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DIFFERENTIAL SANDWICH THEOREMS USING A MULTIPLIER TRANSFORMATION AND RUSCHEWEYH DERIVATIVE

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Abstract. In this work we define a new operator using the multiplier transformation and Ruscheweyh derivative. Denote by $IR^{m,n}_{\lambda,l}$ the Hadamard product of the multiplier transformation $I(m,\lambda,l)$ and Ruscheweyh derivative R^n, given by $IR^{m,n}_{\lambda,l} : \mathcal{A} \to \mathcal{A}$, $IR^{m,n}_{\lambda,l} f(z) = (I(m,\lambda,l) * R^n) f(z)$ and $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots, z \in U\}$ is the class of normalized analytic functions with $\mathcal{A}_1 = \mathcal{A}$. The purpose of this paper is to derive certain subordination and superordination results involving the operator $IR^{m,n}_{\lambda,l}$ and we establish differential sandwich-type theorems.

1. Introduction

Let $\mathcal{H}(U)$ be the class of analytic function in the open unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{H}(a,n)$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots$.

Let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots,\ z \in U\}$ and $\mathcal{A} = \mathcal{A}_1$.

Let the functions f and g be analytic in U. We say that the function f is subordinate to g, written $f \prec g$, if there exists a Schwarz function w, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$, for all $z \in U$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h be an univalent function in U. If p is analytic in U and satisfies the second order differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \quad \text{for } z \in U, \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1.1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all

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dominants q of (1.1) is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of U.

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ and h analytic in U. If p and $\psi\left(p\left(z\right), zp'\left(z\right), z^2p''\left(z\right); z\right)$ are univalent and if p satisfies the second order differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \qquad z \in U, \tag{1.2}$$

then p is a solution of the differential superordination (1.2) (if f is subordinate to F, then F is called to be superordinate to f). An analytic function q is called a subordinant if $q \prec p$ for all p satisfying (1.2). An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant.

Miller and Mocanu [8] obtained conditions h, q and ψ for which the following implication holds

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

For two functions $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ and $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$ analytic in the open unit disc U, the Hadamard product (or convolution) of $f(z)$ and $g(z)$, written as $(f * g)(z)$ is defined by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j.$$

Definition 1.1. [6] *For $f \in \mathcal{A}$, $m \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, the multiplier transformation $I(m, \lambda, l) f(z)$ is defined by the following infinite series*

$$I(m, \lambda, l) f(z) := z + \sum_{j=2}^{\infty} \left(\frac{1 + \lambda(j-1) + l}{1 + l} \right)^m a_j z^j.$$

Remark 1.1. *We have*

$$(l+1) I(m+1, \lambda, l) f(z) = (l + 1 - \lambda) I(m, \lambda, l) f(z) + \lambda z \left(I(m, \lambda, l) f(z)\right)', \quad z \in U.$$

Remark 1.2. *For $l = 0$, $\lambda \geq 0$, the operator $D_\lambda^m = I(m, \lambda, 0)$ was introduced and studied by Al-Oboudi ([3]), which reduced to the Sălăgean differential operator $S^m = I(m, 1, 0)$ ([11]) for $\lambda = 1$.*

Definition 1.2. *(Ruscheweyh [10]) For $f \in \mathcal{A}$ and $n \in \mathbb{N}$, the Ruscheweyh derivative R^n is defined by $R^n : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= z f'(z) \\ &\cdots \\ (n+1) R^{n+1} f(z) &= z\left(R^n f(z)\right)' + n R^n f(z), \quad z \in U. \end{aligned}$$

Remark 1.3. *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$ for $z \in U$.*

The purpose of this paper is to derive the several subordination and superordination results involving a differential operator. Furthermore, we studied the results of Selvaraj and Karthikeyan [12], Shanmugam, Ramachandran, Darus and Sivasubramanian [13] and Srivastava and Lashin [14].

In order to prove our subordination and superordination results, we make use of the following known results.

Definition 1.3. [9] *Denote by* Q *the set of all functions* f *that are analytic and injective on* $\overline{U}\backslash E(f)$, *where* $E(f) = \{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\}$, *and are such that* $f'(\zeta) \neq 0$ *for* $\zeta \in \partial U\backslash E(f)$.

