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# A NEW POINT OF VIEW ON $(1+3)$ THREADING OF SPACETIME

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ABSTRACT. We present a new method for the study of the (1+3) threading of a spacetime $(M, g)$. The new approach is based on the theory of horizontal tensor fields and on the Riemannian horizontal connection. We obtain, in a covariant form, the fully general $3D$ equations of motion in $(M, g)$. We define and study a $3D$ force and obtain a new identity satisfied by geodesics on $(M, g)$. Finally, we apply the method developed in the paper to the study of motions in a Friedmann-Robertson-Walker universe and in a Kerr black hole.

## 1. INTRODUCTION

The decomposition of spacetime into "space plus time" ((3+1) slicing), or into "time plus space" ((1+3) threading) lead to the splitting of spacetime tensors and of basic equations into some counterparts, which of course are more familiar to our Newtonian thinking. As it is well known, the (3+1) slicing of spacetime is based on the existence of a foliation by spacelike hypersurfaces, and it was used for solving initial value problems of general relativity (cf. Misner, Thorne and Wheeler [16], p.484, Wald [22], p.252). On the other hand, the (1+3) threading of spacetime is based on the existence of a foliation by timelike curves, and has been applied to: the parametrization- dependent definition of spatial gravitational forces (cf. Møller [17]), the splitting of Einstein equations (cf. Zel'manov [23]), the discussion of gyroscope precession (cf. Massa and Zordan [15]), the study of gravitoelectromagnetism (cf. Mashhoon, McClune and Quevedo [14], Jantzen, Carini and Bini [11]), etc. We should stress that the (3+1) slicing of spacetime exists only under suitable conditions on the geometry of spacetimes, while the (1+3) threading exists on any spacetime, and therefore the latter can be applied to any cosmological model.

The present paper is the first in a series of papers we would like to devote to a new approach of (1+3) threading of spacetime. The motivation of our work comes from a somehow cumbersome presentation of (1+3) threading, compared with the (3+1) slicing. This justifies the small amount of research on (1+3) threading of spacetime. We simply start with the foliation of curves (congruence of curves), and develop the study in the framework of coordinate systems that are naturally induced by this foliation. Then we

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consider the horizontal distribution that is orthogonal to this foliation and introduce some basic horizontal tensor fields. Also, we construct the Riemannian horizontal connection, which is a metric connection on the horizontal distribution. These geometric objects enable us to obtain simple covariant expressions for the $3D$ equations of motion and to define the $3D$ force in a spacetime. The $3D$ force identity stated along any geodesic seems to have a great impact on the dynamics in a spacetime. This can be already seen from its role in the short study we present on the motions in a Friedmann-Robertson-Walker (FRW) universe and in the spacetime of a Kerr black hole.

Now, we outline the content of the paper. In Section 2, by using the special coordinate systems introduced by threading of the spacetime $(M, g)$ we construct the threading frame and coframe fields. Then, in Section 3 we introduce the concept of horizontal tensor field on $(M, g)$ and construct the horizontal tensor fields which have an important role in the study. The Riemannian horizontal connection is constructed in Section 4, and together with the horizontal tensor fields enables us to decompose the Levi-Civita connection on $(M, g)$ (see (4.11)). These geometric objects lead us to simple forms of the $3D$ equations of motion (cf.(5.5a)) and to geometric characterizations of two classes of geodesics in $(M, g)$. In Section 6, by using the Riemannian horizontal connection we define the $3D$ force in $(M, g)$ and show that it is orthogonal to the $3D$ velocity. Also, we derive the $3D$ force identity and show that all the $3D$ forces along a geodesic induce the same identity (6.10). In Sections 7 and 8 we apply the new method of study to the dynamics in a FRW universe and in a Kerr black hole, respectively. We find the explicit equations of null geodesics in a FRW universe (M(k, f), g) and, in particular, we show that a geodesic which is tangent at one point to a slice of this spacetime should leave that slice at the later times. The Kerr black hole is an example of spacetime with non integrable horizontal distribution. The Kerr metric has a very simple form with respect to a threading coframe filed (see (8.4)). Contrary to the case of FRW model, we show that if a geodesic in the Kerr black hole is tangent to the horizontal distribution at one point, then it remains tangent to it at all later times. Finally, we obtain geometric characterizations of geodesics in a Kerr black hole whose geometric conserved energy is positive, negative or equal to zero.

## 2. Threading (1+3) decomposition of spacetime

Let $(M, g)$ be a $4D$ spacetime, where $M$ is a time-oriented connected 4-dimensional smooth manifold and g is a Lorentz metric on $M$ of signature (+, +, +, -). Then there exists a timelike vector field $\xi$ that is globally defined on $M$ (cf. O'Neill [18], p.149). Denote by $VM$ the line distribution spanned by $\xi$, and by $HM$ the complementary orthogonal distribution to $VM$ in the tangent bundle $TM$ of $M$. Hence we have the Whitney orthogonal decomposition

$$TM = HM \oplus VM . \tag{2.1}$$

We call $HM$ and $VM$ the *horizontal distribution* and the *vertical distribution*, respectively. According to the signature of $g$ and taking into account that $VM$ is a timelike distribution, we conclude that $HM$ is a spacelike distribution. Note that $HM$ is not necessarily an integrable distribution.

Throughout the paper we use the ranges of indices: $\alpha, \beta, \gamma, ... \in \{1, 2, 3\}$ and $i, j, k, ... \in \{0, 1, 2, 3\}$. Also, for any vector bundle $E$ over $M$ denote by $\Gamma(E)$ the $\mathcal{F}(M)$-module of smooth sections of $E$, where $\mathcal{F}(M)$ is the algebra of smooth functions on $M$.

Next, we consider the foliation by curves of $M$ (congruence of curves) determined by the integrable distribution $VM$. Then, around each point of $M$ there exists a coordinate system $(x^i)$, such that $\xi = \partial/\partial x^0$. Moreover, if $(\tilde{x}^i)$ is a another coordinate system induced by this foliation, we have:

$$\widetilde{x}^\alpha = \widetilde{x}^\alpha(x^1, x^2, x^3); \qquad \widetilde{x}^0 = \widetilde{x}^0(x^0, x^1, x^2, x^3).$$

By using these transformations, we deduce that

$$\frac{\partial}{\partial x^0} = \frac{\partial \tilde{x}^0}{\partial x^0}\frac{\partial}{\partial \tilde{x}^0}.$$

As $\partial/\partial x^0$ and $\partial/\partial \tilde{x}^0$ represent the same vector field $\xi$, we must have $\partial\tilde{x}^0/\partial x^0 = 1$. Hence the above transformations should have the special form:

$$\widetilde{x}^\alpha = \widetilde{x}^\alpha(x^1, x^2, x^3); \qquad \widetilde{x}^0 = x^0 + f(x^1, x^2, x^3). \tag{2.2}$$

Thus, the natural frame and coframe fields on $M$ obey the following transformations with respect to (2.2):

$$(a) \quad \frac{\partial}{\partial x^\alpha} = \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial}{\partial \tilde{x}^\gamma} + \frac{\partial f}{\partial x^\alpha}\frac{\partial}{\partial \tilde{x}^0}, \quad (b) \quad \frac{\partial}{\partial x^0} = \frac{\partial}{\partial \tilde{x}^0}, \tag{2.3}$$

and

$$(a) \quad d\tilde{x}^\gamma = \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}dx^\alpha, \quad (b) \quad d\tilde{x}^0 = \frac{\partial f}{\partial x^\alpha}dx^\alpha + dx^0, \tag{2.4}$$

respectively.

According to the decomposition (2.1), for each $\partial/\partial x^\alpha$ there exist a unique $\delta/\delta x^\alpha \in \Gamma(HM)$ and a unique function $A_\alpha$, such that

$$\frac{\delta}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - A_\alpha \frac{\partial}{\partial x^0}. \tag{2.5}$$

This enables us to consider the local frame field $\{\delta/\delta x^\alpha, \partial/\partial x^0\}$ on $M$, which we call the *threading frame field* on $M$. This name comes from the threading point of view on splitting of spacetime, which was first introduced by Landau and Lifshitz [13]. Also, $\{dx^\alpha, \delta x^0\}$ is called a *threading coframe field* on $M$, where we put

$$\delta x^0 = dx^0 + A_\alpha dx^\alpha. \tag{2.6}$$

Now, by direct calculations using (2.3)-(2.6), we obtain the transformations:

$$(a) \quad \frac{\delta}{\delta x^\alpha} = \frac{\partial \widetilde{x}^\gamma}{\partial x^\alpha}\frac{\delta}{\delta \tilde{x}^\gamma}, \quad (b) \quad \delta\tilde{x}^0 = \delta x^0, \quad (c) \quad A_\alpha = \widetilde{A}_\gamma \frac{\partial \widetilde{x}^\gamma}{\partial x^\alpha} + \frac{\partial f}{\partial x^\alpha}, \tag{2.7}$$

with respect to (2.2). From (2.7) we see that $\{\delta/\delta x^\alpha\}$ are transformed exactly as $\{\partial/\partial x^\alpha\}$ on a 3-dimensional manifold with local coordinates $x^\alpha$. On the contrary, by using (2.7) we conclude that, in general, $\{A_\alpha\}$ do not satisfy some $3D$ tensorial transformations. However, in the whole literature published so far, $\{A_\alpha\}$ have been considered as local components of the so called "shift 1-form".

