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MOMENTS OF ORDER STATISTICS OF THE POISSON-LOMAX DISTRIBUTION

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ABSTRACT. The Poisson-Lomax distribution has been proposed as a useful reliability model for analyzing lifetime data. For this distribution, some recurrence relations are established for the single moments and product moments of order statistics. Using these recurrence relations, the means, variances and covariances of all order statistics can be computed for all sample sizes in a simple and efficient recursive manner.

1. INTRODUCTION

Order statistics arise naturally in many life applications. The use of recurrence relations for the moments of order statistics is quite well known in statistical literature (see for example Arnold et al. [2], Malik et al. [6]). For improved form of these results, Samuel and Thomes [7] have reviewed many recurrence relations and identities for the moments of order statistics arising from several specific continuous distributions such as normal, Cauchy, logistic, gamma and exponential. Balakrishnan et. al [11] and Balakrishnan et. al [8] studied recurrence relations and identities for moments of order statistics for specific continuous distributions. Recurrence relations for the expected values of certain functions of two order statistics have been considered by Ali and Khan [1] and Khan et. al [5]. The moments of order statistics have some important applications in inferential methods. For an elaborate treatment on the theory, methods and applications of order statistics, interested readers may refer to Balakrishnan and Rao [9] and [10].

The Poisson-Lomax (PL) distribution, proposed recently by Al- Zahrani and Sagor [3], is a useful model for modeling lifetime data. The distribution is a compound distribution of the zero-truncated Poisson and the Lomax distributions. See also, Al-Awadhi and Ghitany [12] for a discrete extension of this model.

Definition 1.1. *We say that a random variable X with range of values $(0,\infty)$ has a Poisson-Lomax distribution, and write $X \sim PL(\alpha,\beta,\lambda)$, if the survival function*

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(sf) is

$$\bar{F}(x;\alpha,\beta,\lambda)=\frac{1-e^{-\lambda(1+\beta x)^{-\alpha}}}{1-e^{-\lambda}},\quad x>0,\ \alpha,\ \beta,\ \lambda>0. \tag{1.1}$$

The probability density function (pdf) associated with (1.1) is expressed in a closed form and is given by

$$f(x;\alpha,\beta,\lambda)=\frac{\alpha\beta\lambda\,(1+\beta x)^{-(\alpha+1)}\,e^{-\lambda(1+\beta x)^{-\alpha}}}{1-e^{-\lambda}},\quad x>0,\ \alpha,\ \beta,\ \lambda>0. \tag{1.2}$$

It is easy to observe that $f(x)$ and $F(x)=1-\bar{F}(X)$ satisfy the following characterizing relations:

$$f(x)=c_1(1+\beta x)^{-(\alpha+1)}F(x)+c_2(1+\beta x)^{-(\alpha+1)}, \tag{1.3}$$

where $c_1=\alpha\beta\lambda$ and $c_2=c_1e^{-}\lambda/(1-e^{-\lambda})$. The density function given by (1.2) can be interpreted as a compound of the zero-truncated Poisson distribution and the Lomax distribution. Mathematical properties of this distribution can found in Al-Zahrani and Sagor [3]. Here, we will study the distribution of order statistics and establish some recurrence relations for the single and product moments of order statistics for the Poisson-Lomax distribution. These recurrence relations will enable the computation of the means, variances and covariances of all order statistics for all sample sizes in a simple and efficient recursive manner.

2. Distribution of order statistics

Let $X_1,X_2,\cdots,X_n$ be a random sample of size n from the PL distribution in (1.1) and let $X_{1:n},\cdots,X_{n:n}$ denote the corresponding order statistics. Then, the pdf of $X_{r:n}$, $1\le r\le n$, is given by (see David and Nagaraja [4] and Arnold et al. [2])

$$f_{r:n}(x)=C_{r,n}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x),\quad 0<x<\infty, \tag{2.1}$$

where $C_{r,n}=[B(r,n-r+1)]^{-1}$, with $B(a,b)$ being the complete beta function.