Lemma 1.1. [9] *Let the function* q *be univalent in the unit disc* U *and* θ *and* ϕ *be analytic in a domain* D *containing* $q(U)$ *with* $\phi(w) \neq 0$ *when* $w \in q(U)$. *Set* $Q(z) = zq'(z)\phi(q(z))$ *and* $h(z) = \theta(q(z)) + Q(z)$. *Suppose that*

1. Q *is starlike univalent in* U *and*
2. $Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$ *for* $z \in U$.

If p *is analytic with* $p(0) = q(0)$, $p(U) \subseteq D$ *and*

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then $p(z) \prec q(z)$ *and* q *is the best dominant.*

Lemma 1.2. [5] *Let the function* q *be convex univalent in the open unit disc* U *and* ν *and* ϕ *be analytic in a domain* D *containing* $q(U)$. *Suppose that*

1. $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) > 0$ *for* $z \in U$ *and*
2. $\psi(z) = zq'(z)\phi(q(z))$ *is starlike univalent in* U.

If $p(z) \in \mathcal{H}[q(0),1] \cap Q$, *with* $p(U) \subseteq D$ *and* $\nu(p(z)) + zp'(z)\phi(p(z))$ *is univalent in* U *and*

$$\nu(q(z)) + zq'(z)\phi(q(z)) \prec \nu(p(z)) + zp'(z)\phi(p(z)),$$

then $q(z) \prec p(z)$ *and* q *is the best subordinant.*

2. Main results

Definition 2.1. *Let* $\lambda, l \geq 0$ *and* $n, m \in \mathbb{N}$. *Denote by* $IR_{\lambda,l}^{m,n} : \mathcal{A} \to \mathcal{A}$ *the operator given by the Hadamard product of the multiplier transformation* $I(m,\lambda,l)$ *and the Ruscheweyh derivative* R^n,

$$IR_{\lambda,l}^{m,n} f(z) = (I(m,\lambda,l) * R^n) f(z),$$

for any $z \in U$ *and each nonnegative integers* m, n.

Remark 2.1. *If* $f \in \mathcal{A}$ *and* $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, *then*
$IR_{\lambda,l}^{m,n} f(z) = z + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2 z^j$, $z \in U$.

Remark 2.2. *For* $l = 0$, $\lambda \geq 0$, *we obtain the Hadamard product* $IR_{\lambda,0}^{m,n} f(z) = DR_{\lambda}^{m,n} f(z)$, *which was introduced in* [4].

For $l = 0$ *and* $\lambda = 1$ *we obtain the operator* $IR_{1,0}^{m,n} f(z) = SR^{m,n} f(z)$, *which was introduced in* [7].

For $m = n$, *we obtain the Hadamard product* $IR_{\lambda,l}^{m}$ *which was studied in* [1], [2].

Using simple computation one obtains the next result.

Proposition 2.1. *For $m,n\in\mathbb{N}$ and $\lambda\geq 0$ we have*

$$IR_{\lambda,l}^{m+1,n}f(z)=\frac{1+l-\lambda}{l+1}IR_{\lambda,l}^{m,n}f(z)+\frac{\lambda}{l+1}z\left(IR_{\lambda,l}^{m,n}f(z)\right)' \tag{2.1}$$

Proof. We have

$$\begin{aligned}IR_{\lambda,l}^{m+1,n}f(z)&=z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1}\frac{(n+j-1)!}{n!(j-1)!}a_j^2z^j\\&=z+\sum_{j=2}^{\infty}\frac{1+\lambda(j-1)+l}{l+1}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}\frac{(n+j-1)!}{n!(j-1)!}a_j^2z^j\\&=z+\frac{1+l-\lambda}{l+1}\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}\frac{(n+j-1)!}{n!(j-1)!}a_j^2z^j\\&+\frac{\lambda}{l+1}\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}\frac{(n+j-1)!}{n!(j-1)!}ja_j^2z^j\\&=\frac{1+l-\lambda}{l+1}\left[z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}\frac{(n+j-1)!}{n!(j-1)!}a_j^2z^j\right]\\&+\frac{\lambda}{l+1}z\left[1+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}\frac{(n+j-1)!}{n!(j-1)!}ja_j^2z^{j-1}\right]\\&=\frac{1+l-\lambda}{l+1}IR_{\lambda,l}^{m,n}f(z)+\frac{\lambda}{l+1}z\left(IR_{\lambda,l}^{m,n}f(z)\right)'.\end{aligned}$$