At this point we should mention that Zel'manov [24] has considered the so called "chronometric transformations":

$$\tilde{x}^\alpha = x^\alpha, \quad \tilde{x}^0 = \tilde{x}^0(x^i).$$

By (2.7) we see that $\{\delta/\delta x^\alpha\}$ are invariant with respect to these transformations, that is, they are chronometric invariant vector fields on $M$. Moreover, if we consider the chronometric transformations deduced from (2.2), then (2.7) becomes:

$$A_\alpha = \widetilde{A}_\alpha + \frac{\partial f}{\partial x^\alpha}. \tag{2.8}$$

Due to (2.8) we are entitled to call $A_\alpha \in \{1,2,3\}$, as 3D *gauge potentials* on $M$.

Next, suppose that the line element of the Lorentz metric g is given by

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + 2g_{\alpha 0}dx^\alpha dx^0 - \Phi^2(dx^0)^2, \tag{2.9}$$

where we put

$$\begin{aligned} &(a) \quad g_{\alpha\beta} = g\left(\tfrac{\partial}{\partial x^\alpha}, \tfrac{\partial}{\partial x^\beta}\right), \quad (b) \quad g_{\alpha 0} = g\left(\tfrac{\partial}{\partial x^\alpha}, \tfrac{\partial}{\partial x^0}\right), \\ &(c) \quad g\left(\tfrac{\partial}{\partial x^0}, \tfrac{\partial}{\partial x^0}\right) = -\Phi^2, \quad \Phi \neq 0, \end{aligned} \tag{2.10}$$

Then, by using (2.5), (2.10b) and (2.10c) and taking into account that

$$g(\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial x^0}) = 0, \tag{2.11}$$

we deduce that

$$A_\alpha = -\Phi^{-2}g_{\alpha 0}. \tag{2.12}$$

Now, denote by $\bar{g}$ the Riemannian metric induced by $g$ on $HM$, and put

$$\bar{g}_{\alpha\beta} = \bar{g}\left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\right) = g\left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\right). \tag{2.13}$$

Then, by using (2.13), (2.5), (2.10) and (2.12), we obtain

$$\bar{g}_{\alpha\beta} = g_{\alpha\beta} + \Phi^2 A_\alpha A_\beta. \tag{2.14}$$

Thus the line element (2.9) is expressed in terms of the threading coframe $\{dx^\alpha, \delta x^0\}$ as follows

$$ds^2 = \bar{g}_{\alpha\beta}dx^\alpha dx^\beta - \Phi^2(\delta x^0)^2. \tag{2.15}$$

It is noteworthy that $\bar{g}_{\alpha\beta}$ and the entries $\bar{g}^{\alpha\beta}$ of the inverse of the matrix $[\bar{g}_{\alpha\beta}]$ are transformed exactly like $3D$ tensor fields, that is, we have

$$(a) \quad \bar{g}_{\alpha\beta} = \widetilde{\bar{g}}_{\mu\nu}\frac{\partial \widetilde{x}^\mu}{\partial x^\alpha}\frac{\partial \widetilde{x}^\nu}{\partial x^\beta}, \quad (b) \quad \widetilde{\bar{g}}^{\mu\nu} = \bar{g}^{\alpha\beta}\frac{\partial \widetilde{x}^\mu}{\partial x^\alpha}\frac{\partial \widetilde{x}^\nu}{\partial x^\beta}, \tag{2.16}$$

with respect to (2.2).

The configuration of the spacetime presented here by using the foliation determined by timelike curves,is known in literature as (1+3) *threading of spacetime* (cf. Zel'manov [24], Ehlers [6], Møller [17], Ellis [7], Ellis and Bruni [9], Jantzen and Carini [10], Boersma and Dray [4], Bini, Chicone and Mashhoon [3]). This is dual to the (3+1) slicing of spacetime (or to the so called "ADM formalism"), which is based on the existence of a foliation of spacetime by spacelike hypersurfaces (cf. Misner, Thorne and Wheeler, [16], p.506).

**Remark 2.1.** *There is an important difference between the above two splittings of spacetime. This is because any spacetime admits a (1+3)-threading, while the (3+1) slicing exists only on spacetimes for which the horizontal distribution is integrable (example:Robertson-Walker spacetime).*

Our approach on threading of spacetime is totally different from what is known in literature. This is because our threading frame and coframe fields are directly constructed from special coordinate systems introduced by the threading of $M$. Also, the horizontal tensor fields and the Riemannian horizontal connection, which we construct in the next two sections, will have an important role in our study. More precisely, by using these geometric objects, we obtain for the first time in literature, the fully general $3D$ equations of motion in $M$, classify the motions of $(M, g)$, and discover a $3D$ extra force that is acting along the geodesics of spacetime. Also , by using the 3D extra force , we obtain what we call the 3D force identity.

## 3. Horizontal tensor fields on a spacetime

First, we define some $\mathcal{F}(M)$-multi linear operators which have an important role in our study. To this end, we denote by $h$ and $v$ the projection morphisms of $TM$ on $HM$ and $VM$, respectively. Then, we define the $\mathcal{F}(M)$-bilinear mapping

$$A : \Gamma(HM) \times \Gamma(HM) \to \Gamma(VM),$$

$$A(hX, hY) = -v[hX, hY], \quad \forall \;\; X, Y \in \Gamma(TM), \tag{3.1}$$

where $[,]$ stands for the Lie bracket of vector fields. As $HM$ is integrable distribution if and only if $A = 0$, we call $A$ *the integrability tensor field* of $HM$. Also, we define the $\mathcal{F}(M)$-3-linear mapping

$$H : \Gamma(HM)^2 \times \Gamma(VM) \to \mathcal{F}(M),$$

$$\begin{aligned} H(hX, hY, vZ) = \tfrac{1}{2}\{&vZ\bar{g}(hX, hY) - \bar{g}(h[vZ, hX], hY) \\ &-\bar{g}(h[vZ, hY], hX)\}, \quad \forall X, Y, Z \in \Gamma(TM). \end{aligned} \tag{3.2}$$

The Riemannian metric $\bar{g}$ on $HM$ enables us to define an $\mathcal{F}(M)$-bilinear mapping, still denoted by $H$, but given by

$$H : \Gamma(HM) \times \Gamma(VM) \to \Gamma(HM),$$

$$\bar{g}(hX, H(hY, vZ)) = H(hX, hY, vZ). \tag{3.3}$$

Now, as our purpose is to apply the above operators to physics, we need to express them by their local component. We must stress that *throughout the paper, all the local components of the geometric objects involved in the study, will be considered only with respect to the threading frame and coframe fields.* First, by direct calculations using (2.5), we obtain

$$(a) \quad \left[\frac{\delta}{\delta x^{\beta}}, \frac{\delta}{\delta x^{\alpha}}\right] = A_{\alpha\beta}\frac{\partial}{\partial x^0}, \quad (b) \quad \left[\frac{\delta}{\delta x^{\alpha}}, \frac{\partial}{\partial x^0}\right] = B_{\alpha}\frac{\partial}{\partial x^0}, \tag{3.4}$$

where we put

$$
(3.5)\qquad \begin{aligned} &(a)\quad A_{\alpha\beta} = \tfrac{\delta A_\beta}{\delta x^\alpha} - \tfrac{\delta A_\alpha}{\delta x^\beta} = \tfrac{\partial A_\beta}{\partial x^\alpha} - \tfrac{\partial A_\alpha}{\partial x^\beta} - A_\alpha B_\beta + A_\beta B_\alpha,\\ &(b)\quad B_\alpha = \tfrac{\partial A_\alpha}{\partial x^0}. \end{aligned}
$$

Then, by using (3.1) and (3.4a), we deduce that

$$
(3.6)\qquad A\left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\right) = A_{\alpha\beta}\frac{\partial}{\partial x^0},
$$

that is, $A_{\alpha\beta}$ given by (3.5a) are the local components of $A$ with respect to the threading field $\{\delta/\delta x^\alpha, \partial/\partial x^0\}$. Next, we put:

$$
(3.7)\qquad (a)\quad H\left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}, \frac{\partial}{\partial x^0}\right) = H_{\alpha\beta}, \quad (b)\quad H\left(\frac{\delta}{\delta x^\beta}, \frac{\partial}{\partial x^0}\right) = H^\gamma_\beta\frac{\delta}{\delta x^\gamma}.
$$