Theorem 2.1. *Let $F(x)$ and $f(x)$ be the cdf and pdf of a Poisson-Lomax distribution for a random variable X. The density of the rth order statistic, say $f_{(r)}(x)$ is given by*

$$\begin{aligned} f_{r:n}(x) &= \alpha\beta\lambda C_{r,n}\sum_{i=0}^{r-1}\sum_{j=0}^{n-r+i}\binom{r-1}{i}\binom{n-r+i}{j} \\ &\quad\times\frac{(-1)^{i+j}(1+\beta x)^{-(\alpha+1)}\ e^{-\lambda(j+1)(1+\beta x)^{-\alpha}}}{(1-e^{-\lambda})^{n-r+i+1}}. \end{aligned} \tag{2.2}$$

Proof. First it should be noted that (2.1) can be written as follows:

$$f_{r:n}(x)=C_{r,n}\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i[\bar{F}(x)]^{n-r+i}f(x), \tag{2.3}$$

then the proof follows by replacing the sf, $\bar{F}(x)$, and the pdf, $f(x)$, of $X\sim PL(\alpha,\beta,\lambda)$ which are obtained from (1.1) and (1.2), respectively, substituting them into relation (2.3), and expanding the term $(1-e^{-\lambda(1+\beta x)^{-\alpha}})^{n-r+i}$ using the binomial expansion. □

The distributions of the extreme order statistics are always of great interest. Taking $r = 1$ in equation (2.2), yeilds the pdf of the minimum order statistic

$$f_{1:n}(x) = \frac{n\alpha\beta\lambda\,(1+\beta x)^{-(\alpha+1)}\,e^{-\lambda(1+\beta x)^{-\alpha}}}{(1-e^{-\lambda})^n}\left[1-e^{-\lambda(1+\beta x)^{-\alpha}}\right]^{n-1},$$

and if we take $r = n$ in equation (2.2), then this yeilds the pdf of the maximum order statistic.

The joint pdf of $X_{r:n}$ and $X_{s:n}$ for $1 \le r < s \le n$ is given by (see e.g. Arnold et al. [2])

$$f_{r,s:n}(x,y) = C_{r,s,n}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y), \tag{2.4}$$

where

$$C_{r,s,n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}, \quad -\infty < x < y < \infty.$$

Substituting (1.1) and (1.2) into (2.4), one can obtain the joint pdf of the rth and the sth order statistics from the Poisson-Lomax distribution. It is as follows:

$$\begin{aligned} f_{r,s:n}(x,y) &= \frac{C_{r,s,n}(\alpha\beta\lambda)^2}{(1-e^{-\lambda})^n}\,[xy\beta^2+(x+y)\beta+1]^{-(\alpha+1)}e^{-\lambda\{(1+\beta x)^{-\alpha}+(1+\beta y)^{-\alpha}\}} \\ &\quad\times\left[e^{-\lambda(1+\beta x)^{-\alpha}}-e^{-\lambda}\right]^{r-1}\left[e^{-\lambda(1+\beta y)^{-\alpha}}-e^{-\lambda(1+\beta x)^{-\alpha}}\right]^{s-r-1} \\ &\quad\times\left[1-e^{-\lambda(1+\beta y)^{-\alpha}}\right]^{n-s}. \end{aligned}$$

3. Single moments

In this section we first give a closed form expression which is derived easily for the kth, $k = 1, 2, \cdots$, moment of the ith order statistic from the Poisson-Lomax distribution. This formula will be useful in the phase of computation of the identity given.

