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GROWTH AND DISTORTION THEOREMS FOR GENERALIZED p-VALENT CLOSE-TO-CONVEX HARMONIC FUNCTIONS

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ABSTRACT. A harmonic function f in the unit open disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ can be written as a sum of an analytic and anti-analytic functions: $f = h(z) + \overline{g(z)}$; here, $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$, and called the analytic part and co-analytic part of f, respectively.

Growth (the bounds of the modulus of a function) theorems and distortion (the bounds of the modulus of the derivative of a function) theorems play an important role in the study of the geometric function theory, because this theorems give the compactness of the corresponding classes.

In this paper we consider both of these cases with shear construction method for the sense-preserving generalized $p-$valent harmonic mappings in the unit open disc $\mathbb{D}$.

1. INTRODUCTION

Let Ω be the family of functions $\varphi(z)$ regular in the open unit disc $\mathbb{D}$ and satisfy the conditions $\varphi(0) = 0$, $|\varphi(z)| < 1$ for all $z \in \mathbb{D}$. Denote by $\mathcal{P}(p,n)(p \geq 1, n \geq 1)$ the family of functions $p(z) = p + p_n z^n + p_{n+1} z^{n+1} + p_{n+2} z^{n+2} + \cdots$ which are regular in $\mathbb{D}$ and satisfy the conditions $p(0) = p$, $\text{Re}p(z) > 0$ for every $z \in \mathbb{D}$.

Next, let $\mathcal{A}(p,n)$ be the class of all functions of the form $s(z) = z^p + a_{np+1} z^{np+1} + a_{np+2} z^{np+2} + \cdots$ which are analytic in $\mathbb{D}$. In particular, $\mathcal{A}(p,1)$ is the class of standard p-valent analytic functions and $\mathcal{A}(1,n)$ is the class of all analytic functions for which the first n coefficients are zero, and $\mathcal{A}(1,1)$ is the class of all analytic functions in the standard form. Let $F(z) = z + \alpha_2 z^2 + \alpha_3 z^3 + \cdots$ and $G(z) = z + \beta_2 z^2 + \beta_3 z^3 + \cdots$ be analytic functions in $\mathbb{D}$, if there exist $\varphi(z) \in \Omega$, such that $F(z) = G(\varphi(z))$ for all $z \in \mathbb{D}$, then we say that $F(z)$ is subordinate to $G(z)$ and we write $F \prec G$. We also note that if $F \prec G$, then $F(\mathbb{D}) \subset G(\mathbb{D})$ ([6], [5]).

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Moreover, let $s(z)$ be an element of $\mathcal{A}(p,n)$, if $s(z)$ satisfies the condition $\text{Re}\Big(1 + z\frac{s''(z)}{s'(z)}\Big) > 0$, then $s(z)$ is called generalized p-valent convex function. The class of such functions is denoted by $\mathcal{C}(p,n)$. Let $\phi(z)$ be an element of $\mathcal{A}(p,n)$. If there exist a function $s(z) \in \mathcal{C}(p,n)$ such that $\text{Re}\Big(\frac{\phi'(z)}{s'(z)}\Big) > 0$, then we say that $\phi(z)$ is generalized p-valent close-to-convex function. The class of such functions is denoted by $\mathcal{K}(p,n)$ ([6], [5]).

Finally, generalized p-valent harmonic function f in the disc $\mathbb{D}$ has the representation $f = h(z) + \overline{g(z)}$, where $h(z) = z^p + a_{np+1}z^{np+1} + a_{np+2}z^{np+2} + \cdots$ and $g(z) = b_{np}z^{np} + b_{np+1}z^{np+1} + \cdots$ are analytic in $\mathbb{D}$, and called analytic part and co-analytic part of f, respectively. If $J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0$, then f is called sense-preserving generalized p-valent harmonic function. The class of such functions is denoted by $\mathcal{S}_H(p,n)$ with $|b_{np}| < 1$, and the class of sense-preserving generalized p-valent harmonic functions is denoted by $\mathcal{S}_H^0(p,n)$ with $b_{np} = 0$. We also note that $w(z) = g'(z)/h'(z)$ is analytic second dilatation of f and $|w(z)| < 1$ for every $z \in \mathbb{D}$ ([5]).