Then, by using (3.7), (3.2), (2.13), (3.4a) and (3.3), we infer that

$$
(3.8)\qquad (a)\quad H_{\alpha\beta} = \frac{1}{2}\frac{\partial \bar{g}_{\alpha\beta}}{\partial x^0}, \quad (b)\quad H^\gamma_\beta = \bar{g}^{\gamma\alpha}H_{\alpha\beta} = \frac{1}{2}\bar{g}^{\gamma\alpha}\frac{\partial \bar{g}_{\alpha\beta}}{\partial x^0}.
$$

Also, we define locally the functions

$$
(3.9)\qquad C_\alpha = \Phi^{-1}\frac{\delta\Phi}{\delta x^\alpha}.
$$

Now, by using (3.6), (3.7), (2.7a) and (2.3b) , we obtain the following:

$$
(3.10)\qquad \begin{aligned} &(a)\quad A_{\alpha\beta} = \widetilde{A}_{\mu\nu}\tfrac{\partial \tilde{x}^\mu}{\partial x^\alpha}\tfrac{\partial \tilde{x}^\nu}{\partial x^\beta}, \quad (b)\quad H_{\alpha\beta} = \widetilde{H}_{\mu\nu}\tfrac{\partial \tilde{x}^\mu}{\partial x^\alpha}\tfrac{\partial \tilde{x}^\nu}{\partial x^\beta},\\ &(c)\quad H^\gamma_\alpha\tfrac{\partial \tilde{x}^\mu}{\partial x^\gamma} = \widetilde{H}^\mu_\nu\tfrac{\partial \tilde{x}^\nu}{\partial x^\alpha}, \end{aligned}
$$

with respect to the coordinate transformations (2.2). Also, taking derivative with respect to $x^0$ in (2.7c), and using (2.7a) into (3.9) we deduce that

$$
(3.11)\qquad (a)\quad B_\alpha = \widetilde{B}_\gamma\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}, \quad (b)\quad C_\alpha = \widetilde{C}_\gamma\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha},
$$

with respect to (2.2).

Next, inspired by (2.16), (3.10) and (3.11), we suppose that on each coordinate neighborhood in $M$, there exist $3^{p+q}$ functions $T^{\gamma_1\cdots\gamma_p}_{\alpha_1\cdots\alpha_q}(x^i)$ satisfying

$$
T^{\gamma_1\cdots\gamma_p}_{\alpha_1\cdots\alpha_q}\frac{\partial \tilde{x}^{\mu_1}}{\partial x^{\gamma_1}}\cdots\frac{\partial \tilde{x}^{\mu_p}}{\partial x^{\gamma_p}} = \tilde{T}^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_q}\frac{\partial \tilde{x}^{\nu_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial \tilde{x}^{\nu_q}}{\partial x^{\alpha_q}},
$$

with respect to the transformations (2.2). Then we say that these functions are the local components of a *horizontal tensor field* $T$ of type $(p,q)$ on $M$. Thus, by (2.16), (3.10) and (3.11) we conclude that $\{\bar{g}_{\alpha\beta}, A_{\alpha\beta}, H_{\alpha\beta}\}$, $\{\bar{g}^{\alpha\beta}\}, \{H^\gamma_\alpha\}$ and $\{B_\alpha, C_\alpha\}$ define horizontal tensor fields of types (0,2), (2,0), (1,1) and (0,1) respectively. Throughout the paper we use $\bar{g}_{\alpha\beta}$ and $\bar{g}^{\alpha\beta}$ for lowering and raising greek indices, as in the following examples:

$$
(a)\quad A^\gamma_\beta = \bar{g}^{\gamma\alpha}A_{\alpha\beta}, \quad (b)\quad B^\gamma = \bar{g}^{\gamma\alpha}B_\alpha, \quad (c)\quad C^\gamma = \bar{g}^{\gamma\alpha}C_\alpha.
$$

Finally, we should note that in the previous studies of (1+3) threading of spacetime (cf. Ehlers [6], Ellis [8], van Elst and Uggla [21], Bini, Carini and Jantzen [2]) there have been used projections of tensor fields in $M$ on both $HM$ and $VM$. Moreover, the local components of such projections were considered with respect to the natural field of frames $\{\partial/\partial x^i\}$ on $M$. Our approach is based on the horizontal tensor fields which behave exactly as tensor fields on a 3-dimensional manifold. However, their local components with respect to treading frame and coframe fields are defined on coordinate neighbourhoods in $M$, and therefore they depend , in general, on all four coordinates in $M$.

## 4. The Riemannian horizontal connection

The purpose of this section is to construct a metric connection $\bar{\nabla}$ on the horizontal distribution, and to show that the Levi-Civita connection on the spacetime $(M, g)$ is completely determined by $\bar{\nabla}$, the horizontal tensor fields defined in the previous section, and the function

$$\Psi = \Phi^{-1}\frac{\partial \Phi}{\partial x^0}. \tag{4.1}$$

Also, in the next sections we shall see that $\bar{\nabla}$ enables us to express in a covariant form the $3D$ equations of motion, and to define and study a $3D$ force which leads us to the $3D$ force identity. First, denote by $\nabla$ the Levi-Civita connection on $(M, g)$ given by (cf.O'Neill [18], p.61)

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &+ g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y), \end{aligned} \tag{4.2}$$

for all $X, Y, Z \in \Gamma(TM)$. Then it is easy to check that $\bar{\nabla}$ given by

$$(a)\quad \bar{\nabla}_{hX} hY = h\nabla_{hX} hY, \quad (b)\quad \bar{\nabla}_{vX} hY = h[vX, hY] + H(hY, vX), \tag{4.3}$$

is a metric connection on $HM$, that is, we have

$$(\bar{\nabla}_X \bar{g})(hY, hZ) = 0, \quad \forall X, Y, Z \in \Gamma(TM). \tag{4.4}$$

We call $\bar{\nabla}$ the *Riemannian horizontal connection* on $M$. In order to find the local coefficients of $\bar{\nabla}$ with respect to the threading frame field $\{\delta/\delta x^\alpha, \partial/\partial x^0\}$ we put:

$$(a)\quad \bar{\nabla}_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha} = \bar{\Gamma}_\alpha{}^\gamma{}_\beta \frac{\delta}{\delta x^\gamma}, \quad (b)\quad \bar{\nabla}_{\frac{\partial}{\partial x^0}} \frac{\delta}{\delta x^\alpha} = \bar{\Gamma}_\alpha{}^\gamma{}_0 \frac{\delta}{\delta x^\gamma}. \tag{4.5}$$

Then, we take $X = \delta/\delta x^\beta, Y = \delta/\delta x^\alpha, Z = \delta/\delta x^\mu$ in (4.2), and using (4.3a), (4.5a) (2.13), (2.11) and (3.4a), we obtain

$$\bar{\Gamma}_\alpha{}^\gamma{}_\beta = \frac{1}{2}\bar{g}^{\gamma\mu}\left\{\frac{\delta \bar{g}_{\mu\alpha}}{\delta x^\beta} + \frac{\delta \bar{g}_{\mu\beta}}{\delta x^\alpha} - \frac{\delta \bar{g}_{\alpha\beta}}{\delta x^\mu}\right\}. \tag{4.6}$$

Also, by direct calculations, using (4.5b), (4.3b), (3.4b) and (3.7b), we deduce that

$$\bar{\Gamma}_\alpha{}^\gamma{}_0 = H_\alpha^\gamma. \tag{4.7}$$

Next, we consider a horizontal tensor field $T$ of type $(p, q)$. Then it is important to note that the covariant derivatives $\bar{\nabla}_{hX} T$ and $\bar{\nabla}_{vX} T$ define horizontal tensor of type $(p, q + 1)$ and $(p, q)$ respectively. This can be seen more clearly if we consider the local components $T_\alpha^\gamma$ of a horizontal tensor field of type (1,1). In this case the above covariant derivatives are given by

$$T^\gamma_{\alpha|\beta} = \frac{\delta T^\gamma_\alpha}{\delta x^\beta} + T^\mu_\alpha \bar{\Gamma}_\mu{}^\gamma{}_\beta - T^\gamma_\mu \bar{\Gamma}_\alpha{}^\mu{}_\beta, \tag{4.8}$$

and

$$T^{\gamma}_{\alpha|_0} = \frac{\partial T^{\gamma}_{\alpha}}{\partial x^0} + T^{\mu}_{\alpha} H^{\gamma}_{\mu} - T^{\gamma}_{\mu} H^{\mu}_{\alpha}, \tag{4.9}$$

respectively. The formulas (4.8) and (4.9) can be easy extended to any type of horizontal tensor field. As $\delta/\delta x^{\beta}$ (resp. $\partial/\partial x^0$) is spacelike (resp.timelike) vector field, we call the (4.8) (resp. (4.9)) the *spacelike* (resp. *timelike*) *covariant derivative* of $T$. In particular, since $\bar{\nabla}$ is a metric connection on $HM$, we have

$$(a)\quad \bar{g}_{\alpha\beta|_\gamma} = 0, \quad (b)\quad \bar{g}^{\alpha\beta}{}_{|_\gamma} = 0, \quad (c)\quad \bar{g}_{\alpha\beta|_0} = 0, \quad (d)\quad \bar{g}^{\alpha\beta}{}_{|_0} = 0. \tag{4.10}$$

