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INCLUSION PROPERTIES OF CERTAIN SUBCLASS OF MULTIVALENT FUNCTIONS DEFINED BY GENERALIZED DERIVATIVE OPERATOR

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Abstract. By making use of generalized Ruscheweyh derivative a new class of multivalent analytic function is introduced. In the present paper various inclusion relations of the newly defined functions class is determined. The results generalized the works due to Aghalary et. al.(cf.[1], J. Inequal. Pure & Appl. Math., 5(2),Art. 31, (2004), pp. 1-11.)

1. Introduction and definitions

Let $\mathcal{A}$ be the class of functions analytic in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and $\mathcal{A}_p$ be the subclass of $\mathcal{A}$ consisting of functions of the following form:

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k, \ n, p \in \mathbb{N} := \{1, 2, 3, ...\}$$

where f is analytic and p-valent in $\mathbb{U}$.

Recalling subordination (cf.[2, 10]) of two analytic functions f and g in $\mathbb{U}$, we say that f is subordinate to g in $\mathbb{U}$ and written as

$$f(z) \prec g(z), \qquad (z \in \mathbb{U}),$$

if there exists a function $w(z)$ analytic in $\mathbb{U}$ with

$$w(0) = 0, \text{ and } |w(z)| < 1$$

such that

$$f(z) = g(w(z)), \qquad (z \in \mathbb{U}).$$

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It follows that

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}) \Longrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

In particular, if g is univalent in $\mathbb{U}$, we have following equivalence:

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, f is said to be subordinate to g in the disk $\mathbb{U}_r$, if the function $f_r(z) = f(rz)$ is subordinate to $g_r = g(rz)$ in $\mathbb{U}$. Hence, if $f \prec g$ in $\mathbb{U}$, then $f \prec g$ in $\mathbb{U}_r$ for every r $(0 < r < 1)$.

Formulated in terms of subordination, the function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{K}_n(p, \delta)$ $(0 \le \delta < p)$ consisting of p-valent convex functions of order δ (cf.[8], also see [3]) if

$$\frac{1}{p-\delta}\left(1 + \frac{zf''(z)}{f'(z)} - \delta\right) \prec \frac{1+z}{1-z} \qquad (z \in \mathbb{U}).$$

Furthermore the function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{S}_n^*(p, \delta)$ $(0 \le \delta < p)$, consisting of p-valent starlike functions of order δ (cf.[6], also see [8]) if

$$p\int_0^z \frac{f(t)}{t}dt \in \mathcal{K}_n(p, \delta).$$

Equivalently,

$$f \in \mathcal{K}_n(p, \delta) \Longleftrightarrow zf' \in \mathcal{S}_n^*(p, \delta) \qquad (\forall n \in \mathbb{N}).$$

For function $q \in \mathcal{A}$, normalized by $q(0) = 1$ of the form

$$q(z) = 1 + c_1 z + c_2 z^2 + \ldots$$

is said to be in $\mathcal{P}(1, b)$ if

$$q(z) \prec 1 + bz, \qquad b > 0. \tag{1.1}$$

Or, equivalently

$$|q(z) - 1| < b, \qquad b > 0.$$

The class $\mathcal{P}(1, b)$ is introduced and studied by Janowski [5].

In our investigation we also need following definitions of fractional derivative operator defined by Srivastava [12], and Srivastava and Saxena [13]:

Definition 1.1. Let f is an analytic function in a simply connected region of the z-plane containing the region, and the multiplicity of $(z-\zeta)^{-}$ is removed by requiring $log(z-\zeta)$ to be real when $z - \zeta > 0$. Then the generalized fractional derivative of order l is defined for a function $f(z)$ by

$$J_{0,z}^{l,\mu,\nu} f(z) = \begin{cases} \frac{1}{\Gamma(1-l)}\frac{d}{dz}\left\{z^{l-\mu}\int_0^z (z-\zeta)^{-l}.\ {}_2F_1\left(\mu - l, -\nu; 1-l; 1-\frac{\zeta}{z}\right) f(\zeta)d\zeta\right\}, & (0 \le l < 1), \\ \frac{d^n}{dz^n} J_{0,z}^{l-n,\mu,\nu} f(z), & (n \le l < n+1,\ n \in \mathbb{N}) \end{cases}$$

provided further that

$$f(z) = O(|z|^k), \ (z \to 0;\ k > \max\{0, \mu - \nu - 1\} - 1).$$

It follows from the above definition that

$$J_{0,z}^{l,\mu,\nu} f(z) = \Omega_z^l f(z), \ (0 \le l < 1), \tag{1.2}$$

where Ω_z^l is the fractional derivative operator of order l. In terms of Gamma function, we have

$$J_{0,z}^{l,\mu,\nu} z^\rho = \frac{\Gamma(\rho+1)\Gamma(\rho-\mu+\nu+2)}{\Gamma(\rho-\mu+1)\Gamma(\rho-l+\nu+2)} z^{\rho-\mu}, \ (0 \le l < 1, \rho > \max\{0, \mu-\nu-1\}-1).$$

