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SOME ADDITION TO PARSEVAL'S RELATION AND TO OTHER PROBLEMS

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Abstract. In this paper we consider some addition to Parseval's relation and analytic representation of some classes of distributions. Also we consider analytic representation of some density functions in the theory of probability.

1. Introduction

With $L^1(R^n)$ we denote the space of all Lebesgue integrable functions on R^n.

If $\chi = (x_1, \ \dots, \ x_n)$ and $t = (t_1, \ \dots, \ t_n)$ are two points of R^n then its inner product is

$$\chi t \ = x_1 t_1 + \cdots + x_n t_n.$$

With $C_0(R^n)$ we denote the Banach space which consists of all continuous functions defined on R^n which vanish at infinity.

The dual space $C_0(R^n)'$ is the space of all complex Borel measure μ defined on R^n. If $f \in L^1(R^n)$, then it is well known that the function

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(t) e^{iTx} dx$$

is called a Fourier transform of f.

The function $\hat{f}$ belongs to the space $C_0(R^n)$ and

$$|\hat{f}(t)| \leq \int_{-\infty}^{\infty} |f(x) e^{itx}| dx \leq ||f||_1,$$

where $||f||_1$ is norm in $L^1(R^n)$.

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Now, let μ be a complex Borel measure defined on R^n. With $\hat{\mu}(t)$ we denote the Fourier transform of the measure μ defined by

$$\hat{\mu}(t) = \int_{-\infty}^{\infty} e^{itx} d\mu(x) = \int_{-\infty}^{\infty} e^{itx} h(x) d|\mu|(x),$$

where $h(x)$ is a Borel function such that $|h(x)| = 1$ ([2], p.124)

For the sake of simplicity, the further exposition will be on the real line.

First we will prove that the function $\hat{\mu}(t)$ is continuous and bounded. Let (t_n) be a sequence of real numbers such that $t_n \to t$ as $n \to \infty$. Since $|e^{it_n x} h(x)| \le 1$ and since the function lis $|\mu|$ integrable, we may apply Lebesgue's dominated convergence theorem and obtain

$$\begin{aligned}\lim_{n\to\infty} \hat{\mu}(t_n) &= \lim_{n\to\infty} \int_{-\infty}^{\infty} e^{it_n x} h(x) d|\mu|(x) \\ &= \int_{-\infty}^{\infty} \lim_{n\to\infty} e^{it_n x} h(x) d|\mu|(x) \ = \int_{-\infty}^{\infty} e^{itx} h(x) d|\mu|(x) \ = \hat{\mu}(t).\end{aligned}$$

Thus we proved that $\hat{\mu}$ is continuous function.

From the estimate

$$|\hat{\mu}(t)| \le \int_{-\infty}^{\infty} |e^{itx} h(x)\ | d|\mu|(x) \ = ||\mu||$$

it follows that $\hat{\mu}$ is bounded function, where the norm $||\mu|| = |\mu|(R)$ is the total variation of μ.

Thus from the above we may conclude that, if $f \in L^1(R)$ then also $f \cdot \hat{\mu}$ belongs to $L^1(R)$.

Theorem 1.1. *If $f \in L^1(R)$ and μ is a complex Borel measure on R, then*

$$\int_{-\infty}^{\infty} \hat{f}(x) d\mu(x) = \int_{-\infty}^{\infty} f(t)\hat{\mu}(t) dt. \tag{1.1}$$

Proof. Since the function $\hat{f}(x)$ is continuous and bounded, the integral

$$\int_{-\infty}^{\infty} \hat{f}(x) d\mu(x)$$

exists, and we have that

$$\begin{aligned}\int_{-\infty}^{\infty} \hat{f}(x) d\mu(x) &= \int_{-\infty}^{\infty} d\mu(x) \int_{-\infty}^{\infty} f(t) e^{itx} dt \\ &= \int_{-\infty}^{\infty} h(x) d|\mu|(x) \ \int_{-\infty}^{\infty} f(t) e^{itx} dt.\end{aligned}$$

Since both integrals exist we may apply Fubini's theorem and receive

$$\begin{aligned}\int_{-\infty}^{\infty} \hat{f}(x) d\mu(x) &= \int_{-\infty}^{\infty} f(t) dt \ \int_{-\infty}^{\infty} e^{itx} h(x) d|\mu|(x) \\ &= \int_{-\infty}^{\infty} f(t)\hat{\mu}(t) dt.\end{aligned}$$