Finally, by direct calculations using (4.2) and the geometric objects introduced in Sections 3 and 4, we deduce that the Levi-Civita connection $\nabla$ on the spacetime $(M, g)$ is expressed as follows:

$$\begin{aligned}
&(a)\quad \nabla_{\frac{\delta}{\delta x^{\beta}}} \tfrac{\delta}{\delta x^{\alpha}} = \bar{\Gamma}_{\alpha}{}^{\gamma}{}_{\beta} \tfrac{\delta}{\delta x^{\gamma}} + \left(\Phi^{-2} H_{\alpha\beta} + \tfrac{1}{2} A_{\alpha\beta}\right) \tfrac{\partial}{\partial x^0},\\
&(b)\quad \nabla_{\frac{\partial}{\partial x^{0}}} \tfrac{\delta}{\delta x^{\alpha}} = \left(H^{\gamma}_{\alpha} + \tfrac{1}{2}\Phi^2 A^{\gamma}_{\alpha}\right) \tfrac{\delta}{\delta x^{\gamma}} + \left(C_{\alpha} - B_{\alpha}\right) \tfrac{\partial}{\partial x^0},\\
&(c)\quad \nabla_{\frac{\delta}{\delta x^{\alpha}}} \tfrac{\partial}{\partial x^{0}} = \left(H^{\gamma}_{\alpha} + \tfrac{1}{2}\Phi^2 A^{\gamma}_{\alpha}\right) \tfrac{\delta}{\delta x^{\gamma}} + C_{\alpha} \tfrac{\partial}{\partial x^0},\\
&(d)\quad \nabla_{\frac{\partial}{\partial x^{0}}} \tfrac{\partial}{\partial x^{0}} = \Phi^2 \left(C^{\gamma} - B^{\gamma}\right) \tfrac{\delta}{\delta x^{\gamma}} + \Psi \tfrac{\partial}{\partial x^0}.
\end{aligned} \tag{4.11}$$

**Remark 4.1.** *From (4.11) we deduce that any physical theory based on the Levi-Civita connection of the* $4D$ *spacetime can be equivalently developed in terms of the Riemannian horizontal connection and the horizontal tensor fields introduced in the previous section.*

## 5. $3D$ Equations of motion in a $4D$ spacetime

In this section we apply the new approach we developed on the (1+3) threading of $(M, g)$ to the study of equations of motion. More precisely, we obtain in a covariant form, the fully general $3D$ equations of motion induced by the equations of motion in $(M, g)$. The geometric configuration we introduce on $(M, g)$ enables us to study the motions in $(M, g)$ with respect to the geometry of the horizontal distribution. In particular, we show that the geodesics of $(M, g)$ which are tangent to $HM$, must be autoparallels of the Riemannian horizontal connection constructed in Sect.4.

Let $C$ be a smooth curve in $M$ given by parametric equations

$$\begin{aligned}
&(a)\quad x^{\alpha} = x^{\alpha}(t), \quad (b)\quad x^0 = x^0(t),\\
&t \in [a, b],\ \alpha \in \{1, 2, 3\},
\end{aligned} \tag{5.1}$$

where $(x^{\alpha}, x^0)$ are the local coordinates introduced by the (1+3) threading of $(M, g)$. Note that $t$ does not necessarily represent the time in $(M, g)$. Taking into account the decomposition (2.1) and using (2.5),we deduce that the tangent vector field $d/dt$ to $C$ is expressed as follows

$$\frac{d}{dt} = \frac{dx^{\alpha}}{dt} \frac{\delta}{\delta x^{\alpha}} + \frac{\delta x^0}{\delta t} \frac{\partial}{\partial x^0}, \tag{5.2}$$

where we put

$$\frac{\delta x^0}{\delta t} = \frac{dx^0}{dt} + A_\alpha \frac{dx^\alpha}{dt}.$$

Next, by direct calculations, using (4.11) and (5.2), we deduce that

$$(5.3)\quad \begin{aligned} &(a)\ \ \nabla_{\frac{d}{dt}} \frac{\delta}{\delta x^\alpha} = \left\{ \bar{\Gamma}_{\alpha\ \beta}^{\ \gamma} \tfrac{dx^\beta}{dt} + \tfrac{\delta x^0}{\delta t} \left( H_\alpha^\gamma + \tfrac{1}{2}\Phi^2 A_\alpha^\gamma \right) \right\} \tfrac{\delta}{\delta x^\gamma} \\ &\qquad + \left\{ \left( \Phi^{-2} H_{\alpha\beta} + \tfrac{1}{2} A_{\alpha\beta} \right) \tfrac{dx^\beta}{dt} + \tfrac{\delta x^0}{\delta t} \left( C_\alpha - B_\alpha \right) \right\} \tfrac{\partial}{\partial x^0}, \\ &(b)\ \ \nabla_{\frac{d}{dt}} \frac{\partial}{\partial x^0} = \left\{ \left( H_\alpha^\gamma + \tfrac{1}{2}\Phi^2 A_\alpha^\gamma \right) \tfrac{dx^\alpha}{dt} + \Phi^2 \tfrac{\delta x^0}{\delta t} \left( C^\gamma - B^\gamma \right) \right\} \tfrac{\delta}{\delta x^\gamma} \\ &\qquad + \left\{ C_\alpha \tfrac{dx^\alpha}{dt} + \Psi \tfrac{\delta x^0}{\delta t} \right\} \tfrac{\partial}{\partial x^0}. \end{aligned}$$

Then, by using (5.2) and (5.3), and taking into account that $A = (A_{\alpha\beta})$ is a skew-symmetric horizontal tensor field, we obtain

$$(5.4)\quad \begin{aligned} \bar{\nabla}_{\frac{d}{dt}} \frac{d}{dt} &= \left\{ \tfrac{d^2x^\gamma}{dt^2} + \bar{\Gamma}_{\alpha\ \beta}^{\ \gamma} \tfrac{dx^\alpha}{dt} \tfrac{dx^\beta}{dt} + 2 \tfrac{\delta x^0}{\delta t} H_\alpha^\gamma \tfrac{dx^\alpha}{dt} + \Phi^2 \tfrac{\delta x^0}{\delta t} A_\alpha^\gamma \tfrac{dx^\alpha}{dt} \right. \\ &\left. + \Phi^2 \left( \tfrac{\delta x^0}{\delta t} \right)^2 \left( C^\gamma - B^\gamma \right) \right\} \tfrac{\delta}{\delta x^\gamma} + \left\{ \tfrac{d}{dt}\left( \tfrac{\delta x^0}{\delta t} \right) + \Phi^{-2} H_{\alpha\beta} \tfrac{dx^\alpha}{dt} \tfrac{dx^\beta}{dt} \right. \\ &\left. + \tfrac{\delta x^0}{\delta t} (2C_\alpha - B_\alpha) \tfrac{dx^\alpha}{dt} + \left( \tfrac{\delta x^0}{\delta t} \right)^2 \Psi \right\} \tfrac{\partial}{\partial x^0}. \end{aligned}$$

Taking into account that motions in $(M, g)$ are curves of acceleration zero, and using (5.4), we deduce that the fully general equations of motion are expressed by the following two groups of equations:

$$(5.5)\quad \begin{aligned} &(a)\ \ \tfrac{d^2x^\gamma}{dt^2} + \bar{\Gamma}_{\alpha\ \beta}^{\ \gamma} \tfrac{dx^\alpha}{dt} \tfrac{dx^\beta}{dt} + \tfrac{\delta x^0}{\delta t} \left( 2H_\alpha^\gamma + \Phi^2 A_\alpha^\gamma \right) \tfrac{dx^\alpha}{dt} \\ &\qquad + \Phi^2 \left( \tfrac{\delta x^0}{\delta t} \right)^2 \left( C^\gamma - B^\gamma \right) = 0, \\ &(b)\ \ \tfrac{d}{dt}\left( \tfrac{\delta x^0}{\delta t} \right) + \Phi^{-2} H_{\alpha\beta} \tfrac{dx^\alpha}{dt} \tfrac{dx^\beta}{dt} \\ &\qquad + \tfrac{\delta x^0}{\delta t} (2C_\alpha - B_\alpha) \tfrac{dx^\alpha}{dt} + \Psi \left( \tfrac{\delta x^0}{\delta t} \right)^2 = 0, \end{aligned}$$

where $t$ is an affine parameter on the geodesics of $(M, g)$. It is noteworthy that equations (5.5a) are invariant with respect to the coordinate transformations (2.2). We call (5.5a) the $3D$ *equations of motion* in the $4D$ spacetime $(M, g)$.

