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VISIBILITY IN PROXIMAL DELAUNAY MESHES AND STRONGLY NEAR WALLMAN PROXIMITY

JAMES F. PETERS

Dedicated to the Memory of Som Naimpally

ABSTRACT. This paper introduces a visibility relation v, leading to the strongly visible relation $\stackrel{\wedge\wedge}{v}$ on proximal Delaunay meshes. Two main results in this paper are that the visibility relation v is equivalent to Wallman proximity and the *strongly near* proximity $\stackrel{\wedge\wedge}{\delta}$ is a Wallman proximity. In addition, a Delaunay triangulation region endowed with the visibility relation v has a local Leader uniform topology.

1. INTRODUCTION

Delaunay triangulations, introduced by B.N Delone [Delaunay] [3], represent pieces of a continuous space. A *triangulation* is a collection of triangles, which includes the edges and vertices of the triangles in the collection.

A 2D *Delaunay triangulation* of a set of sites (generators) $S \subset \mathbb{R}^2$ is a triangulation of the points in S. The set of vertices (called sites) in a Delaunay triangulation define a *Delaunay mesh*. A Delaunay mesh endowed with a nonempty set of proximity relations is a *proximal Delaunay mesh*. A proximal Delaunay mesh is an example of a proximal relator space [20], which is an extension of a Száz relator space [21, 22, 23].

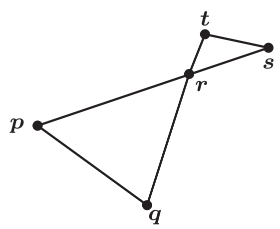

FIGURE 1. Visibility

Let $S \subset \mathbb{R}^2$ be a set of distinguished points called *sites* (mesh generating points), $p, q \in S$, $\overline{pq}$ a straight line segment in the Euclidean plane. A site p in a straight line segment $\overline{pq}$ is *visible* to another site q in the same straight line segment, provided there is no other site between p and q. New forms of proximity are found via the geometry of visibility.

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Example 1.1. Visible Points.
Let $p, q, r, s, t \in S$, a set of sites. A pair of Delaunay triangles $\triangle(pqr), \triangle(rst)$ are shown in Fig. 1. Points r, q are visible from p, since $\overline{pq}$ and $\overline{pr}$ are straight line segments with no other sites in between the endpoints. However, in the straight line segment $\overline{ps}$, s is not visible from p, since site r is blocking p's view of s. Similarly, points p, r are visible from q but in the straight line segment $\overline{qt}$, t is not visible from q. From r, points p, q, s, t are visible. For more about visibility, see [7].

A straight edge connecting sites p and q is a *Delaunay edge* if and only if the Voronoï region of p [6, 18] and Voronoï region of q intersect along a common line segment [5, §I.1, p. 3]. For example, in Fig. 2, the intersection of Voronoï regions V_p, V_q is a common edge, *i.e.*, $V_p \cap V_q = \overline{xy}$, and p and q are connected by the straight edge $\overline{pq}$. Hence, $\overline{pq}$ is a Delaunay edge in Fig. 2.

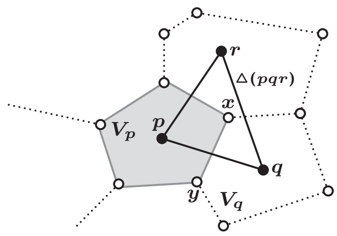

FIGURE 2. Delaunay triangle $\triangle(pqr)$

A triangle with vertices $p, q, r \in S$ is a *Delaunay triangle* (denoted $\triangle(pqr)$ in Fig. 2), provided the edges in the triangle are Delaunay edges. This paper introduces proximal Delaunay triangulation regions derived from the sites of Voronoï regions [18], which are named after the Ukrainian mathematician Georgy Voronoï [25].

A nonempty set A of a space X is a *convex set*, provided $\alpha A + (1-\alpha)A \subset A$ for each $\alpha \in [0, 1]$ [1, §1.1, p. 4] (see, also, [10]). A *simple convex set* is a closed half plane (all points on or on one side of a line in R^2 [6]) The edges in a Delaunay mesh are examples of convex sets. A closed set S in the Euclidean space E^n is *convex* if and only if to each point in E^n there corresponds a unique nearest point in S. In this paper, E denotes a normed linear space and the set of sites S is a subset of E. For $z \in S$, a closed set in $\mathbb{R}^n$,

$$S_z = \left\{ x \in E : \|x - z\| = \inf_{y \in S} \|x - y\| \right\},$$

which is a *convex cone*[24].