Theorem 3.1. *Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from the Poisson-Lomax distribution, and let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ denote the corresponding order statistics. Then the kth moment of the rth order statistic for $k = 1, 2, \cdots$, denoted by $\mu^{(k)}_{r:n}$, is given as*

$$\begin{aligned} \mu^{(k)}_{r:n} = E[X^k_{r:n}] &= \alpha\,C_{r:n}\sum_{j=0}^{n-r}\sum_{i=0}^{k}\sum_{l=0}^{r+j}\sum_{m=0}^{\infty}\binom{n-r}{j}\binom{k}{i}\binom{r+j}{l}(-1)^{j+i+l+m+1} \\ &\quad\times\left(\frac{\lambda^{m+1}(r+j-l)^m\,e^{-\lambda l}}{m!\beta^k\,(1-e^{-\lambda})^{r+j}\,(k-i-\alpha-\alpha m)}\right) \\ &\quad+\frac{\alpha e^{-\lambda}C_{r:n}}{1-e^{-\lambda}}\sum_{j=0}^{n-r}\sum_{i=0}^{k}\sum_{l=0}^{r+j-1}\sum_{m=0}^{\infty}\binom{n-r}{j}\binom{k}{i}\binom{r+j-1}{l} \\ &\quad\times(-1)^{j+i+l+m+1}\left(\frac{\lambda^{m+1}(r+j-1-l)^m\,e^{-\lambda l}}{m!\beta^k\,(1-e^{-\lambda})^{r+j-1}\,(k-i-\alpha-\alpha m)}\right). \end{aligned} \tag{3.1}$$

for $k < i + \alpha + \alpha m$.

Proof. We know that

$$
\begin{aligned}
\mu_{r:n}^{(k)} &= \int_0^\infty x^k f_{r:n}(x)dx \\
&= C_{r:n}\int_0^\infty x^k\left[F(x)\right]^{r-1}\left[1-F(x)\right]^{n-r} f(x)dx \\
&= C_{r:n}c_1\int_0^\infty x^k(1+\beta x)^{-(\alpha+1)}\left[F(x)\right]^{r}\left[1-F(x)\right]^{n-r} dx \\
&\quad +C_{r:n}c_2\int_0^\infty x^k(1+\beta x)^{-(\alpha+1)}\left[F(x)\right]^{r-1}\left[1-F(x)\right]^{n-r} dx. \qquad (3.2)
\end{aligned}
$$

For ease of notation, we write

$$\mu_{r:n}^{(k)} = C_{r:n}c_1I_1 + C_{r:n}c_2I_2, \qquad (3.3)$$

where I_1 can be worked out as follows:

$$
\begin{aligned}
I_1 &= \int_0^\infty x^k(1+\beta x)^{-(\alpha+1)}\left[F(x)\right]^{r}\left[1-F(x)\right]^{n-r} dx \\
&= \int_0^\infty x^k(1+\beta x)^{-(\alpha+1)}\left[F(x)\right]^{r}\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j\left[F(x)\right]^j dx \\
&= \sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j\int_0^\infty x^k(1+\beta x)^{-(\alpha+1)}\left(1-\frac{1-e^{-\lambda(1+\beta x)^{-\alpha}}}{1-e^{-\lambda}}\right)^{r+j} dx \\
&= \sum_{j=0}^{n-r}\sum_{i=0}^{k}\sum_{l=0}^{r+j}\sum_{m=0}^{\infty}\binom{n-r}{j}\binom{k}{i}\binom{r+j}{l}(-1)^{j+i+l+m+1} \\
&\quad\times\left(\frac{\lambda^m(r+j-l)^m e^{-\lambda l}}{m!\beta^k\left(1-e^{-\lambda}\right)^{r+j}(k-i-\alpha-\alpha m)}\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_2 &= \frac{\alpha e^{-\lambda}C_{r:n}}{1-e^{-\lambda}}\sum_{j=0}^{n-r}\sum_{i=0}^{k}\sum_{l=0}^{r+j-1}\sum_{m=0}^{\infty}\binom{n-r}{j}\binom{k}{i}\binom{r+j-1}{l}(-1)^{j+i+l+m+1} \\
&\quad\times\left(\frac{\lambda^m(r+j-1-l)^m e^{-\lambda l}}{m!\beta^k\left(1-e^{-\lambda}\right)^{r+j-1}(k-i-\alpha-\alpha m)}\right).
\end{aligned}
$$

Substituting I_1 and I_2 into (3.3) yields (3.1). □

Now we derive recurrence relation for single moments.