Recently, Goyal and Goyal [4] (also see [9]) defined a generalized Ruscheweyh derivative $\mathbb{J}_p^{l,\mu} f, \ \mu > -1$ as follows:

$$\mathbb{J}_p^{l,\mu} f(z) = \frac{\Gamma(\mu-l+\nu+2)}{\Gamma(\nu+2)\Gamma(\mu+1)} z^p J_{0,z}^{l,\mu,\nu}(z^{\mu-p} f(z)) = z^p + \sum_{k=n+p}^{\infty} B_p^{l,\mu}(k) a_k z^k, \tag{1.3}$$

where

$$B_p^{l,\mu}(k) = \frac{\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-l)\Gamma(k+\nu-p+2)}{\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-l)\Gamma(\nu+2)\Gamma(1+\mu)} \tag{1.4}$$

For $\mu = l$, this generalized Ruscheweyh derivative get reduced to Ruscheweyh derivative of $f(z)$ of order $l > -1$ (see, e.g. [11]) as follows:

$$\begin{aligned} D^l f(z) &= \frac{z^p}{\Gamma(l+1)} \frac{d^l}{dz^l}(z^{l-p} f(z)) \\ &= z^p + \sum_{k=n+p}^{\infty} \frac{\Gamma(l+k-p+1)}{\Gamma(l+1)\Gamma(k-p+1)} a_k z^k. \end{aligned} \tag{1.5}$$

For $p = 1$, (1.5) reduces to ordinary Ruscheweyh derivative for univalent functions [11]. By making use of the above generalized Ruscheweyh derivative operator we define following:

Definition 1.2. For $\mu > -1, \ n \ge 1$ and $\lambda \ge 0$ and we define a new class of functions $\mathcal{S}_{p,\lambda}^{l,\mu}(1,b)$ subclass of $\mathcal{P}(1,b)$ consisting of functions $q(f)$ such that

$$q(f(z)) = (1-p\lambda)\frac{(\mathbb{J}_p^{l,\mu} f(z))}{z^p} + \lambda \frac{(\mathbb{J}_p^{l,\mu} f(z))'}{z^{p-1}}.$$

It may note that, for $\mu = l$, the class $\mathcal{S}_{p,\lambda}^{l,\mu}(1,b)$ reduced to the class $\mathcal{S}_{p,\lambda}^{l}(1,b)$ consisting of the functions $q(f)$ such that

$$q(f(z)) = (1-p\lambda)\frac{(D^l f(z))}{z^p} + \lambda \frac{(D^l f(z))'}{z^{p-1}}.$$

For $p = 1$, the above class is further reduced to the class $\mathcal{S}_{\lambda}^{l}(1,b)$ defined by Aghalary et al. [1].

Following Lemma due to Miller and Mocanu [7] play key role to prove our main results:

Lemma 1.1. *Let $q(z) = 1 + q_n z^n + ...,\ (n \ge 1)$ be analytic in $\mathbb{U}$ and let $h(z)$ be convex univalent in $\mathbb{U}$ with $h(0) = 1$. If $q(z) + \frac{1}{c} z q'(z) \prec h(z)$ for $c > 0$, then*

$$q(z) \prec \frac{c}{n} z^{-c/n} \int_0^z h(t) t^{\frac{c}{n}-1} dt.$$

2. Main Results

We have following properties of the family $\mathcal{S}_{p,\lambda}^{l,\mu}(1,b)$:

Theorem 2.1. *If* $q(f) \in \mathcal{S}_{p,\lambda}^{l,\mu}(1,b)$ *then*

$$z^{-p}\mathbb{J}_p^{l,\mu}f(z) \in \mathcal{P}\left(1, \frac{b}{1+\lambda n}\right).$$