Lemma 1.1. [6, §2.1, p. 9] *The intersection of convex sets is convex.*

Proof. Let $A, B \subset \mathbb{R}^2$ be convex sets and let $K = A \cap B$. For every pair of points $x, y \in K$, the line segment $\overline{xy}$ connecting x and y belongs to K, since this property holds for all points in A and B. Hence, K is convex. $\square$

2. Preliminaries

Delaunay triangles are defined on a finite-dimensional normed linear space E that is topological. For simplicity, E is the Euclidean space $\mathbb{R}^2$. The *closure* of $A \subset E$ (denoted cl$A$) is defined by

$$\mathrm{cl}(A) = \{x \in X : D(x, A) = 0\}, \text{ where}$$
$$D(x, A) = inf\,\{\|x - a\| : a \in A\},$$

i.e., cl(A) is the set of all points x in X that are close to A ($D(x, A)$ is the Hausdorff distance [10, §22, p. 128] between x and the set A and $\|x - a\|$ is the Euclidean distance between x and a).

Let A^c denote the complement of A (all points of E not in A). The *boundary* of A (denoted bdy$A$) is the set of all points that are near A and near A^c [14, §2.7, p. 62]. An important structure is the *interior* of A (denoted int$A$), defined by int$A$ = cl$A$ − bdy$A$. For example, the interior of a Delaunay edge $\overline{pq}$ are all of the points in the segment, except the endpoints p and q. Notice that the interior of a Delaunay triangle is empty.

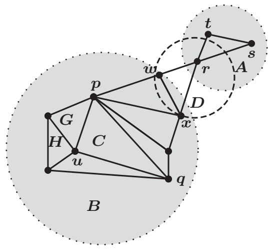

Figure 3. Far

In general, a *relator* is a nonvoid family of relations $\mathcal{R}$ on a nonempty set X. The pair $(X, \mathcal{R})$ is called a relator space. Let E be endowed with the relator $\mathcal{R}_\delta$ defined by

$$\mathcal{R}_\delta = \left\{\delta, \overset{\wedge\wedge}{\delta}, \not\delta, \overset{\wedge\wedge}{\not\delta}\right\},$$

called a *proximal relator* (*cf.* [20]), containing the the proximities $\delta, \overset{\wedge\wedge}{\delta}, \not\delta, \overset{\wedge\wedge}{\not\delta}$. The Delaunay tessellated space E endowed with the proximal relator $\mathcal{R}_\delta$ (briefly, $\mathcal{R}$) is a *Delaunay proximal relator space.*

The proximity relations δ (near), $\overset{\wedge\wedge}{\delta}$ (*strongly near*) and their counterparts $\not\delta$ (far) and $\overset{\wedge\wedge}{\not\delta}$ (*strongly far*) facilitate the description of properties of Delaunay edges, triangles, triangulations and regions. The strongly near proximity $\overset{\wedge\wedge}{\delta}$ was introduced in [17].

Remark 2.1. *The notation $\overset{\wedge\wedge}{\not\delta}$ for the strongly far proximity was suggested by C. Guadagni* [9]. *For the use of $\not\delta$ in local proximity spaces, see* [8, §2.2, p. 7]. *The notation for the* far *proximity $\not\delta$ is commonly used* (*see,* e.g., [4, 12]). *The variant notation $\underline{\delta}$ for the* far *proximity is also used* [14]. *For various forms of proximity, see* [4, 8, 15, 12, 13, 14, 16].

Let $A, B \subset E$. The set A is near B (denoted $A\ \delta\ B$), provided cl$A\cap$cl$B \neq \emptyset$ [4] (closure axiom). The Wallman proximity δ (named after H. Wallman [26]) satisfies the closure axiom as well as the four Čech proximity axioms [2, §2.5, p. 439] and is central in near set theory [14, 15]. Sets A, B are *far* apart (denoted $A\ \not\delta\ B$), provided cl$A \cap$ cl$B = \emptyset$. For example, Delaunay edges $\overline{pq}\ \delta\ \overline{qr}$ are near, since the edges have a common point, *i.e.*, $q \in \overline{pq} \cap \overline{qr}$ (see, *e.g.*, $\overline{pq}\ \delta\ \overline{qr}$ in Fig. 2). By contrast, edges $\overline{pr}, \overline{xy}$ have no points in common in Fig. 2, *i.e.*, $\overline{pr}\ \not\delta\ \overline{xy}$.

Voronoï regions V_p, V_q are strongly near (denoted $V_p \ \overset{\wedge\wedge}{\delta}\ V_q$) if and only if the regions have a common edge.