□

We begin with the following

Theorem 2.2. *Let $\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta}\in\mathcal{H}(U)$ and let the function $q(z)$ be analytic and univalent in U such that $q(z)\neq 0$, for all $z\in U$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. Let*

$$Re\left(1+\frac{\xi}{\beta}q(z)+\frac{2\mu}{\beta}(q(z))^2-\frac{zq'(z)}{q(z)}+\frac{zq''(z)}{q'(z)}\right)>0, \tag{2.2}$$

for $\delta,\alpha,\xi,\mu,\beta\in\mathbb{C}$, $\beta\neq 0$, $z\in U$ and

$$\psi_{\lambda,l}^{m,n}(\delta,\alpha,\xi,\mu,\beta;z):=\alpha+\xi\left[\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right]^{\delta}+ \tag{2.3}$$

$$\mu\left[\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right]^{2\delta}+\frac{\beta\delta(l+1)}{\lambda}\left[\frac{IR_{\lambda,l}^{m+2,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z)}-1\right]$$

If q satisfies the following subordination

$$\psi_{\lambda,l}^{m,n}(\delta,\alpha,\xi,\mu,\beta;z)\prec\alpha+\xi q(z)+\mu(q(z))^2+\beta\frac{zq'(z)}{q(z)}, \tag{2.4}$$

for $\delta, \alpha, \xi, \mu, \beta \in \mathbb{C}$, $\beta, \delta \neq 0$, *then*

$$\left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{z} \right)^{\delta} \prec q(z), \quad \delta \in \mathbb{C},\ \delta \neq 0,$$

and q *is the best dominant.*

Proof. Let the function p be defined by $p(z) := \left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{z} \right)^{\delta}$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$. We have $p'(z) = \delta \left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{z} \right)^{\delta-1} \cdot \frac{z\left(IR_{\lambda,l}^{m+1,n} f(z)\right)' - IR_{\lambda,l}^{m+1,n} f(z)}{z^2}$. Then $\frac{zp'(z)}{p(z)} = \delta \left[\frac{z\left(IR_{\lambda,l}^{m+1,n} f(z)\right)'}{IR_{\lambda,l}^{m+1,n} f(z)} - 1 \right]$.

By using the identity (2.1), we obtain

$$\frac{zp'(z)}{p(z)} = \frac{\delta(l+1)}{\lambda} \left[\frac{IR_{\lambda,l}^{m+2,n} f(z)}{IR_{\lambda,l}^{m+1,n} f(z)} - 1 \right]. \tag{2.5}$$

By setting $\theta(w) := \alpha + \xi w + \mu w^2$ and $Q(w) := \frac{\beta}{w}$, it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Also, by letting $Q(z) = zq'(z)\phi(q(z)) = \beta \frac{zq'(z)}{q(z)}$ and $h(z) = \theta(q(z)) + Q(z) = \alpha + \xi q(z) + \mu(q(z))^2 + \beta \frac{zq'(z)}{q(z)}$, we find that $Q(z)$ is starlike univalent in U.

We have $h'(z) = \xi + q'(z) + 2\mu q(z) q'(z) + \beta \frac{\left(q'(z) + zq''(z)\right)q(z) - z\left(q'(z)\right)^2}{(q(z))^2}$ and $\frac{zh'(z)}{Q(z)} = \frac{zh'(z)}{\beta \frac{zq'(z)}{q(z)}} = 1 + \frac{\xi}{\beta} q(z) + \frac{2\mu}{\beta}(q(z))^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}$.

We deduce that $Re\left(\frac{zh'(z)}{Q(z)}\right) = Re\left(1 + \frac{\xi}{\beta} q(z) + \frac{2\mu}{\beta}(q(z))^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0$.

By using (2.5), we obtain

$$\alpha + \xi p(z) + \mu(p(z))^2 + \beta \frac{zp'(z)}{p(z)} = \alpha + \xi \left[\frac{IR_{\lambda,l}^{m+1,n} f(z)}{z} \right]^{\delta} + \mu \left[\frac{IR_{\lambda,l}^{m+1,n} f(z)}{z} \right]^{2\delta} + \frac{\beta\delta(l+1)}{\lambda} \left[\frac{IR_{\lambda,l}^{m+2,n} f(z)}{IR_{\lambda,l}^{m+1,n} f(z)} - 1 \right].$$

By using (2.4), we have $\alpha + \xi p(z) + \mu(p(z))^2 + \beta \frac{zp'(z)}{p(z)} \prec \alpha + \beta q(z) + \mu(q(z))^2 + \beta \frac{zq'(z)}{q(z)}$.