Proof. Let $q(f) \in \mathcal{S}_{p,\lambda}^{l,\mu}(1,b)$. Taking $g(z) = z^{-p}\mathbb{J}_p^{l,\mu}f(z)$. Upon differentiation and application of (1.1), yields

$$g(z) + \lambda z g'(z) = q(f(z)) \prec 1 + bz.$$

Now putting $\lambda = 1/c$ and applying Lemma 1.1, we have

$$g(z) \prec \frac{1}{n\lambda} z^{-1/n\lambda} \int_0^z (1+bt)t^{\frac{1}{n\lambda}-1}dt = 1 + \frac{b}{1+\lambda n}z.$$

By principle of subordination, we have for $|w(z)| \leq |z|^n$

$$g(z) = z^{-p}\mathbb{J}_p^{l,\mu}f(z) = 1 + \frac{b}{1+\lambda n}w(z).$$

Thus, the theorem follows from the condition (1.1).

The estimates in Theorem 2.1 are sharp for $q(f)$ where f is given by

$$z^{-p}\mathbb{J}_p^{l,\mu}f(z) = 1 + \frac{b}{1+\lambda n}z^n.$$

This completes the proof of Theorem 2.1. □

Corollary 2.1. *If* $q(f) \in \mathcal{S}_{p,\lambda}^{l,\mu}(1,b)$, *then*

$$\left|z^{-p}\mathbb{J}_p^{l,\mu}f(z) - 1\right| \leq \frac{b}{1+\lambda n}|z|^n.$$

Putting $\mu = l = 0$ in Theorem 2.1, we have following Corollary:

Corollary 2.2. *If* $\left|(1-p\lambda)\frac{f(z)}{z^p} + \lambda\frac{f'(z)}{z^{p-1}} - 1\right| < b$, *then*

$$\frac{f(z)}{z^p} \prec 1 + \frac{b}{1+\lambda n}z.$$

On replacing $f(z) \longrightarrow z^p f'(z)$, Corollary 2.2 further reduced to the following:

Corollary 2.3. *If* $|f'(z) + \lambda z f''(z) - 1| < b$, *then*

$$f'(z) \prec 1 + \frac{b}{1+\lambda n}z.$$

In the next two theorems, we give the inclusion results for the functions class $\mathcal{S}_{p,\lambda}^{l,\mu}$:

Theorem 2.2. *For* $0 \leq \lambda_1 < \lambda$ *and* $l \geq 0$, *let* $b_1 = \frac{1+n\lambda_1}{1+n\lambda}b$. *Then*

$$\mathcal{S}_{p,\lambda}^{l,\mu}(1,b) \subset \mathcal{S}_{p,\lambda_1}^{l,\mu}(1,b_1)$$

Proof. The case for $\lambda_1 = 0$ is trivial as $b_1 = \frac{1}{1+n\lambda}b$. Let $q(f) \in \mathcal{S}_{p,\lambda}^{l,\mu}(1,b)$. Therefore

$$(1-p\lambda_1)\frac{\mathbb{J}_p^{l,\mu}f(z)}{z^p} + \lambda_1\frac{(\mathbb{J}_p^{l,\mu}f(z))'}{z^{p-1}} = \frac{\mathbb{J}_p^{l,\mu}f(z)}{z^p}.$$

Application of Theorem 2.1, yields

$$\frac{\mathbb{J}_p^{l,\mu}f(z)}{z^p} \prec 1 + \frac{1}{1+n\lambda}b = 1 + b_1.$$

This shows that $q(f) \in \mathcal{S}_{p,\lambda_1}^{l,\mu}(1,b_1)$.

For $\lambda_1 \neq 0$, suppose that $q(f) \in \mathcal{S}_{p,\lambda}^{l,\mu}(1,b)$. Therefore we have

$$\begin{aligned}(1-p\lambda_1)&\frac{\mathbb{J}_p^{l,\mu}f(z)}{z^p} + \lambda_1\frac{(\mathbb{J}_p^{\lambda,\mu}f(z))'}{z^{p-1}}\\ &= \frac{\lambda_1}{\lambda}\left[(1-p\lambda)\frac{\mathbb{J}_p^{l,\mu}f(z)}{z^p} + \lambda\frac{(\mathbb{J}_p^{l,\mu}f(z))'}{z^{p-1}}\right] + \left(1-\frac{\lambda_1}{\lambda}\right)\frac{\mathbb{J}_p^{l,\mu}f(z)}{z^p}.\end{aligned}$$