According to the decomposition (2.1), the geometry of motions in $(M, g)$ is strongly dependent on their positions with respect to $HM$ and $VM$. Here, we develop a local study of the motions in $(M, g)$, that is, we refer to the behaviour of geodesics in a coordinate neighbourhood $\mathcal{U}$ of $M$. First, we say that a curve $C$ given by (5.1) is a *horizontal curve*, if one of the following conditions is satisfied:

$$(5.6)\qquad (a)\quad \frac{\delta x^0}{\delta t} = 0, \quad \text{or} \quad (b)\quad \frac{d}{dt} = \frac{dx^\alpha}{dt} \frac{\delta}{\delta x^\alpha}.$$

Similarly, we say that $C$ is a *vertical curve*, if one of the following conditions is satisfied:

$$(5.7)\qquad (a)\quad x^\alpha = c^\alpha, \quad \alpha \in \{1, 2, 3\}, \quad \text{or} \quad (b)\quad \frac{d}{dt} = \frac{dx^0}{dt} \frac{\partial}{\partial x^0},$$

where $c^\alpha$ are constants. Thus, by using (5.5) and (5.6a) we deduce that *$C$ is a horizontal geodesic of $(M, g)$, if and only if,* (5.6a) *and the following equations are satisfied*:

$$
\text{(5.8)} \qquad \begin{aligned} &(a) \quad \frac{d^2x^\gamma}{dt^2} + \bar{\Gamma}_{\alpha\ \beta}^{\ \gamma} \frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0, \\ &(b) \quad H_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0. \end{aligned}
$$

Also, by using (5.5) (5.7a), (3.5) and (3.9), we infer that *$C$ is a vertical geodesic, if and only if, we have*

$$
x^\alpha = c^\alpha, \quad \alpha \in \{1, 2, 3\}, \qquad \frac{dx^0}{dt} = \frac{1}{c + \int \Psi(c^\alpha, x^0)dt},
$$

$$
\frac{\partial \Phi}{\partial x^\alpha} = \frac{\partial}{\partial x^0}(\Phi A_\alpha),
$$

where $c$ is a constant.

It is important to note that the system (5.8) is tightly related to the geometry of the horizontal distribution. To emphasize this we give some definitions. First, we say that a curve $C$ in $M$ is an *autoparallel* for the Riemannian horizontal connection $\bar{\nabla}$, if it is a horizontal curve satisfying

$$
\text{(5.9)} \qquad \bar{\nabla}_{\frac{d}{dt}} \frac{d}{dt} = 0,
$$

where $d/dt$ is given by (5.6b). Then by using (5.6b) and (4.5a) into (5.9) we deduce that *$C$ is an autoparallel for $\bar{\nabla}$, if and only if*, (5.6a) *and* (5.8a) *are satisfied.* Next, from (4.11a) we deduce that

$$
\text{(5.10)} \qquad K_{\alpha\beta} = \Phi^{-2}H_{\alpha\beta} + \frac{1}{2}A_{\alpha\beta},
$$

can be thought as *second fundamental form* of $HM$. Then we say that a curve $C$ in $M$ is an *asymptotic line* for $HM$, if it is a horizontal curve satisfying

$$
\text{(5.11)} \qquad K_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0.
$$

Taking into account that $A_{\alpha\beta}$ define a skew-symmetric horizontal tensor field and using (5.10) into (5.11) we infer that *$C$ is an asymptotic line for $HM$, if and only if*, (5.6a) *and* (5.8b) *are satisfied.* Summing up these results, we conclude that *a curve $C$ is a horizontal geodesic, if and only if, the following conditions are satisfied*:

(i) *$C$ is an autoparallel for the Riemannian horizontal connection.*

(ii) *$C$ is an asymptotic line for the horizontal distribution.*

## 6. 3$D$ Force identity along a geodesic in a 4$D$ spacetime

The Riemannian horizontal connection introduced in Section 4 enables us to define a 3$D$ force along a geodesic in $(M, g)$. It is important to note that this force is orthogonal to the 3$D$ velocity along a geodesic in $(M, g)$, and thus it brings into the study a new identity which we call the 3$D$ force identity (see (6.10)). Actually, this 3$D$ force is a consequence of the existence of the fourth dimension (time), and therefore represents one of the major differences between Newtonian gravity and Einsten's general relativity.

First, we consider a geodesic $C$ in $(M, g)$ satisfying the system (5.5), and denote by $U(t)$ the projection of its tangent field $d/dt$ on $HM$. Then by (5.2) we have

$$U(t) = U^{\alpha}(t)\frac{\delta}{\delta x^{\alpha}} = \frac{dx^{\alpha}}{dt}\frac{\delta}{\delta x^{\alpha}}. \tag{6.1}$$

We call $U(t)$ the $3D$ *velocity* along $C$. By using $U(t)$ and the Riemannian metric $\bar{g}$ on $HM$, we define the $3D$ *arc length parameter* $\bar{s}$ on $C$ by

$$\bar{s} = \int_a^t \bar{g}(U(t), U(t))^{1/2} dt = \int_a^t \left(\bar{g}_{\alpha\beta}(x^i(t))\frac{dx^{\alpha}}{dt}\frac{dx^{\beta}}{dt}\right)^{1/2} dt,$$

and obtain

$$d\bar{s}^2 = \bar{g}_{\alpha\beta} dx^{\alpha} dx^{\beta}. \tag{6.2}$$

Thus we have

$$\bar{g}(U(\bar{s}), U(\bar{s})) = 1, \tag{6.3}$$

where we put

$$U(\bar{s}) = \frac{dx^{\alpha}}{d\bar{s}}\frac{\delta}{\delta x^{\alpha}}. \tag{6.4}$$

We call $U(\bar{s})$ the *unit* $3D$ *velocity* along the geodesic $C$. Note that $d\bar{s}/dt$ is positive, and therefore $\bar{s}$ defines a new parametrization on $C$. However, *$\bar{s}$ is not necessarily an affine parameter on the geodesic $C$*. As we shall see later in this section, this happens for horizontal geodesics.

Next, we consider $\bar{s}$ as parameter on the geodesic $C$ and define the $3D$ force along $C$ as the horizontal vector field $F(\bar{s})$ given by

$$F(\bar{s}) = \bar{\nabla}_{\frac{d}{d\bar{s}}} U(\bar{s}), \tag{6.5}$$

where $\bar{\nabla}$ is the Riemannian horizontal connection on $HM$, and $d/d\bar{s}$ is the tangent vector field to $C$ expressed as follows

$$\frac{d}{d\bar{s}} = \frac{dx^{\alpha}}{d\bar{s}}\frac{\delta}{\delta x^{\alpha}} + \frac{\delta x^0}{\delta \bar{s}}\frac{\partial}{\partial x^0}. \tag{6.6}$$

If $s^{\star}$ is another $3D$ arc length parameter on $C$ then we have $s^{\star} = \bar{s} + c$, and therefore $F(s^{\star}) = F(\bar{s})$. Moreover, by using (4.4), (6.3) and (6.5), we deduce that $F(\bar{s})$ is orthogonal to $U(\bar{s})$. As $U(t)$ and $U(\bar{s})$ are parallel horizontal vector fields, we conclude that *the $3D$ force $F(\bar{s}(t))$ is orthogonal to the $3D$ velocity $U(t)$ too.*

Now, we put

$$F(\bar{s}) = F^{\gamma}(\bar{s})\frac{\delta}{\delta x^{\gamma}},$$

and by using (6.5), (6.4), (6.6), (4.5) and (4.7) we obtain

$$F^{\gamma}(\bar{s}) = \frac{d^2 x^{\gamma}}{d\bar{s}^2} + \bar{\Gamma}_{\alpha}{}^{\gamma}{}_{\beta}\frac{dx^{\alpha}}{d\bar{s}}\frac{dx^{\beta}}{d\bar{s}} + \frac{\delta x^0}{\delta \bar{s}} H_{\alpha}^{\gamma}\frac{dx^{\alpha}}{d\bar{s}}. \tag{6.7}$$

By using the affine parameter $t$ on the geodesic $C$ and taking into account that

$$\frac{d^2 t}{d\bar{s}^2} = -\frac{d^2 \bar{s}}{dt^2}\left(\frac{d\bar{s}}{dt}\right)^{-3},$$

from (6.7) we deduce that

$$F^{\gamma}(t) = F^{\gamma}(\bar{s}(t)) = \left\{ \tfrac{d^2x^{\gamma}}{dt^2} + \bar{\Gamma}_{\alpha}{}^{\gamma}{}_{\beta}\tfrac{dx^{\alpha}}{dt}\tfrac{dx^{\beta}}{dt} + \tfrac{\delta x^0}{\delta t}H^{\gamma}_{\alpha}\tfrac{dx^{\alpha}}{dt} \right. \tag{6.8}$$
$$\left. -\left(\tfrac{d\bar{s}}{dt}\right)^{-1}\tfrac{d^2\bar{s}}{dt^2}\tfrac{dx^{\gamma}}{dt} \right\} \left(\tfrac{d\bar{s}}{dt}\right)^{-2}.$$