Thus the relation (1.1) is proved. (see [1], p.145) □

In particular, if $d\mu(x) = g(x)dx$ where $g \in L^1(R)$, then we obtain

$$\int_{-\infty}^{\infty} \hat{f}(x)d\mu(x) = \int_{-\infty}^{\infty} \hat{f}(x)g(x)dx$$
$$= \int_{-\infty}^{\infty} f(t)dt \int_{-\infty}^{\infty} e^{itx}g(x)dx = \int_{-\infty}^{\infty} f(t)\hat{g}(t)dt.$$

This relation is known as Parseval's relation. For this reason one may take the relation (1.1) as some *generalization of the Parseval relation.*

Example 1.1. *If δ is the Dirac measure, then*

$$\int_{-\infty}^{\infty} \hat{f}(x)d\delta(x) = \hat{f}(0) = \int_{-\infty}^{\infty} f(t)dt,$$

since

$$\hat{\delta}(t) = \int_{-\infty}^{\infty} e^{itx}d\delta(x) = 1.$$

The Riesz representation theorem for the bounded linear functionals on the Banach space $C_0(R)$ asserts that if μ and λ are two measures such that $\mu \neq \lambda$, then the functionals defined by μ and λ, respectively, are different, and since the set of all functions $\hat{f}$ is dense in $C_0(R)$ from the relation (1.1) follows that $\hat{\mu} \neq \hat{\lambda}$. Further, in the space of all complex Borel measures on R, i.e., on $C_0(R)'$, the convolution of measures is defined. Namely, if μ and λ are two Borel measures, then there exists measure $v = \mu * \lambda$ defined by means of the product of measures which is also complex Borel measure. The measure v is called a convolution of the measures μ and λ.

We know that

$$\int_{-\infty}^{\infty} fd(\mu * \lambda) = \int\int f(x + y)d\mu d\lambda, \tag{1.2}$$

from where we receive

$$\int_{-\infty}^{\infty} e^{itx}d(\mu * \lambda)\ (x) = \int\int_{R^2} e^{it(x+y)}d\mu(x)d\lambda(y)$$
$$= \int_{-\infty}^{\infty} e^{itx}d\mu(x) \int_{-\infty}^{\infty} e^{ity}d\lambda(y) = \hat{\mu}(t)\hat{\lambda}(t).$$

For example, if $\lambda = \delta$, then from (1.2) we have

$$\int_{-\infty}^{\infty} e^{itx}d(\mu * \delta) = \int_{-\infty}^{\infty} e^{itx}d\mu(x) \int_{-\infty}^{\infty} e^{ity}d\delta(y)$$
$$= \int_{-\infty}^{\infty} e^{itx}d\mu(x)\ 1 = \hat{\mu}(t).$$

This means that $\mu * \delta = \mu$, i.e., δ is the unit element with respect to the convolution of measures. This is similarly as a convolution of distribution with the distribution $\delta = 1$. From (1.2) we also obtain that

$$||\mu * \lambda|| \leq ||\mu||||\lambda||.$$

Note that the Banach space $C_0(R)'$ with the operation of convolution of measures is a commutative Banach algebra with unit element δ.

Let now define mapping h from $C_0(R)'$ into $L^\infty(R)$ as follows

$$h(\mu) = \hat{\mu}.$$

The mapping h is homomorphism form $C_0(R)'$ into $L^\infty(R)$. To realize this, first note that h is linear and since

$$h(\mu * \lambda) = (\widehat{\mu * \lambda}) = \hat{\mu}\hat{\lambda} = h(\mu)h(\lambda),$$

it follows that h is homomorphism.

Since

$$|h(\mu)(t)| = |\hat{\mu}(t)| \leq ||\mu||$$

we have that

$$||\hat{\mu}||_\infty \leq ||\mu||,$$

and h is continuous.