Which on application of Theorem 2.1, gives

$$\begin{aligned}\left|(1-p\lambda_1)\frac{\mathbb{J}_p^{l,\mu}}{z^p} + \lambda_1\frac{(\mathbb{J}_p^{l,\mu}f(z))'}{z^{p-1}} - 1\right| &\leq \frac{\lambda_1}{\lambda}|q(f)-1| + \left(1-\frac{\lambda_1}{\lambda}\right)\left|\frac{\mathbb{J}_p^{l,\mu}}{z^p} - 1\right|\\ &< \frac{\lambda_1}{\lambda}b + \left(1-\frac{\lambda_1}{\lambda}\right)\left(\frac{b}{1+n\lambda}\right)\\ &= b\left(\frac{1+n\lambda_1}{1+n\lambda}\right) = b_1.\end{aligned}$$

Which shows that $q(f) \in \mathcal{S}_{p,\lambda_1}^{l,\mu}(1,b_1)$. This completes the proof of Theorem 2.2. □

Theorem 2.3. *For $l \geq 0$, let $b_1 = \frac{b(1+\mu)}{n+1+\mu}$. Then*

$$\mathcal{S}_{p,\lambda}^{l+1,\mu+1}(1,b) \subset \mathcal{S}_{p,\lambda}^{l,\mu}(1,b_1).$$

Proof. Suppose that $q_1(f) \in \mathcal{S}_{p,\lambda}^{l+1,\mu+1}(1,b)$. Therefore we have

$$q_1(f(z)) = (1-p\lambda)\frac{\mathbb{J}_p^{l+1,\mu+1}f(z)}{z^p} + \lambda\frac{(\mathbb{J}_p^{l+1,\mu+1}f(z))'}{z^{p-1}} \prec 1 + bz. \tag{2.1}$$

Taking

$$q_2(f(z)) = (1-p\lambda)\frac{\mathbb{J}_p^{l,\mu}f(z)}{z^p} + \lambda\frac{(\mathbb{J}_p^{l,\mu}f(z))'}{z^{p-1}}. \tag{2.2}$$

Using (1.4), we find that

$$\frac{B_p^{l+1,\mu+1}(k)}{B_p^{l,\mu}(k)} = \frac{k-p+\mu+1}{\mu+1}. \tag{2.3}$$

Setting $q_1(f(z)) = q_2(f(z)) + czq_2'(f(z))$, solving using (1.3) and (2.2) and making use of (2.3), we get $c = \frac{1}{\mu+1}$. Therefore, we have

$$q_1(f(z)) = q_2(f(z)) + \frac{1}{\mu+1}zq_2'(f(z)). \tag{2.4}$$

In the view of (2.1), we have

$$q_1(f(z)) = q_2(f(z)) + \frac{1}{\mu+1} z q_2'(f(z)) \prec 1 + bz.$$

Hence an application of Lemma 1.1 gives

$$q_2(f(z)) \prec \frac{\mu+1}{n} z^{\frac{-(\mu+1)}{n}} \int_0^z h(t) t^{\frac{(\mu+1)}{n}-1} dt = 1 + \frac{(\mu+1)bz}{\mu+1+n} = 1 + b_1 z.$$

Thus we conclude that

$$q_1(f) \in \mathcal{S}_{p,\lambda}^{l,\mu}(1, b_1) \text{ implies } q_2(f) \in \mathcal{S}_{p,\lambda}^{l,\mu}(1, b_1).$$

This completes the proof of Theorem 2.3. □

Letting $\mu = l$ in the Theorem 2.3 and using (1.2), we obtain the following:

Corollary 2.4. *For $l \geq 0$, let $b_1 = \frac{b(1+l)}{n+1+l}$. Then*

$$\mathcal{S}_{p,\lambda}^{l+1}(1, b_1) \subset \mathcal{S}_{p,\lambda}^{l}(1, b_1).$$

For $p = 1$, Corollary 2.4 reduces to the recently established result due to Aghalary et. al.[1, Theorem 3.2] as our special case.

Further putting $l = 0$ and $q_2(f(z)) = f'(z)$ in Theorem 2.3 and with suitable applications of (2.2) and (2.4) we get the following:

Corollary 2.5. *If $f'(z) + zf_n''(z) \in \mathcal{P}(1, b)$, then $(1+p\lambda)\frac{f(z)}{z^p} + \lambda\frac{f'(z)}{z^{p-1}} \in \mathcal{P}(1, \frac{b}{n+1})$.*

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