Finally, by using the $3D$ equations of motion (5.5a) into (6.8) we find

$$F^{\gamma}(t) = \left\{ -\tfrac{\delta x^0}{\delta t}\left(H^{\gamma}_{\alpha} + \Phi^2 A^{\gamma}_{\alpha}\right)\tfrac{dx^{\alpha}}{dt} + \Phi^2\left(\tfrac{\delta x^0}{\delta t}\right)^2 (B^{\gamma} - C^{\gamma}) \right. \tag{6.9}$$
$$\left. -\left(\tfrac{d\bar{s}}{dt}\right)^{-1}\tfrac{d^2\bar{s}}{dt^2}\tfrac{dx^{\gamma}}{dt} \right\} \left(\tfrac{d\bar{s}}{dt}\right)^{-2}.$$

Taking into account that $F(t)$ and $U(t)$ are orthogonal horizontal vector fields, and using (6.9), (6.1) and (6.2) we obtain the identity

$$\frac{\delta x^0}{\delta t}H_{\alpha\beta}\frac{dx^{\alpha}}{dt}\frac{dx^{\beta}}{dt} + \Phi^2\left(\frac{\delta x^0}{\delta t}\right)^2 (C_{\alpha} - B_{\alpha})\frac{dx^{\alpha}}{dt} + \frac{d\bar{s}}{dt}\frac{d^2\bar{s}}{dt^2} = 0. \tag{6.10}$$

**Remark 6.1.** *The identity (6.10) must be satisfied along any geodesic in the* $4D$ *spacetime. Therefore, it can be used for a study of motions in* $(M, g)$*, and even for solving the equations of motion in* $(M, g)$*.*

Next, we need to discuss the uniqueness of the identity (6.10). As we have seen, the $3D$ force $F$ was defined by using the Riemannian horizontal connection $\bar{\nabla}$ on $HM$. Then by using the orthogonality between $F$ and $U$ we obtain the identity (6.10). Taking into account that there are some other metric connections on $HM$ (an example being the projection of Levi-Civita connection $\nabla$ on $HM$), it is natural to raise the question: *Do the* $3D$ *forces constructed by means of all metric connections on* $HM$ *induce the same identity* (6.10)? *It is noteworthy that the answer is in the affirmative, as we show in what follows.*

Let $\tilde{\nabla}$ be another metric connection on $HM$ given by

$$\tilde{\nabla}_{\frac{\delta}{\delta x^{\beta}}}\frac{\delta}{\delta x^{\alpha}} = \tilde{\Gamma}_{\alpha}{}^{\gamma}{}_{\beta}\frac{\delta}{\delta x^{\gamma}}, \quad \text{and} \quad \tilde{\nabla}_{\frac{\partial}{\partial x^0}}\frac{\delta}{\delta x^{\alpha}} = \tilde{H}^{\gamma}_{\alpha}\frac{\delta}{\delta x^{\gamma}},$$

and put:

$$\bar{\Gamma}_{\alpha}{}^{\gamma}{}_{\beta} - \tilde{\Gamma}_{\alpha}{}^{\gamma}{}_{\beta} = D_{\alpha}{}^{\gamma}{}_{\beta}, \quad H^{\gamma}_{\alpha} - \tilde{H}^{\gamma}_{\alpha} = D^{\gamma}_{\alpha}. \tag{6.11}$$

Then by using (4.10a) and (4.10c) for both metric connection $\bar{\nabla}$ and $\tilde{\nabla}$, and taking into account (6.11), we obtain

$$D_{\alpha\beta\gamma} + D_{\beta\alpha\gamma} = 0, \quad D_{\alpha\beta} + D_{\beta\alpha} = 0, \tag{6.12}$$

where we put $D_{\alpha\beta\gamma} = \bar{g}_{\beta\mu}D_{\alpha}{}^{\mu}{}_{\gamma}$ and $D_{\alpha\beta} = \bar{g}_{\beta\mu}D^{\mu}_{\alpha}$. Now, by using (6.11) into (6.8) we deduce that the $3D$ forces $\tilde{F}$ and $F$ defined by $\tilde{\nabla}$ and $\bar{\nabla}$ are related by

$$F^{\gamma}(t) = \tilde{F}^{\gamma}(t) + \left(D_{\alpha}{}^{\gamma}{}_{\beta}\frac{dx^{\alpha}}{dt}\frac{dx^{\beta}}{dt} + \frac{\delta x^0}{\delta t}D^{\gamma}_{\alpha}\frac{dx^{\alpha}}{dt}\right)\left(\frac{d\bar{s}}{dt}\right)^{-2}. \tag{6.13}$$

Finally, by using (6.13) and (6.12), and taking into account that both $3D$ forces $F(t)$ and $\tilde{F}(t)$ are orthogonal to the $3D$ velocity $U(t)$, we infer that

$$\bar{g}_{\gamma\mu}F^{\gamma}(t)\frac{dx^{\mu}}{dt} = \bar{g}_{\gamma\mu}\tilde{F}^{\gamma}(t)\frac{dx^{\mu}}{dt} = 0.$$

As a conclusion, we proved that *all the 3D forces along a geodesic in a 4D spacetime induce the same identity* (6.10). For this reason we call (6.10) the *3D force identity.* As we shall see in the next two sections, (6.10) together with (5.5) bring a lot of information on the FRW universes and on Kerr black holes.

## 7. 3D Force identity for a FRW universe

Let $S$ be a connected 3-dimensional manifold endowed with a Riemannian metric $h = (h_{\alpha\beta})$, of constant curvature $k = -1, 0$, or 1. Also, suppose that $f$ is a positive smooth function on an open interval $I$ in $R$. Then, a *Robertson-Walker (RW) spacetime* is a pair $(M(k, f), g)$, where $M(k, f)$ is the warped product $I \times_f S$ and $g$ is a Lorentz metric with line element given by

$$ds^2 = -(dx^0)^2 + f^2(x^0)h_{\alpha\beta}(x^\mu)dx^\alpha dx^\beta. \tag{7.1}$$

Comparing (7.1) with (2.9) and using (2.12), (3.5), (3.9) and (4.1), we obtain

$$\Phi = 1, A_\alpha = 0, A_{\alpha\beta} = 0, B_\alpha = 0, C_\alpha = 0, \Psi = 0. \tag{7.2}$$

Hence the horizontal distribution is integrable, and its induced metric $\bar{g}$ is given by (cf. (2.14))

$$\bar{g}_{\alpha\beta} = f^2(x^0)h_{\alpha\beta}(x^\mu). \tag{7.3}$$

Also, we have

$$\frac{\delta x^0}{\delta t} = \frac{dx^0}{dt}. \tag{7.4}$$

Note that $\delta/\delta x^\alpha = \partial/\partial x^\alpha$, and therefore, in this section the local components of all geometric objects are expressed with respect to the natural frame field $\{\partial/\partial x^\alpha, \partial/\partial x^0\}$, where $\partial/\partial x^\alpha \in \Gamma(HM(k, f))$ and $\partial/\partial x^0 \in \Gamma(VM(k, f))$.

Now, by using (3.8) and taking into account (7.3) we deduce that

$$(a) \quad H_{\alpha\beta} = f\frac{df}{dx^0}h_{\alpha\beta}, \quad H_\beta^\gamma = H\delta_\beta^\gamma, \tag{7.5}$$

where $H$ is the *Hubble parameter* given by

$$H(x^0) = \frac{1}{f(x^0)}\frac{df}{dx^0}. \tag{7.6}$$

Finally, the local coefficients $\bar{\Gamma}_{\alpha\ \beta}^{\ \gamma}$ of the Riemannian horizontal connection $\bar{\nabla}$ are equal to the local coefficients of the Levi-Civita connection on $S$ given by

$$\Gamma_{\alpha\ \beta}^{\ \gamma} = \frac{1}{2}h^{\gamma\mu}\left\{\frac{\partial h_{\mu\alpha}}{\partial x^\beta} + \frac{\partial h_{\mu\beta}}{\partial x^\alpha} - \frac{\partial h_{\alpha\beta}}{\partial x^\mu}\right\}. \tag{7.7}$$

Next, by using (7.2)-(7.5), (7.7) and (6.2) into (5.5) and (6.10), we deduce that in a RW spacetime $(M(k, f), g)$ the equations of motion and the 3D force identity are given by

$$\begin{aligned} &(a) \quad \tfrac{d^2x^\gamma}{dt^2} + \Gamma_{\alpha\ \beta}^{\ \gamma}\tfrac{dx^\alpha}{dt}\tfrac{dx^\beta}{dt} + 2H\tfrac{dx^0}{dt}\tfrac{dx^\gamma}{dt} = 0, \\ &(b) \quad \tfrac{d^2x^0}{dt^2} + H\left(\tfrac{d\bar{s}}{dt}\right)^2 = 0, \end{aligned} \tag{7.8}$$

and

$$H\frac{d\bar{s}}{dt}\frac{dx^0}{dt}+\frac{d^2\bar{s}}{dt^2}=0, \tag{7.9}$$

respectively. Integrate (7.9) and infer that the $3D$ force identity is given by

$$f(x^0(t))\frac{d\bar{s}}{dt}=k, \tag{7.10}$$

where k is a positive constant. Suppose that the slices $S(x^0)$ of $HM(k,f)$ are expanding, that is $H>0$ along each geodesic. Then from (7.8b) we deduce that *the time coordinate* $x^0$ *on* $M(k,f)$ *can not be an affine parameter on a geodesic.* In particular, from (7.8b) we see that do not exist (even locally) horizontal geodesics. This means that *a geodesic of* $(M(k,f),g)$ *which is tangent to a slice* $S(x^0)$ *at the point* $P$ *can not lie in a neighbourhood of* $P$ *in* $S(x^0)$. Thus, from now on we may suppose, without loss of generality, that $dx^0/dt>0$ along any geodesic. Now, by using (7.10) and (7.6) into (7.8b) we obtain

$$\frac{d^2x^0}{dt^2}+\frac{k^2}{f^3(x^0)}\frac{df}{dx^0}=0. \tag{7.11}$$

Multiplying (7.11) by $dx^0/dt$ and then integrating, we infer that

$$f(x^0(t))\left(\left(\frac{dx^0}{dt}\right)^2+c\right)^{1/2}=k, \tag{7.12}$$

where $c$ is a constant. Two cases have to be studied.