Example 2.1. Near and Strongly Near Sets.
In the Delaunay mesh in Fig. 3, let C, G, H be mesh triangles. $H\ \delta\ C$, since each these pairs of triangles have a common vertex. $H\ \overset{\wedge\wedge}{\delta}\ G$ and $G\ \overset{\wedge\wedge}{\delta}\ C$ (the triangles are strongly near), since these triangles have a common edge. Similarly, in Fig. 2, $V_p\ \overset{\wedge\wedge}{\delta}\ V_q$. In general, strongly near *Delaunay triangles have a common edge. Delaunay triangles $\triangle(pqr)$ and $\triangle(qrt)$ are strongly near in Fig. 4, since edge $\overline{qr}$ is common to both triangles. In that case, we write $\triangle(pqr)\ \overset{\wedge\wedge}{\delta}\ \triangle(qrt)$.*

Nonempty sets $A\ \overset{\wedge\wedge}{\not\delta}\ C$ are strongly far apart (denoted A, C), provided $C \subset \mathrm{int}(\mathrm{cl}B)$ and $A \not\delta B$.

Example 2.2. Far and Strongly Far Sets.
In the Delaunay mesh in Fig. 3, sets A and B have no points in common. Hence, $A\ \not\delta\ B$ (A is far from B). Also in Fig. 3, let $C = \{\triangle(pqu)\}$. Consequently, $C \subset int(clB)$, such that triangle $\triangle(pqu)$ lies in the interior of the closure of B. Hence, $A \overset{\wedge\wedge}{\not\delta} C$.

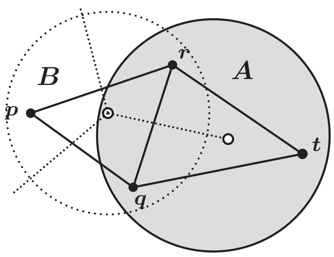

FIGURE 4. Strongly Visible Sets $A\ \overset{\wedge\wedge}{v}\ B$

Let A, B be subsets in a Delaunay mesh, $\triangle(pqr) \in B, \triangle(qrt) \in A$. Subsets A, B in a Delaunay mesh are *visible* to each other (denoted AvB), provided at least one triangle vertex $x \in \mathrm{cl}A \cap \mathrm{cl}B$. That is, if there is at least one site in A visible to at least one site in B, then AvB. A, B are *strongly visible* to each other (denoted $A \overset{\wedge\wedge}{v} B$), provided at least one triangle edge is common to A and B.

Example 2.3. Visibility in Delaunay Meshes.
*In the Delaunay mesh in Fig. 3, $A\ v\ D$, since A and D have one triangle vertex is common, namely, vertex r. Sets B and D in Fig. 3 are strongly visible (*i.e., *$B \overset{\wedge\wedge}{v} D$), since edge $\overline{wx}$ is common to B and D. In Fig. 3, let $C = \{\triangle(pqu)\}$. Then $C \overset{\wedge\wedge}{v} B$, since $C \subset B$. In Fig. 4, edge $\overline{qr}$ is common to A and B. $\overline{qr}$ is visible from $p \in B$ and from $t \in A$. Hence, $A \overset{\wedge\wedge}{v} B$.*

Subsets A, B in a Delaunay mesh are *invisible* to each other (denoted $A \not{v} B$), provided $\mathrm{cl}A \cap \mathrm{cl}B = \emptyset$, *i.e.*, A and B have no triangle vertices in common. A, B are *strongly* *invisible* to each other (denoted $A \overset{\wedge\wedge}{\not v} B$), provided $C \not v A$ for all sets of mesh triangles $C \subset B$.

Example 2.4. Invisible and Strongly Invisible Subsets in a Delaunay Mesh. *In the Delaunay mesh in Fig. 3, A and B are not visible to each other, since $clA \cap clB = \emptyset$,* i.e., *A and B have no triangle vertices in common. In Fig. 3, let $C = \{\triangle(pqu)\}$. Then $A \overset{\wedge\wedge}{\not v} B$ (A and B are strongly invisible to each other), since $C \not v A$ for all sets of mesh triangles $C \subset B$.*

3. Main Results

The Delaunay visibility relation v is equivalent to the proximity δ.