By an application of Lemma 1.1, we have $p(z) \prec q(z)$, $z \in U$, i.e. $\left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{z} \right)^{\delta} \prec q(z)$, $z \in U$ and q is the best dominant. □

Corollary 2.3. *Let* $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. *Assume that (2.2) holds. If* $f \in \mathcal{A}$ *and*

$$\psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z) \prec \alpha + \xi \frac{1+Az}{1+Bz} + \mu \left(\frac{1+Az}{1+Bz} \right)^2 + \beta \frac{(A-B)z}{(1+Az)(1+Bz)},$$

for $\delta, \alpha, \xi, \mu, \beta \in \mathbb{C}$, $\beta, \delta \neq 0$, $-1 \leq B < A \leq 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.3), then*

$$\left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{z} \right)^{\delta} \prec \frac{1+Az}{1+Bz}, \quad \delta \in \mathbb{C},\ \delta \neq 0,$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Proof. For $q(z)=\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$ in Theorem 2.2 we get the corollary. □

Corollary 2.4. *Let $m,n\in\mathbb{N}$, $\lambda,l\ge 0$. Assume that (2.2) holds. If $f\in\mathcal{A}$ and*

$$\psi_{\lambda,l}^{m,n}(\delta,\alpha,\xi,\mu,\beta;z)\prec\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma}+\frac{2\beta\gamma z}{1-z^2},$$

for $\delta,\alpha,\xi,\mu,\beta\in\mathbb{C}$, $0<\gamma\le 1$, $\beta\ne 0$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.3), then

$$\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta}\prec\left(\frac{1+z}{1-z}\right)^{\gamma},\ \ \text{for } \delta\in\mathbb{C},\ \delta\ne 0,$$

and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ is the best dominant.

Proof. Corollary follows by using Theorem 2.2 for $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $0<\gamma\le 1$. □

Theorem 2.5. *Let q be analytic and univalent in U such that $q(z)\ne 0$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in U. Assume that*

$$Re\left(\frac{2\mu}{\beta}(q(z))^2+\frac{\xi}{\beta}q(z)\right)>0,\ \text{for } \xi,\mu,\beta\in\mathbb{C},\ \beta\ne 0. \tag{2.6}$$

If $f\in\mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q$ and $\psi_{\lambda,l}^{m,n}(\delta,\alpha,\xi,\mu,\beta;z)$ is univalent in U, where $\psi_{\lambda,l}^{m,n}(\delta,\alpha,\xi,\mu,\beta;z)$ is as defined in (2.3), then

$$\alpha+\xi q(z)+\mu(q(z))^2+\beta\frac{zq'(z)}{q(z)}\prec\psi_{\lambda,l}^{m,n}(\delta,\alpha,\xi,\mu,\beta;z) \tag{2.7}$$

implies

$$q(z)\prec\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta},\quad \delta\in\mathbb{C},\ \delta\ne 0,\ z\in U,$$

and q is the best subordinant.

Proof. Let the function p be defined by $p(z):=\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta}$, $z\in U$, $z\ne 0$, $f\in\mathcal{A}$.

By setting $\nu(w):=\alpha+\xi w+\mu w^2$ and $\phi(w):=\frac{\beta}{w}$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\ne 0$, $w\in\mathbb{C}\backslash\{0\}$.

Since $\frac{\nu'(q(z))}{\phi(q(z))}=\frac{q'(z)[\xi+2\mu q(z)]q(z)}{\beta}$, it follows that

$$Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right)=Re\left(\frac{2\mu}{\beta}(q(z))^2+\frac{\xi}{\beta}q(z)\right)>0,$$

for $\mu,\xi,\beta\in\mathbb{C}$, $\beta\ne 0$.

By using (2.5) and (2.7) we obtain

$$\alpha+\xi q(z)+\mu(q(z))^2+\beta\frac{zq'(z)}{q(z)}\prec\alpha+\xi p(z)+\mu(p(z))^2+\beta\frac{zp'(z)}{p(z)}.$$

Using Lemma 1.2, we have

$$q(z)\prec p(z)=\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta},\quad z\in U,\delta\in\mathbb{C},\delta\ne 0,$$

and q is the best subordinant. □

Corollary 2.6. *Let $m,n\in\mathbb{N}$, $\lambda,l\geq 0$. Assume that (2.6) holds. If $f\in\mathcal{A}$,*

$$\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q$$

and

$$\alpha+\xi\frac{1+Az}{1+Bz}+\mu\left(\frac{1+Az}{1+Bz}\right)^2+\beta\frac{(A-B)z}{(1+Az)(1+Bz)}\prec\psi_{\lambda,l}^{m,n}(\delta,\alpha,\xi,\mu,\beta;z),$$

for $\delta,\alpha,\xi,\mu,\beta\in\mathbb{C}$, $\beta,\delta\neq 0$, $-1\leq B<A\leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.3), then

$$\frac{1+Az}{1+Bz}\prec\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta},\quad\delta\in\mathbb{C},\ \delta\neq 0,$$

and $\frac{1+Az}{1+Bz}$ is the best subordinant.