**Case 1.** c = 0. In this case (7.12) becomes

$$\frac{dx^0}{dt}=\frac{k}{f}, \tag{7.13}$$

and the geodesic is a photon, that is, a future-pointing null geodesic. This follows from

$$\left(\frac{ds}{dt}\right)^2=\left|-\left(\frac{dx^0}{dt}\right)^2+\left(\frac{d\bar{s}}{dt}\right)^2\right|, \tag{7.14}$$

by using (7.10) and (7.13). We have to note that (7.13) has an important role in a relativistic explanation for the cosmological redshift for a Robertson-Walker spacetime.

**Case 2.** $c\neq 0$. In this case, by using (7.10) and (7.12) in (7.14), we deduce that the geodesic is either spacelike or timelike, and its arc length parameter $s$ is an affine parameter. Taking $s$ as affine parameter on the geodesic, we deduce that $c=1$ (resp. $c=-1$) for a spacelike (resp. timelike) geodesic. Thus (7.12) becomes

$$\frac{dx^0}{ds}=\left(\frac{k^2}{f^2}-1\right)^{1/2}, \tag{7.15}$$

in case of a spacelike geodesic, and

$$\frac{dx^0}{ds}=\left(\frac{k^2}{f^2}+1\right)^{1/2},$$

in case of a timelike geodesic. From (7.15) we see that the warping function along a spacelike geodesic of a RW spacetime must be bounded above by k.

Now, we recall the three FRW cosmological models for dust (p = 0) (cf.[18] p.351):
(A) Einstein-de Sitter cosmological model is the RW spacetime
$M(0,(x^0)^{2/3}) = \mathbf{R}^+ \times_{(x^0)^{2/3}} \mathbf{R}^3$.
(B) The RW spacetime $M(1,f) = (0,\pi A) \times_{f(x^0)} \mathbf{S}^3$, where $A > 0$ and

$$(7.16) \qquad (a) \quad f = \frac{1}{2}A\,(1-\cos\eta)\,, \quad (b) \quad x^0 = \frac{1}{2}A\,(\eta - \sin\eta)\,,\ \eta \in (0,2\pi).$$

(C) The RW spacetime $M(-1,f) = \mathbf{R}^+ \times_{f(x^0)} \mathbf{H}^3$, where

$$(7.17) \qquad (a) \quad f = \frac{1}{2}A\,(\cosh\eta - 1)\,, \quad (b) \quad x^0 = \frac{1}{2}A\,(\sinh\eta - \eta)\,.$$

The method we developed in this paper enables us to write down explicitly the equations of null geodesics in all the above models. First, taking into account that $f(x^0) = (x^0)^{2/3}$ for the model (A), and using (7.13) we obtain

$$(7.18) \qquad x^0 = \left(\frac{5k}{3}t + a\right)^{3/5}, \quad t > -\frac{3a}{5k}, \quad a \in R.$$

Then, as $\Gamma_{\alpha\ \beta}^{\ \gamma} = 0$ for all $\alpha,\beta,\gamma \in \{1,2,3\}$, (7.8a) becomes

$$(7.19) \qquad \frac{d^2x^\gamma}{dt^2} + 2f^{-1}\frac{df}{dt}\frac{dx^\gamma}{dt} = 0.$$

Integrate (7.19), and deduce that

$$(7.20) \qquad x^\gamma = \frac{3}{k}\left(\frac{5k}{3}t + a\right)^{1/5} c^\gamma + b^\gamma,$$

where $b^\gamma \in R$ and $c^\gamma \neq 0$ for any $\gamma \in \{1,2,3\}$. Thus any null geodesic of Einstein-de Sitter model is given by equations (7.18) and (7.20). In Case (B), by using (7.13) and (7.16) we infer that the affine parameter $t$ on the geodesic and $\eta$ are related by

$$(7.21) \qquad \frac{d\eta}{dt} = \frac{k}{f^2}.$$

Then we consider (7.8a) in the form

$$\frac{d^2x^\gamma}{dt^2} + \Gamma_{\alpha\ \beta}^{\ \gamma}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} + \frac{2}{f}\frac{df}{dt}\frac{dx^\gamma}{dt} = 0,$$

which by means of (7.21) becomes

$$\frac{d^2x^\gamma}{d\eta^2} + \Gamma_{\alpha\ \beta}^{\ \gamma}\frac{dx^\alpha}{d\eta}\frac{dx^\beta}{d\eta} = 0.$$

Thus any null geodesic in case of $M(1,f)$ is given by the pair $(x^0(\eta), x^\gamma(\eta))$ where $x^0(\eta)$ is given by (7.16b) and $x^\gamma(\eta)$ represent the equations of a geodesic in $\mathbf{S}^3$. In a similar way we deduce that the null geodesics in $M(-1,f)$ are given by the pair $(x^0(\eta), x^\gamma(\eta))$ where $x^0(\eta)$ is given by (7.17b) and $(x^\gamma(\eta))$ are given by the equations of geodesics of $\mathbf{H}^3$.

## 8. 3$D$ Force identity for Kerr black hole

The most general stationary black hole solutions of the vacuum Einstein equations are described by the family of metrics defined in 1963 by Kerr [12]. Since then, these solutions are called *Kerr black holes*. The geometry of geodesics of Kerr black holes is presented in detail in the excellent monographs of Chandrasekhar [5] and O'Neill [19]. However, by using the new method of study developed in the present paper, we find new properties of geodesics of a Kerr black hole. First, we deduce a simple form of the line element (see (8.4)) and show that a Kerr black hole is an example wherein the horizontal distribution is not integrable. Then we prove the existence of geodesics in a Kerr black hole that are tangent to the horizontal distribution at any of their points. Finally, we present characterizations of geodesics along which the conserved energy is positive, negative or equal to zero.

The family of Kerr black hole solutions depend on two parameters: the total mass $m$ and the angular momentum $J$. The line element of the spacetime $(M, g)$ around a Kerr black hole is given by

$$\begin{aligned} ds^2 &= -\left(1-\tfrac{2mr}{\Sigma}\right)\left(dx^0\right)^2 - \tfrac{4amr\sin^2\theta}{\Sigma}dx^0 d\varphi \\ &+ \left(r^2+a^2+\tfrac{2a^2mr\sin^2\theta}{\Sigma}\right)\sin^2\theta d\varphi^2 + \tfrac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \end{aligned} \tag{8.1}$$

where $(x^0, r, \theta, \varphi)$ are the Boyer-Lindquist coordinates, and we put

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2mr + a^2, \qquad a = \frac{J}{m}.$$

Comparing (8.1) with (2.9) and using (2.12) we obtain

$$(a)\quad \Phi^2 = 1 - \frac{2mr}{\Sigma}, \quad (b)\quad A_\varphi = \frac{2amr\sin^2\theta}{\Sigma - 2mr}, \quad (c)\quad A_r = A_\theta = 0. \tag{8.2}$$

Then by using (2.5) and (8.2c) we deduce that the horizontal distribution $HM$ is locally spanned by

$$\frac{\delta}{\delta\varphi} = \frac{\partial}{\partial\varphi} - A_\varphi\frac{\partial}{\partial x^0}, \qquad \frac{\delta}{\delta r} = \frac{\partial}{\partial r}, \qquad \frac{\delta}{\delta\theta} = \frac{\partial}{\partial\theta}.$$

By using (8.2) into (3.5a) we infer that the only non zero local components of the horizontal tensor field $A$ (see (3.5)) are the following:

$$\begin{aligned} A_{r\varphi} &= \tfrac{2am(a^2\cos^2\theta - r^2)\sin^2\theta}{(\Sigma-2mr)^2}, \\ A_{\theta\varphi} &= \tfrac{2amr\Delta\sin 2\theta}{(\Sigma-2mr)^2}. \end{aligned}$$

Thus the *horizontal distribution on the spacetime of a Kerr black hole is not integrable.*

