$, reflect the sizes of the impulsive withdrawals from the biomass upon reaching the upper barrier constant;
- m_0 is the amount of biomass at the initial moment $t = 0$. The inequalities $0 < m_0 < m_{maxi}$ are valid.

We assume that the biomass $m = m(t)$ of the isolated species whose development is described by impulsive problem (4.1), (4.2), (4.3) is optimal, if the following restrictions $0 < m(t) < m_{maxi}$ are satisfied for $t_{i-1} < t < t_i$. We pay attention to the fact that the constants 0 and $\exp\left(\frac{r}{\gamma}\right)$ are zeros on the right hand side of equation (4.2) (the first of these zeros in the boundary form). Therefore, they are specific points and they are unattainable, if the initial point m_0 is between them.

In the model above, the discrete effects are realized when the biomass of the isolated population becomes equal to the upper barrier constant. The moments of impulsive effects are denoted by $t_1, t_2, ...$ and the following inequalities are valid $0 = t_0 < t_1 < t_2 < ....$ If $m(t_i) = m_{maxi}$ is fulfilled at moment t_i, then the impulsive withdrawal of biomass is carried out with size $I_i\big(m(t_i)\big) = I_i(m_{maxi}) < 0$, $i = 1, 2, ...$ As we said, the purpose of these discreet interventions is the biomass to be maintained within the optimal limits. We will pay attention to the following fact. Since the right hand side of equation (4.1) is positive for $0 < m < \exp\left(\frac{r}{\gamma}\right)$, then the amount of biomass increases between two neighboring impulsive moments in the model of Gompertz.

We will demonstrate that this model satisfies the conditions of Theorem 3.1. Indeed, the conditions H1 and H3 are valid. Furthermore, since $f \in C[G, R]$, $G = \left(0, \exp\left(\frac{r}{\gamma}\right)\right)$ and

$$\lim_{m \to 0} f(m) = \lim_{m \to 0} m(r - \gamma \ln m) = 0,$$
$$\lim_{m \to \exp(\frac{r}{\gamma})} f(m) = \lim_{m \to \exp(\frac{r}{\gamma})} m(r - \gamma \ln m) = 0,$$

it follows that f is bounded in the domain (open interval) G. Therefore, condition H2 is satisfied. Through concrete computations, we obtain constant

$$C_{f_i} = C_f = \gamma \exp\left(\frac{r - \gamma}{\gamma}\right) > 0.$$

Lipschitz constants for condition H4 is

$$C_{Lip\varphi_i} = C_{Lip\varphi} = 1.$$

Finally, for definiteness, we assume that

$$I_i(m) = \frac{1}{2} m_{max(i+1)} - m_{maxi}, \ i = 1, 2, ...$$

Then

$$\left(\forall m \in \Phi_i \ \Leftrightarrow \ m = m_{maxi}\right) \Rightarrow \left|\varphi_{i+1}\big(J_i(m)\big)\right| = \frac{1}{2}m_{max(i+1)} = C_{J_i}, \ i = 1, 2, ...$$

Thereby, it is shown that condition H5 is valid.

From Theorem 3.1, it follows that if the series $\sum_{i=1}^{\infty} m_{maxi}$ is divergent then the switching points for system (4.1), (4.2) do not have a point of compression, i.e. $\lim_{i\to\infty} t_i = \infty$.

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