Proof. For $q(z)=\frac{1+Az}{1+Bz}$, $-1\leq B<A\leq 1$ in Theorem 2.5 we get the corollary. □

Corollary 2.7. *Let $m,n\in\mathbb{N}$, $\lambda,l\geq 0$. Assume that (2.6) holds. If $f\in\mathcal{A}$,*

$$\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q$$

and

$$\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma}+\frac{2\beta\gamma z}{1-z^2}\prec\psi_{\lambda,l}^{m,n}(\delta,\alpha,\xi,\mu,\beta;z),$$

for $\delta,\alpha,\xi,\mu,\beta\in\mathbb{C}$, $\beta,\delta\neq 0$, $0<\gamma\leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.3), then

$$\left(\frac{1+z}{1-z}\right)^{\gamma}\prec\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta},\quad\delta\in\mathbb{C},\ \delta\neq 0,$$

and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ is the best subordinant.

Proof. For $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $0<\gamma\leq 1$ in Theorem 2.5 we get the corollary. □

Combining Theorem 2.2 and Theorem 2.5, we state the following sandwich theorem.

Theorem 2.8. *Let q_1 and q_2 be analytic and univalent in U such that $q_1(z)\neq 0$ and $q_2(z)\neq 0$, for all $z\in U$, with $\frac{zq_1'(z)}{q_1(z)}$ and $\frac{zq_2'(z)}{q_2(z)}$ being starlike univalent. Suppose that q_1 satisfies (2.2) and q_2 satisfies (2.6). If $f\in\mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q$ and $\psi_{\lambda,l}^{m,n}(\delta,\alpha,\xi,\mu,\beta;z)$ is as defined in (2.3) univalent in U, then*

$$\alpha+\xi q_1(z)+\mu(q_1(z))^2+\beta\frac{zq_1'(z)}{q_1(z)}\prec\psi_{\lambda,l}^{m,n}(\delta,\alpha,\xi,\mu,\beta;z)$$
$$\prec\alpha+\xi q_2(z)+\mu(q_2(z))^2+\beta\frac{zq_2'(z)}{q_2(z)},$$

for $\delta, \alpha, \xi, \mu, \beta \in \mathbb{C}$*,* $\beta, \delta \neq 0$*, implies*

$$q_1(z) \prec \left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{z}\right)^{\delta} \prec q_2(z), \quad \delta \in \mathbb{C},\ \delta \neq 0,$$

and q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

For $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$, $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 2.9. *Let* $m, n \in \mathbb{N}$*,* $\lambda, l \geq 0$*. Assume that (2.2) and (2.6) hold. If* $f \in \mathcal{A}$*,* $\left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{z}\right)^{\delta} \in \mathcal{H}[q(0), 1] \cap Q$ *and*

$$\alpha + \xi \frac{1+A_1 z}{1+B_1 z} + \mu \left(\frac{1+A_1 z}{1+B_1 z}\right)^2 + \beta \frac{(A_1 - B_1) z}{(1+A_1 z)(1+B_1 z)} \prec \psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z)$$

$$\prec \alpha + \xi \frac{1+A_2 z}{1+B_2 z} + \mu \left(\frac{1+A_2 z}{1+B_2 z}\right)^2 + \frac{(A_2 - B_2) z}{(1+A_2 z)(1+B_2 z)},$$

for $\delta, \alpha, \xi, \mu, \beta \in \mathbb{C}$*,* $\beta, \delta \neq 0$*,* $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$*, where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.3), then*

$$\frac{1+A_1 z}{1+B_1 z} \prec \left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{z}\right)^{\delta} \prec \frac{1+A_2 z}{1+B_2 z},$$

hence $\frac{1+A_1 z}{1+B_1 z}$ *and* $\frac{1+A_2 z}{1+B_2 z}$ *are the best subordinant and the best dominant, respectively.*

For $q_1(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_1}$, $q_2(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, where $0 < \gamma_1 < \gamma_2 \leq 1$, we have the following corollary.