Lemma 3.1. *Let A, B be subsets in a Delaunay mesh. $A\ \delta\ B$ if and only if $A\ v\ B$.*

Proof. $A\ \delta\ B \Leftrightarrow \mathrm{cl}A \cap \mathrm{cl}B \neq \emptyset \Leftrightarrow A$ and B have a triangle vertex in common if and only if $A\ v\ B$. □

Theorem 3.1. *The visibility relation v is a Wallman proximity.*

Proof. Immediate from Lemma 3.1. □

Lemma 3.2. *Let A, B be subsets in a Delaunay mesh. $A \overset{\wedge\wedge}{v} B$ if and only if $A\ v\ B$.*

Proof. $A \overset{\wedge\wedge}{v} B \Leftrightarrow \overline{pq}$ for some triangle edge common to A and $B \Leftrightarrow AvB$, since $\overline{pq}$ is visible from a vertex in A and from a vertex in B and A and B have vertices in common. □

Theorem 3.2. *The strong visibility relation $\overset{\wedge\wedge}{v}$ is a Wallman proximity.*

Proof. Immediate from Theorem 3.1 and Lemma 3.2. □

Theorem 3.3. *$\overset{\wedge\wedge}{\delta}$ is a Wallman proximity.*

Proof. Immediate from Lemma 3.1, Lemma 3.2 and Theorem 3.2. □

Remark 3.1. *From Theorem 3.3, $\overset{\wedge\wedge}{\delta}$ is a strongly near Wallman proximity.*

Theorem 3.4. *Let A, B be subsets in a Delaunay mesh. Then*

1^o *$A \overset{\wedge\wedge}{\not v} B$ implies $A \not v B$.*

2^o *$A \overset{\wedge\wedge}{\not v} B$ if and only if $A \overset{\wedge\wedge}{\not\delta} B$.*

Proof.
1^o: Given $A \overset{\wedge\wedge}{\not v} B$, then A and B have no triangle vertices in common. Hence, $A \not v B$.
2^o: $A \overset{\wedge\wedge}{\not v} B$ if and only if A and B have no triangles in common if and only if $A \overset{\wedge\wedge}{\not\delta} B$. □

Theorem 3.5 is an extension of Theorem 3.1 in [19], which results from Theorem 3.1.

Theorem 3.5. *The following statements are equivalent.*

1^o *$\triangle(pqr)$ is a Delaunay triangle.*

2^o *Circumcircle $\bigcirc(pqr)$ has center $u = clV_p \cap clV_q \cap clV_r$.*

3^o $V_p \overset{\wedge\!\!\wedge}{v} V_q \overset{\wedge\!\!\wedge}{v} V_r$.
4^o $\triangle(pqr)$ *is the union of convex sets.*

Let P be a polygon. Two points $p, q \in P$ are *visible*, provided the line segment $\overline{pq}$ is in int$P$ [7]. Let $p, q \in S$, L a finite set of straight line segments and let $\overline{pq} \in L$. Points p, q are visible from each other, which implies that $\overline{pq}$ contains no point of $S - \{p, q\}$ in its interior and $\overline{pq}$ shares no interior point with a constraining line segment in $L - \overline{pq}$. That is, int$\overline{pq} \cap S = \emptyset$ and $\overline{pq} \cap \overline{xy} = \emptyset$ for all $\overline{xy} \in L$ [5, §II, p. 32].

Theorem 3.6. *If points in int $\overline{pq}$ are visible from p, q, then int $\overline{pq}$ $\underline{v}$ $S - \{p, q\}$ and $\overline{pq}$ $\underline{v}$ $\overline{xy} \in L - \overline{pq}$ for all $x, y \in S - \{p, q\}$.*

Proof. Symmetric with the proof of Theorem 3.2 [19]. ☐

A *Delaunay triangulation region* $\mathcal{D}$ is a collection of Delaunay triangles such that every pair triangles in the collection is strongly near. That is, every Delaunay triangulation region is a triangulation of a finite set of sites and the triangles in each region are pairwise strongly near. *Proximal Delaunay triangulation regions* have at least one vertex in common. From Lemma 1.1 and the definition of a Delaunay triangulation region, observe

Lemma 3.3. [19] *A Delaunay triangulation region is a convex polygon.*

Theorem 3.7. [19] *Proximal Delaunay triangulation regions are convex polygons.*

A *local Leader uniform topology* [11] on a set in the plane is determined by finding those sets that are close to each given set.

Theorem 3.8. [19] *Every Delaunay triangulation region has a local Leader uniform topology* (application of [11]).

Theorem 3.9. [19] *A Delaunay triangulation region endowed with the visibility relation v has a local Leader uniform topology.*

Proof. Let $\mathcal{D}$ be a Delaunay triangulation region. From Theorem 3.1 and Theorem 3.8, determine all subsets of $\mathcal{D}$ that are visible from each given subset of $\mathcal{D}$. For each $A \subset \mathcal{D}$, this procedure determines a family of Delaunay triangles that are visible from (near) each A. By definition, this procedure induces a local Leader uniform topology on $\mathcal{D}$. ☐

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Computational Intelligence Laboratory
University of Manitoba
WPG, MB, R3T 5V6, Canada
and
Department of Mathematics
Faculty of Arts and Science
Adiyaman University
Adiyaman, Turkey
E-mail address: James.Peters3@umanitoba.ca