In this paper we investigate some properties of the class $\mathcal{K}(p,n)$ and give the growth and distortion theorems for the class of sense-preserving generalized p-valent harmonic mappings that are convex in the direction of real axis. The second part of this study is based on the shear construction method for harmonic functions. We also note that harmonic shear construction method is based on the following theorem of Clunie and T.Sheil-Small. This theorem says that; A harmonic function $f = h(z) + \overline{g(z)}$, locally univalent in $\mathbb{D}$, is a univalent mapping of $\mathbb{D}$ onto a convex domain in the direction of the real axis if and only if $(h(z) - g(z))$ is a conformal univalent mapping of $\mathbb{D}$ onto a convex domain in the direction of the real axis [2].

2. Main Results

Lemma 2.1. *If $p(z) \in \mathcal{P}(p,n)$, then*

$$p(z) = p\frac{1 + z^n\varphi(z)}{1 - z^n\varphi(z)}, \tag{2.1}$$

where $\varphi(z)$ is an analytic function in $\mathbb{D}$ and satisfies the condition $|\varphi(z)| < 1$ for every $z \in \mathbb{D}$.

Proof. Let $p_1(z)$ be the analytic satisfying the conditions $\text{Re}p_1(z) > 0$ and $p_1(0) = 1$ in $\mathbb{D}$, then $p_1(z)$ can be written in the form

$$p_1(z) = \frac{1 + \varphi_1(z)}{1 - \varphi_1(z)}, \tag{2.2}$$

where $\varphi_1(z)$ is analytic in $\mathbb{D}$ and also has n^{th} zero at the origin. Hence $\varphi_1(z) = z^n\varphi(z)$, where $\varphi(z)$ satisfies the conditions of Schwarz's lemma. Therefore $p_1(z)$ can be written in the form

$$p_1(z) = \frac{1 + z^n\varphi(z)}{1 - z^n\varphi(z)}. \tag{2.3}$$

On the other hand, we consider the function $p(z) = pp_1(z)$, this function is analytic and satisfies the conditions $p(0) = p$, $\text{Re}p(z) > 0$ for every $z \in \mathbb{D}$. Using (2.3) we obtain (2.1). $\square$

Remark 2.1. *We also note that using the subordination principle, we have*

$$p(z) \in \mathcal{P}(p,n) \Leftrightarrow p(z) \prec p\frac{1+z^n}{1-z^n}$$

and the image of $|z| = r$ *under the transformation* $w(z) = \dfrac{1+z^n}{1-z^n}$ *is the disc with the center* $C(r) = \left(\frac{1+r^{2n}}{1-r^{2n}}, 0\right)$ *and radius* $\rho(r) = \dfrac{2r^n}{1-r^{2n}}$.

Corollary 2.1. *Let* $p(z)$ *be an element of* $\mathcal{P}(p,n)$*, then*

$$\operatorname{Re}\left(z\frac{p'(z)}{p(z)}\right) \geq -\frac{2nr^n}{1-r^{2n}} \tag{2.4}$$

for all $z \in \mathbb{D}$.

Proof. Using Lemma 2.1, then we have

$$p(z) = p\left(\frac{1+\psi(z)}{1-\psi(z)}\right),$$

where $\psi(z) = z^n\varphi(z)$ for all z in $\mathbb{D}$. Taking logarithmic derivative we obtain,

$$\frac{p'(z)}{p(z)} = \frac{2\psi'(z)}{1-(\psi'(z))^2} \Rightarrow$$

$$\left|\frac{p'(z)}{p(z)}\right| = \frac{2|\psi'(z)|}{|1-(\psi'(z))^2|} \leq \frac{2|\psi'(z)|}{1-|\psi(z)^2|}. \tag{2.5}$$

On the other hand, "Let $\psi(z) = d_0 + d_n z^n + d_{n+1}z^{n+1} + \cdots$ be analytic and bounded by 1 in $\mathbb{D}$, then

$$|\psi'(z)| \leq \frac{n|z|^{n-1}}{1-|z|^{2n}}(1-|\psi(z)|^2) \tag{2.6}$$

for $z \in \mathbb{D}$ with equality holding only when

$$\psi(z) = \varepsilon\frac{z^n+a}{1+\overline{a}z^n}, \quad |\varepsilon| = 1,\ |a| < 1.$$

This theorem was proved by G.M.Golusin ([4]). Using this theorem in (2.5) then we obtain

$$\left|z\frac{p'(z)}{p(z)}\right| \leq \frac{2nz^n}{1-|z|^{2n}} \quad \text{or} \quad \operatorname{Re}\left(z\frac{p'(z)}{p(z)}\right) \geq -\frac{2nr^n}{1-r^{2n}}.$$

□

Lemma 2.2. *If* $\phi(z) = z^p + c_{np+1}z^{np+1} + c_{np+2}z^{np+2} + \cdots$ *be an element of* $\mathcal{K}(p,n)$*, then*

$$\operatorname{Re}\left((1-z^n)^{\frac{2p}{n}}\frac{\phi'(z)}{z^{p-1}}\right) > 0 \tag{2.7}$$

for every z *in* $\mathbb{D}$.