Next, by using (2.14), (8.1) and (8.2), we deduce that the only non zero entries of the matrix $[\bar{g}_{\alpha\beta}]$ are given by

$$\bar{g}_{\varphi\varphi} = \frac{\Delta\sin^2\theta}{\Phi^2}, \quad \bar{g}_{rr} = \frac{\Sigma}{\Delta}, \quad \bar{g}_{\theta\theta} = \Sigma. \tag{8.3}$$

Hence the metric of a Kerr black hole is simply expressed as follows:

$$ds^2 = -\Phi^2(\delta x^0)^2 + \frac{\Delta\sin^2\theta}{\Phi^2}d\varphi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \tag{8.4}$$

where we put

$$\delta x^0 = dx^0 + A_\varphi d\varphi.$$

Now, from (3.5b), (3.8) and (4.1), we obtain

$$\text{(8.5)} \qquad (a) \quad B_\alpha = 0, \quad (b) \quad H_{\alpha\beta} = 0, \quad (c) \quad H_\alpha^\gamma = 0, \quad (d) \quad \Psi = 0,$$

via (8.2) and (8.3). Also, by using (8.5d) into (3.9), we infer that

$$\text{(8.6)} \qquad C_\alpha \frac{dx^\alpha}{dt} = \Phi^{-1} \frac{d\Phi}{dt},$$

along a geodesic $C$ in the Kerr spacetime $(M, g)$. Due to (8.5) and (8.6), the equation of motion (5.5b) and the $3D$ force identity (6.10) become

$$\text{(8.7)} \qquad \frac{d}{dt}\left(\frac{\delta x^0}{\delta t}\right) + 2\Phi^{-1}\frac{d\Phi}{dt}\frac{\delta x^0}{dt} = 0,$$

and

$$\text{(8.8)} \qquad \left(\frac{\delta x^0}{\delta t}\right)^2 \Phi \frac{d\Phi}{dt} + \frac{d\bar{s}}{dt}\frac{d^2\bar{s}}{dt^2} = 0,$$

respectively. Finally, by using (2.15) and (6.2), we obtain

$$\text{(8.9)} \qquad \left(\frac{ds}{dt}\right)^2 = \left| \left(\frac{d\bar{s}}{dt}\right)^2 - \Phi^2 \left(\frac{\delta x^0}{\delta t}\right)^2 \right|.$$

In order to study (8.7) and (8.8), we consider the following two cases.

**Case 1.** Suppose that

$$\text{(8.10)} \qquad \frac{\delta x^0}{\delta t} = \frac{dx^0}{dt} + A_\varphi \frac{d\varphi}{dt} = 0,$$

that is, $C$ is a horizontal geodesic. Then, (8.7) is identically satisfied and from (8.8) and (8.9) we deduce that both *the* $3D$ *arc length parameters* $\bar{s}$ *and the* $4D$ *arc length parameter* $s$ *are affine parameters on* $C$. Hence, taking into account (5.6a) and (5.8) we can state that $C$ *is a horizontal geodesic in the Kerr spacetime* $(M, g)$*, if and only if,* (8.10) *and the following equations are satisfied*:

$$\frac{d^2x^\gamma}{dt^2} + \bar{\Gamma}_{\alpha\ \beta}^{\ \gamma}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0,$$

where $\bar{\Gamma}_{\alpha\ \beta}^{\ \gamma}$ are local coefficients for the Riemannian horizontal connection $\bar{\nabla}$ (see(4.6)). Since $\bar{g}_{\alpha\beta}$ given by (8.3) do not depend on $x^0$, from (4.6) we infer that $\bar{\Gamma}_{\alpha\ \beta}^{\ \gamma}$ are given by similar formulas for the Christoffel symbols of a $3D$ Riemannian manifold:

$$\bar{\Gamma}_{\alpha\ \beta}^{\ \gamma} = \frac{1}{2}\bar{g}^{\gamma\mu}\left\{\frac{\partial \bar{g}_{\mu\alpha}}{\partial x^\beta} + \frac{\partial \bar{g}_{\mu\beta}}{\partial x^\alpha} - \frac{\partial \bar{g}_{\alpha\beta}}{\partial x^\mu}\right\}.$$

However,since the horizontal distribution is not integrable, $\bar{\Gamma}_{\alpha\ \beta}^{\ \gamma}$ are not the local coefficients of the Levi-Civita connection on a hypersurface in $(M, g)$.

Next, we remark that due to (8.5b) the Kerr spacetime has bundle-like metric with respect to the vertical foliation (cf. Reinhart [20], Bejancu-Farran [1], p.111). Thus we have the following interesting property of geodesics of a Kerr black hole:*If a geodesic of* $(M, g)$ *is tangent to the horizontal distribution at one point, then it remains tangent to it at all later times.*

**Case 2.** Suppose that $\delta x^0/\delta t$ is non zero along the geodesic $C$. Then we integrate (8.7) and obtain

$$\Phi^2 \left| \frac{\delta x^0}{\delta t} \right| = k_1 \,, \tag{8.11}$$

where $k_1$ is a positive constant. Next, by using (8.11) into the $3D$ force identity (8.8) and integrating, we deduce that

$$\left( \frac{d\bar{s}}{dt} \right)^2 = (k_1)^2 \Phi^{-2} + k_2, \tag{8.12}$$

where $k_2$ is a real constant. Due to (6.2) and (8.3), (8.12) becomes

$$\frac{\Delta \sin^2\theta}{\Phi^2} \left( \frac{d\varphi}{dt} \right)^2 + \frac{\Sigma}{\Delta} \left( \frac{dr}{dt} \right)^2 + \Sigma \left( \frac{d\theta}{dt} \right)^2 = (k_1)^2 \Phi^{-2} + k_2. \tag{8.13}$$

Finally, by using (8.11), (8.5c), (8.5d) and (3.9) into (5.5a), we obtain

$$\frac{d^2 x^\gamma}{dt^2} + \bar{\Gamma}_{\alpha\ \beta}^{\ \gamma} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} + k_1 \bar{g}^{\gamma\mu} \left( A_{\mu\alpha} \frac{dx^\alpha}{dt} + k_1 \Phi^{-3} \frac{\partial \Phi}{\partial x^\mu} \right) = 0. \tag{8.14}$$

Thus we may state that any geodesic that is not horizontal, must be a solution of the system formed by (8.11) and (8.14). The equation (8.13), which represents the $3D$ force identity, might have an important role in the dynamics of geodesics in $(M, g)$. By using (8.13) and (8.11) into (8.9) we obtain

$$\left( \frac{ds}{dt} \right)^2 = |k_2|, \tag{8.15}$$

which implies that $k_2 = 0$, $k_2 > 0$ or $k_2 < 0$, according as the geodesic is null, spacelike or timelike, respectively. Also from (8.15) we deduce that *the $4D$ arc length parameter $s$ is an affine parameter.*

We close this section with an interesting characterization of the conserved energy along a geodesic. To state this, we recall that the energy $E$ per unit rest mass for a geodesic $C$ with tangent vector $d/dt$ is a constant given by

$$E = -g\left( \frac{d}{dt}, \frac{\partial}{\partial x^0} \right). \tag{8.16}$$

Then, by using (5.2) and (2.10c) into (8.16), we obtain

$$E = \Phi^2 \frac{\delta x^0}{\delta t}. \tag{8.17}$$

From (8.17) we deduce that *the energy for a geodesic $C$ is equal to zero, if and only if, the geodesic is horizontal.* Moreover, from (8.17) we conclude that *the energy along a geodesic $C$ is positive (resp. negative) according as its tangent vector field*

$$\frac{d}{dt} = \frac{dx^\alpha}{dt} \frac{\delta}{\delta x^\alpha} + \frac{\delta x^0}{\delta t} \frac{\partial}{\partial x^0},$$

has $\delta x^0/\delta t > 0$ (resp. $\delta x^0/\delta t < 0$).

## 9. Conclusions

In the present paper we develop a new method for the study of the (1+3) threading of spacetime and apply it to the dynamics of a FRW universe and of a Kerr black hole. The differences between this method and the earlier approaches on this matter can be summarized as follows. So far, the coordinate systems of the spacetime have not been used in the construction of adapted frame and coframe fields for the (1+3) threading. In this way, some of the geometric objects involved in the study might be incorrectly defined. As a first example we present the three local functions $\{A_\alpha\}$ from (2.5) which by (2.7c) do not define a $3D$ tensor field. However, in the whole literature published so far it is claimed that these functions are the local components of the so called "shift 1-form". Also, the spatial equations of motion ($3D$ equations of motion in our terminology) are given by the equations corresponding to the three spatial indices (cf. Zel'manov [24], p.139). Certainly such equations are not invariant with respect to the transformations of coordinates of the spacetime, and therefore they can not represent a realistic phenomenon. Unfortunately, a similar procedure has been applied for spatial Einstein equations.

The method developed in the present paper opens new perspectives in the study of some other concepts and equations from the geometry and physics of the spacetime. As example, we only mention the splitting of Einstein equations and the study of gravito-electromagnetism. By using the same method, these subjects and some others related to them will be investigated in near future.

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