Example 1.2. *Let consider the following Borel measure*

$$\mu = \sum_{k=1}^{\infty} a_k \delta(t - l_k),$$

where a_k are complex numbers and l_k are real numbers. We know that $\delta(t - l_k)$ is the Dirac measure at the point l_k with $\delta(t - l_k)(\{l_k\}) = 1$. If μ is a complex Borel measure, then it is necessary that

$$\sum_{k=1}^{\infty} |a_k| < \infty$$

and the norm of measure μ is

$$||\mu|| = \sum_{k=1}^{\infty} |a_k|.$$

The Fourier transform is

$$\hat{\mu}(t) = \int_{-\infty}^{\infty} e^{itx} d\mu(x) = \sum_{k=1}^{\infty} a_k \int_{-\infty}^{\infty} e^{itx} d\delta(x - l_k) = \sum_{k=1}^{\infty} a_k e^{itl_k}$$

and consequently, Theorem 1 implies

$$\int_{-\infty}^{\infty} f(t)\hat{\mu}(t)dt \ = \sum_{k=1}^{\infty} a_k \int_{-\infty}^{\infty} f(t) e^{itl_k} dt \ = \sum_{k=1}^{\infty} a_k \hat{f}(l_k).$$

The Riemann-Lebesgue lemma shows that

$$\lim_{R\to\infty} \frac{1}{2R} \int_{|t|\leq R} |\hat{\mu}(t)|^2 dt \ = \sum_{k=1}^{\infty} |a_k|^2.$$

Example 1.3. *If in Example 1.2, the numbers a_k are such that $a_k \geq 0$, for $k = 1, 2, \ldots$ and if*

$$\sum_{k=1}^{\infty} a_k = 1,$$

then the measure μ given with the formula

$$\mu = \sum_{k=1}^{\infty} a_k \delta(t - l_k)$$

is called a probability Borel measure and the function

$$F(t) \ = \sum_{l_k \leq t} a_k t \in R$$

is called a function of distribution. The function $F(t)$ has the following properties:

(*i*) $0 \leq F(t) \leq 1$,

(*ii*) $\lim_{t \to -\infty} F(t) = 0, \ \lim_{t \to \infty} F(t) = 1$,

(*iii*) *$F(t)$ is continuous function on the right hand side at every point t, and has jump equal to l_k in each points l_k, for $k = 1, 2, \ldots$.*

The function X defined on the σ algebra of all Borel sets with the values $l_1, \ l_2, \ldots$ on the real line and with the probabilities $\mu(X = l_k) = a_{k'}$ for $k = 1, 2, \ \ldots$ is called a discrete random variable. In the theory of probability the triple: σ algebra of all Borel sets, probability function (the measure μ) and the Borel function $X(t)$ is called probability space.

Note that the function

$$F(t) \ = \mu(X \leq t)$$

is called a cumulative function for the random variable X which may be discrete or continuous.

If X is continuous random variable, then we know that

$$\mu(a < X \leq b) = F(b) - F(a),$$

where $F(t)$ is function of distribution.

If there exists a function $f(t)$ such that

$$F'(t) = f(t)$$

except possibly at a finite number of points, then the function $f(t)$ is called density function and

$$F(x) \ = \int_{-\infty}^{x} f(t).$$

The density function $f(t)$ has the following properties

(1) $f(t) \geq 0$

(2) *f belongs to the space $L^1(R)$ and*

$$\int_{-\infty}^{\infty} f(t) dt \ = 1.$$

AIso we know that

$$\mu(a < X \leq b) = \int_{a}^{b} f(t) dt.$$

One very important distribution function in the theory of probability is the normal distribution.

For the random variable X we say that has normal distribution if it has a function of distribution

$$F(x) \ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{u^2}{2}\right) du$$

and density function

$$f(x) \ = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

In this case we denote $X \sim N(0,1)$ and it means that X has expected valu zero and variance one.

Note that the function $F(t)$ in Example 1.3 has no density function in ordinary sense, but if we consider it as a distribution, then the distribution derivative is

$$F' = \sum_{k=1}^{\infty} a_k \delta(t - l_k) = \mu.$$

This measure (or distribution) serves as a density function.

Since the distribution function F does not belong neither to L^1 nor to L^2, it has no Cauchy representation as a distribution, but the density function has the Cauchy representation. The Cauchy representation of the density function is the function

$$\hat{\mu}(z) = \sum_{k=1}^{\infty} \frac{a_k}{2\pi i (l_k - z)} z \neq l_k.$$

Example 1.4. *If in Example 1.3 the coefficients a_k are given with*

$$a_k = \frac{\lambda^k}{k!} \exp(-\lambda),$$

where $k = 0, 1, \ldots$ and $\lambda > 0$, then the Borel measure (or distribution)

$$\mu(t) = \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} \delta(t - k)$$

is the density function of the Poisson distribution, and is also one of the very important distribution in probability theory. The Cauchy representation of this distribution is the function

$$\hat{\mu}(z) = \frac{1}{2\pi i} \sum_{k=0}^{\infty} \frac{1}{(k - z)} \frac{\lambda^k}{k!} e^{-\lambda}, \ z \neq k.$$