Theorem 3.2. *For $k > i+\alpha$, and $1 \le r \le n-1$*

$$
\begin{aligned}
\mu_{r:n}^{(k)} &= \sum_{i=0}^{k}\sum_{j=0}^{k-i-\alpha}\frac{1}{k-i-\alpha}\binom{k}{i}\binom{k-i-\alpha}{j}(-1)^i\beta^{j-k-1} \\
&\quad\times\left(c_1r\mu_{r+1:n}^{(j)} - c_1r\mu_{r:n}^{(j)} - c_2(n-r+1)\mu_{r-1:n}^{(j)} + c_2n\mu_{r:n-1}^{(j)}\right). \qquad (3.4)
\end{aligned}
$$

Proof. Again we use equation (3.2)

$$\mu_{r:n}^{(k)} = C_{r:n}c_1 I_1 + C_{r:n}c_2 I_2,$$

where I_1 is as before:

$$I_1 = \int_0^\infty x^k (1+\beta x)^{-(\alpha+1)} \left(F\left(x\right)\right)^r \left(1-F\left(x\right)\right)^{n-r} dx\,.$$

Now using integration by parts, we obtain

$$\begin{aligned} I_1 &= -\sum_{i=0}^{k}\sum_{j=0}^{k-i-\alpha}\frac{(-1)^i\beta^{j-k-1}}{k-i-\alpha}\binom{k}{i}\binom{k-i-\alpha}{j}\Big\{r\int_0^\infty x^j[F(x)]^r[1-F(x)]^{n-r}f(x)dx \\ &\quad -(n-r)\int_0^\infty x^j[F(x)]^r[1-F(x)]^{n-r-1}f(x)dx.\Big\}. \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 &= -\sum_{i=0}^{k}\sum_{j=0}^{k-i-\alpha}\frac{(-1)^i\beta^{j-k-1}}{k-i-\alpha}\binom{k}{i}\binom{k-i-\alpha}{j} \\ &\quad \times\Big\{(r-1)\int_0^\infty x^j[F(x)]^{r-2}[1-F(x)]^{n-r}f(x)dx \\ &\quad -(n-r)\int_0^\infty x^j[F(x)]^{r-1}[1-F(x)]^{n-r-1}f(x)dx.\Big\}. \end{aligned}$$

Substituting I_1 and I_2 into (3.3) yields (3.4). ☐

4. Recurrence relations for product moments

Theorem 4.1. *For, $k_2 > i+\alpha$ and $s-r \geq 2$*

$$\begin{aligned} \mu_{r;s:n}^{(k_1,\,k_2)} &= \sum_{i=0}^{k_2}\sum_{j=0}^{k_2-i-\alpha}\frac{\binom{k_2}{i}\binom{k_2-i-\alpha}{j}}{k_2-i-\alpha}(-1)^i\beta^{j-k_2-1}\Big\{(c_1+c_2)\,n\mu_{r;s:n-1}^{(k_1,j)} \\ &\quad -(c_1+c_2)\,n\mu_{r;s-1:n-1}^{(k_1,j)}+c_1\,(n-s+1)\,\mu_{r;s:n}^{(k_1,j)}-c_1(n-s+1)\mu_{r;s-1:n}^{(k_1,j)}\Big\}. \end{aligned} \tag{4.1}$$

Proof.