Corollary 2.10. *Let* $m, n \in \mathbb{N}$*,* $\lambda, l \geq 0$*. Assume that (2.2) and (2.6) hold. If* $f \in \mathcal{A}$*,* $\left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{z}\right)^{\delta} \in \mathcal{H}[q(0), 1] \cap Q$ *and*

$$\alpha + \xi \left(\frac{1+z}{1-z}\right)^{\gamma_1} + \mu \left(\frac{1+z}{1-z}\right)^{2\gamma_1} + \frac{2\beta\gamma_1 z}{1-z^2} \prec \psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z)$$

$$\prec \alpha + \xi \left(\frac{1+z}{1-z}\right)^{\gamma_2} + \mu \left(\frac{1+z}{1-z}\right)^{2\gamma_2} + \frac{2\beta\gamma_2 z}{1-z^2},$$

for $\delta, \alpha, \xi, \mu, \beta \in \mathbb{C}$*,* $\beta, \delta \neq 0$*,* $0 < \gamma_1 < \gamma_2 \leq 1$*, where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.3), then*

$$\left(\frac{1+z}{1-z}\right)^{\gamma_1} \prec \left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{z}\right)^{\delta} \prec \left(\frac{1+z}{1-z}\right)^{\gamma_2},$$

hence $\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ *and* $\left(\frac{1+z}{1-z}\right)^{\gamma_2}$ *are the best subordinant and the best dominant, respectively.*

We have also

Theorem 2.11. *Let* $\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta} \in \mathcal{H}(U), f \in \mathcal{A}, z \in U, \delta \in \mathbb{C}, \delta \neq 0, m, n \in \mathbb{N},$ $\lambda, l \geq 0$ *and let the function* $q(z)$ *be convex and univalent in* U *such that* $q(0) = 1$, $z \in U$. *Assume that*

$$\operatorname{Re}\left(\frac{\alpha+\beta}{\beta} + \frac{zq''(z)}{q'(z)}\right) > 0, \tag{2.8}$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $z \in U$, *and*

$$\psi_{\lambda,l}^{m,n}(\delta,\alpha,\beta;z) := \left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta}\left[\alpha + \frac{\beta\delta(l+1)}{\lambda}\left(\frac{IR_{\lambda,l}^{m+2,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z)} - 1\right)\right] \tag{2.9}$$

If q *satisfies the following subordination*

$$\psi_{\lambda,l}^{m,n}(\delta,\alpha,\beta;z) \prec \alpha q(z) + \beta zq'(z), \tag{2.10}$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $z \in U$, *then*

$$\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta} \prec q(z), \quad z \in U, \ \delta \in \mathbb{C}, \ \delta \neq 0,$$

and q *is the best dominant.*

Proof. Let the function p be defined by $p(z) := \left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta}$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$. The function p is analytic in U and $p(0) = 1$

We have $zp'(z) = \delta\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta}\left[\frac{z\left(IR_{\lambda,l}^{m+1,n}f(z)\right)'}{IR_{\lambda,l}^{m+1,n}f(z)} - 1\right]$.

By using the identity (2.1), we obtain

$$zp'(z) = \frac{\delta(l+1)}{\lambda}\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta}\left(\frac{IR_{\lambda,l}^{m+2,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z)} - 1\right). \tag{2.11}$$

By setting $\theta(w) := \alpha w$ and $\phi(w) := \beta$, it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Also, by letting $Q(z) = zq'(z)\phi(q(z)) = \beta zq'(z)$, we find that $Q(z)$ is starlike univalent in U.

Let $h(z) = \theta(q(z)) + Q(z) = \alpha q(z) + \beta zq'(z)$.

We have $\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \operatorname{Re}\left(\frac{\alpha+\beta}{\beta} + \frac{zq''(z)}{q'(z)}\right) > 0$.

By using (2.11), we obtain

$$\alpha p(z) + \beta zp'(z) = \left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta}\left[\alpha + \frac{\beta\delta(l+1)}{\lambda}\left(\frac{IR_{\lambda,l}^{m+2,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z)} - 1\right)\right].$$

By using (2.10), we have $\alpha p(z) + \beta zp'(z) \prec \alpha q(z) + \beta zq'(z)$.