Example 1.5. *If $a_k = q^{k-1}p$, where p and q are greater than zero and $p + q = 1$, then the function*

$$\mu = \sum_{k=0}^{\infty} q^{k-1} p \delta(t - k)$$

is density function for the geometric distribution, and the analytic representation of the geometric distribution is the function

$$\hat{\mu}(z) = \frac{1}{2\pi i} \sum_{k=1}^{\infty} \frac{1}{(k - z)} q^{k-1} p, \ z \neq k.$$

2. The spaces S and S'

With S we denote the space of all infinity differentiable complex-valued functions rapidly decreasing at infinity, and with S' we denote the space of all continuous linear functionals on S. The members of S' are called tempered distributions.

It is known that, if $\varphi \in S$, then the Fourier transform is the function

$$\hat{\varphi}(t) = \int_{-\infty}^{\infty} \varphi(x) e^{itx} dx$$

which is also belong to S.

If the sequence (φ_n) tends to zero as $n \to \infty$ in S, then the sequence of Fourier transforms $(\hat{\varphi}_n)$ tends to zero as $n \to \infty$ in S.

This properties enables to define the Fourier transform for a given tempered distribution u. Namely, let u be a given tempered distribution. Then we define distribution $\hat{u}$ by the formula

$$< \hat{u}, \varphi > = < u, \hat{\varphi} > . \tag{2.1}$$

The distribution $\hat{u}$ defined with (2.1) is called a Fourier transform of the distribution u.

The members of the space L^p, for $1 \leq p \leq \infty$, are tempered distributions, and also the set of all complex Borel measures μ are tempered distributions. It is known the last distributions are defined by the formula

$$< \mu,\ \varphi > = \int_{-\infty}^{\infty} \varphi(x) d\mu(x), \quad \varphi \in S.$$

In the next theorem we will determine the Fourier transform $\hat{\mu}$ for the distribution μ by using relation (1.1).

Theorem 2.1. *If μ be a complex Borel measure on R, then its Fourier transform $\hat{\mu}$, as a tempered distribution, is equal to the Fourier transform $\hat{\mu}(t)$ of the measure μ.*

Proof. For the distribution μ, we define distribution $\hat{\mu}$ as follows

$$< \hat{\mu},\ \varphi > = < \mu, \hat{\varphi} > = \int_{-\infty}^{\infty} \hat{\varphi}(x) d\mu(x),$$

and from the relation (1.1) we have

$$\int_{-\infty}^{\infty} \hat{\varphi}(x) d\mu(x) = \int_{-\infty}^{\infty} \varphi(t) \hat{\mu}(t) dt$$

for every $\varphi \in S$. Thus, we may consider the function

$$\hat{\mu}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{itx}} d\mu(x)$$

as a distribution $\hat{\mu}$. □

We know that this is not true for every tempered distribution. In particular, if $d\mu(x) = f(x)dx$ and if $f \in L^1$, then $\hat{\mu}(t) = \hat{f}(t)$.

The Paley-Wiener theorem ([3], p.199) asserts that, if the distribution u has compact support, if u has order N and if

$$f(z) = < u(t), e^{itz} >, \tag{2.2}$$

then f is entire, the restriction of f on the real line is the Fourier transform of u and there is a constant $\gamma < \infty$ such that

$$|f(z)| \leq \gamma(1+|z|)^N \exp(t|\mathrm{Im}\, z|)$$

and

$$f(x) =< u(t), e^{itx} >= \hat{u}.$$

Conversely, if f is an entire function which satisfies condition (2.2), then there exists distribution u with support in $[-r,\ r]$ and u has order N such that (1.2) holds.

The same is true if the measure μ has compact support. Then the function

$$\hat{\mu}(z) = \int_{-\infty}^{\infty} \exp(itz)\ d\mu(t)$$

is entire and the restriction on the real line is

$$\hat{\mu}(x) = \int_{-\infty}^{\infty} \exp(itx)\ d\mu(t),$$

which is the Fourier transform of the distribution μ.

Thus, for every measure μ, the function $\hat{\mu}$ is a regular distribution.

Remark 2.1. *The Fourier transform of positive Borel measures is characterized by the famous Boshner theorem for positively defined functions. ([4]).*

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