$$\begin{aligned} \mu_{r;s:n}^{(k_1,\,k_2)} &= C_{r:s;n}\int_0^\infty\int_x^\infty x^{k_1}y^{k_2}[F\left(x\right)]^{r-1}[F\left(y\right)-F\left(x\right)]^{s-r-1} \\ &\quad \times[1-F\left(y\right)]^{n-s}f\left(x\right)f\left(y\right)dydx, \end{aligned}$$

where $C_{r:s;n} = n!/(r-1)!\,(s-r-1)!\,(n-s)!$. Let I be as follows:

$$\begin{aligned} I &= \int_0^\infty\int_x^\infty x^{k_1}y^{k_2}[F\left(x\right)]^{r-1}[F\left(y\right)-F\left(x\right)]^{s-r-1} \\ &\quad \times[1-F\left(y\right)]^{n-s}f\left(x\right)f\left(y\right)dydx \\ &= \int_0^\infty x^{k_1}[F(x)]^{r-1}f(x)I_x dx, \end{aligned} \tag{4.2}$$

where

$$I_X = \int_x^{\infty} y^{k_2}[F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s} f(y)dy\,.$$

Using (1.3) we have:

$$\begin{aligned} I_X &= c_1 \int_x^{\infty} y^{k_2}(1+\beta y)^{-(\alpha+1)}[F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s} F(y)\,dy \\ &\quad + c_2 \int_x^{\infty} y^{k_2}(1+\beta y)^{-(\alpha+1)}[F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s} dy\,, \end{aligned}$$

or it can be written as:

$$\begin{aligned} I_X &= c_1 \int_x^{\infty} y^{k_2}(1+\beta y)^{-(\alpha+1)}[F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s} dy \\ &\quad - c_1 \int_x^{\infty} y^{k_2}(1+\beta y)^{-(\alpha+1)}[F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s+1} dy \\ &\quad + c_2 \int_x^{\infty} y^{k_2}(1+\beta y)^{-(\alpha+1)}[F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s} dy\,. \end{aligned}$$

By using integration by parts, we obtain:

$$\begin{aligned} I_X &= \sum_{i=0}^{k_2} \sum_{j=0}^{k_2-i-\alpha} \frac{(-1)^i \beta^{j-k_2-1}}{k_2-i-\alpha} \binom{k_2}{i} \binom{k_2-i-\alpha}{j} \\ &\quad \times \Big\{ (c_1+c_2)(n-s) \int_x^{\infty} y^j [F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s} f(y)\,dy \\ &\quad - (c_1+c_2)(s-r-1) \int_x^{\infty} y^j [F(y) - F(x)]^{s-r-2}[1 - F(y)]^{n-s} f(y)\,dy \\ &\quad + c_1 (n-s+1) \int_x^{\infty} y^j [F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s} f(y)\,dy \\ &\quad - c_1 (s-r-1) \int_x^{\infty} y^j [F(y) - F(x)]^{s-r-2}[1 - F(y)]^{n-s} f(y)\,dy \Big\}. \end{aligned}$$

Substituting I_X in (4.2) and then into (4.1) produces the desired result. □

Corollary 4.1.

$$\begin{aligned} \mu_{r;r+1:n}^{(k_1,\,k_2)} &= \sum_{i=0}^{k_2} \sum_{j=0}^{k_2-i-\alpha} \frac{\binom{k_2}{i}\binom{k_2-i-\alpha}{j}}{k_2-i-\alpha} (-1)^i \beta^{j-k_2-1} \Big\{ (c_1+c_2) n \mu_{r;r+1:n-1}^{(k_1,j)} \\ &\quad - c_1(n-r)\mu_{r;r+1:n}^{(k_1,j)} - (c_1+c_2) n \mu_{r;:n-1}^{(k_1+j)} + c_1 (n-r)\, \mu_{r:n}^{(k_1+j)} \Big\}\,. \end{aligned}$$

Proof. The proof follows by substituting $s = r+1$ into equation (4.1), the rest of procedure is similar to that of Theorem 4.1. □

5. Conclusion

In this paper we studied the Poisson-Lomax distribution from the order statistics viewpoint. Also we considered the single and product moment of order statistics from this distribution. We established some recurrence relations for both single and product moments of order statistics. Using these recurrence relations, one can easily compute the means, variances and covariances of all order statistics for all sample sizes in a simple and efficient recursive manner.

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