From Lemma 1.1, we have $p(z) \prec q(z)$, $z \in U$, i.e. $\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta} \prec q(z)$, $z \in U, \delta \in \mathbb{C}, \delta \neq 0$ and q is the best dominant. □

Corollary 2.12. *Let $q(z) = \frac{1+Az}{1+Bz}$, $z \in U$, $-1 \le B < A \le 1$, $m, n \in \mathbb{N}$, $\lambda, l \ge 0$. Assume that (2.8) holds. If $f \in \mathcal{A}$ and*

$$\psi_{\lambda,l}^{m,n}(\delta,\alpha,\beta;z) \prec \alpha\frac{1+Az}{1+Bz} + \beta\frac{(A-B)z}{(1+Bz)^2},$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \le B < A \le 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.9), then

$$\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta} \prec \frac{1+Az}{1+Bz}, \quad \delta \in \mathbb{C},\ \delta \neq 0,$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, in Theorem 2.11 we get the corollary. □

Corollary 2.13. *Let $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $m, n \in \mathbb{N}$, $\lambda, l \ge 0$. Assume that (2.8) holds. If $f \in \mathcal{A}$ and*

$$\psi_{\lambda,l}^{m,n}(\delta,\alpha,\beta;z) \prec \alpha\left(\frac{1+z}{1-z}\right)^{\gamma} + \frac{2\beta\gamma z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma},$$

for $\alpha, \beta \in \mathbb{C}$, $0 < \gamma \le 1$, $\beta \neq 0$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.9), then

$$\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta} \prec \left(\frac{1+z}{1-z}\right)^{\gamma}, \quad \delta \in \mathbb{C},\ \delta \neq 0,$$

and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ is the best dominant.

Proof. Corollary follows by using Theorem 2.11 for $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $0 < \gamma \le 1$. □

Theorem 2.14. *Let q be convex and univalent in U such that $q(0) = 1$. Assume that*

$$\text{(2.12)} \qquad Re\left(\frac{\alpha}{\beta}q'(z)\right) > 0, \text{ for } \alpha, \beta \in \mathbb{C},\ \beta \neq 0.$$

If $f \in \mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q$ and $\psi_{\lambda,l}^{m,n}(\delta,\alpha,\beta;z)$ is univalent in U, where $\psi_{\lambda,l}^{m,n}(\delta,\alpha,\beta;z)$ is as defined in (2.9), then

$$\text{(2.13)} \qquad \alpha q(z) + \beta z q'(z) \prec \psi_{\lambda,l}^{m,n}(\delta,\alpha,\beta;z)$$

implies

$$q(z) \prec \left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta}, \quad \delta \in \mathbb{C},\ \delta \neq 0,\ z \in U,$$

and q is the best subordinant.

Proof. Let the function p be defined by $p(z) := \left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta}$, $z \in U$, $z \neq 0$, $\delta \in \mathbb{C}, \delta \neq 0$, $f \in \mathcal{A}$. The function p is analytic in U and $p(0) = 1$.

By setting $\nu(w) := \alpha w$ and $\phi(w) := \beta$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Since $\frac{\nu'(q(z))}{\phi(q(z))} = \frac{\alpha}{\beta} q'(z)$, it follows that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) = Re\left(\frac{\alpha}{\beta} q'(z)\right) > 0$, for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$.

Now, by using (2.13) we obtain

$$\alpha q(z) + \beta z q'(z) \prec \alpha q(z) + \beta z q'(z), \quad z \in U.$$

From Lemma 1.2, we have

$$q(z) \prec p(z) = \left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{z}\right)^{\delta}, \quad z \in U,\ \delta \in \mathbb{C},\ \delta \neq 0,$$

and q is the best subordinant. □

Corollary 2.15. *Let* $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, $z \in U$, $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. *Assume that (2.12) holds. If* $f \in \mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{z}\right)^{\delta} \in \mathcal{H}[q(0), 1] \cap Q$, $\delta \in \mathbb{C}$, $\delta \neq 0$ *and*

$$\alpha \frac{1+Az}{1+Bz} + \beta \frac{(A-B) z}{(1+Bz)^2} \prec \psi_{\lambda,l}^{m,n}(\delta, \alpha, \beta; z),$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.9), then*

$$\frac{1+Az}{1+Bz} \prec \left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{z}\right)^{\delta}, \delta \in \mathbb{C},\ \delta \neq 0,$$

and $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, in Theorem 2.14 we get the corollary. □

Corollary 2.16. *Let* $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. *Assume that (2.12) holds. If* $f \in \mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{z}\right)^{\delta} \in \mathcal{H}[q(0), 1] \cap Q$ *and*

$$\alpha \left(\frac{1+z}{1-z}\right)^{\gamma} + \frac{2\beta\gamma z}{1-z^2} \left(\frac{1+z}{1-z}\right)^{\gamma} \prec \psi_{\lambda,l}^{m,n}(\delta, \alpha, \beta; z),$$

for $\alpha, \beta \in \mathbb{C}$, $0 < \gamma \leq 1$, $\beta \neq 0$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.9), then*

$$\left(\frac{1+z}{1-z}\right)^{\gamma} \prec \left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{z}\right)^{\delta}, \quad \delta \in \mathbb{C},\ \delta \neq 0,$$

and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best subordinant.*

Proof. Corollary follows by using Theorem 2.14 for $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $0 < \gamma \leq 1$. □

Combining Theorem 2.11 and Theorem 2.14, we state the following sandwich theorem.

Theorem 2.17. *Let* q_1 *and* q_2 *be convex and univalent in* U *such that* $q_1(z) \neq 0$ *and* $q_2(z) \neq 0$, *for all* $z \in U$. *Suppose that* q_1 *satisfies (2.8) and* q_2 *satisfies (2.12).*

If $f\in\mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q$, $\delta\in\mathbb{C},\delta\neq0$ *and* $\psi_{\lambda,l}^{m,n}(\delta,\alpha,\beta;z)$ *is as defined in (2.9) univalent in* U*, then*

$$\alpha q_1(z)+\beta zq_1'(z)\prec\psi_{\lambda,l}^{m,n}(\delta,\alpha,\beta;z)\prec\alpha q_2(z)+\beta zq_2'(z),$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq0$, *implies*

$$q_1(z)\prec\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta}\prec q_2(z),\quad z\in U,\delta\in\mathbb{C},\ \delta\neq0,$$

and q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

For $q_1(z)=\frac{1+A_1z}{1+B_1z}$, $q_2(z)=\frac{1+A_2z}{1+B_2z}$, where $-1\leq B_2<B_1<A_1<A_2\leq1$, we have the following corollary.

Corollary 2.18. *Let* $m,n\in\mathbb{N}$, $\lambda,l\geq0$. *Assume that (2.8) and (2.12) hold for* $q_1(z)=\frac{1+A_1z}{1+B_1z}$ *and* $q_2(z)=\frac{1+A_2z}{1+B_2z}$, *respectively. If* $f\in\mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q$ *and*

$$\alpha\frac{1+A_1z}{1+B_1z}+\beta\frac{(A_1-B_1)z}{(1+B_1z)^2}\prec\psi_{\lambda,l}^{m,n}(\delta,\alpha,\beta;z)$$
$$\prec\alpha\frac{1+A_2z}{1+B_2z}+\beta\frac{(A_2-B_2)z}{(1+B_2z)^2},\quad z\in U,$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq0$, $-1\leq B_2\leq B_1<A_1\leq A_2\leq1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.3), then*

$$\frac{1+A_1z}{1+B_1z}\prec\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta}\prec\frac{1+A_2z}{1+B_2z},\quad z\in U,\ \delta\in\mathbb{C},\ \delta\neq0,$$

hence $\frac{1+A_1z}{1+B_1z}$ *and* $\frac{1+A_2z}{1+B_2z}$ *are the best subordinant and the best dominant, respectively.*

For $q_1(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_1}$, $q_2(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_2}$, where $0<\gamma_1<\gamma_2\leq1$, we have the following corollary.

Corollary 2.19. *Let* $m,n\in\mathbb{N}$, $\lambda,l\geq0$. *Assume that (2.8) and (2.12) hold for* $q_1(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ *and* $q_2(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_2}$, *respectively. If* $f\in\mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q$ *and*

$$\alpha\left(\frac{1+z}{1-z}\right)^{\gamma_1}+\frac{2\beta\gamma_1z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma_1}\prec\psi_{\lambda,l}^{m,n}(\delta,\alpha,\beta;z)$$
$$\prec\alpha\left(\frac{1+z}{1-z}\right)^{\gamma_2}+\frac{2\beta\gamma_2z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma_2},\quad z\in U,$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq0$, $0<\gamma_1<\gamma_2\leq1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.3), then*

$$\left(\frac{1+z}{1-z}\right)^{\gamma_1}\prec\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{z}\right)^{\delta}\prec\left(\frac{1+z}{1-z}\right)^{\gamma_2},\quad z\in U,\ \delta\in\mathbb{C},\ \delta\neq0,$$

hence $\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ *and* $\left(\frac{1+z}{1-z}\right)^{\gamma_2}$ *are the best subordinant and the best dominant, respectively.*

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