Proof. We consider the function

$$s(z)=\int_0^z \frac{\xi^{p-1}}{(1-\xi^n)^{\frac{2p}{n}}}d\xi=\frac{z^p}{p}\ {}_2F_1\left(\frac{p}{n},\frac{2p}{n},\frac{n+p}{n};z^n\right),$$

where ${}_2F_1$ is a Gauss hypergeometric function [1] and z in $\mathbb{D}$. This function, $s(z)$ is an element of $\mathcal{C}(p,n)$. Indeed

$$s'(z)=z^{p-1}(1-z^n)^{\frac{-2p}{n}}\Rightarrow$$

$$\left(1+z\frac{s''(z)}{s'(z)}\right)=p\frac{1+z^n}{1-z^n}\quad (z\in\mathbb{D}). \tag{2.8}$$

Using the definition of class of $\mathcal{K}(p,n)$ we obtain

$$\operatorname{Re}\left(\frac{\phi'(z)}{s'(z)}\right)=\operatorname{Re}\left((1-z^n)^{\frac{2p}{n}}\frac{\phi'(z)}{z^{p-1}}\right)>0\quad (z\in\mathbb{D}).$$

□

Corollary 2.2. *If we give the special values to p and n we obtain following inequalities:*

(i) *for $n=1$,*

$$\operatorname{Re}\left((1-z)^{2p}\frac{\phi'(z)}{z^{p-1}}\right)>0\quad (z\in\mathbb{D}).$$

This result was obtained by T. Umezawa ([8]).

(ii) *for $n=1$, $p=1$,*

$$\operatorname{Re}[(1-z)^2\phi'(z)]>0\quad (z\in\mathbb{D}).$$

This result was obtained by W. Kaplan ([7]).

Theorem 2.1. *The radius of convexity of the family $\mathcal{K}(p,n)$ is less than the smallest positive root of the equation*

$$Q(r)=pr^{2n}-2(p+n)r^n+p=0 \tag{2.9}$$

Proof. Since $\phi(z)\in\mathcal{K}(p,n)$, then $\operatorname{Re}\left[(1-z)^{2p}\frac{\phi'(z)}{z^{p-1}}\right]>0$. So we have

$$(1-z^n)^{\frac{2p}{n}}\frac{\phi'(z)}{z^{p-1}}=p(z), \tag{2.10}$$

where $p(z)\in\mathcal{P}(p,n)$ for all z in $\mathbb{D}$. If we take the logarithmic derivative from (2.10) we get

$$1+z\frac{\phi''(z)}{\phi'(z)}=p\frac{1+z^n}{1-z^n}+z\frac{p'(z)}{p(z)},$$

thus we have

$$\operatorname{Re}\left[1+z\frac{\phi''(z)}{\phi'(z)}\right]\geq\min_{|z|=r}\left(\operatorname{Re}\left(p\frac{1+z^n}{1-z^n}\right)\right)+\min_{\substack{|z|=r\\ p(z)\in\mathcal{P}(n,p)}}\operatorname{Re}\left(z\frac{p'(z)}{p(z)}\right).$$

Using Corollary 2.1 we obtain

$$Re[1+z\frac{\phi''(z)}{\phi'(z)}\geq p\frac{1-r^n}{1+r^n}-\frac{2nr^n}{1-r^{2n}}=\frac{pr^{2n}-2(p+n)r^n+p}{1-r^{2n}}. \tag{2.11}$$

The denominator of the expression on the right-hand side in inequality (2.11) is positive for $0 \le r < 1$, $Q(0) = p$, $Q(1) = -2n < 0$. Thus the smallest positive root r_0 of the equation $Q(r) = pr^{2n} - 2(p+n)r^n + p$ lies between 0 and 1. Therefore the inequality $\mathrm{Re}\big[1 + z\frac{\phi''(z)}{\phi'(z)}\big] > 0$ is valid for $|z| = r < r_0$. Hence the radius of convexity for $\mathcal{K}(p,n)$ is not less than r_0. So we have

$$r_0 = \left(\frac{p}{(p+n) + \sqrt{n^2 + 2pn}}\right)^{\frac{1}{n}}.$$

$\square$

Remark 2.2. *By giving special values to the above radius we obtain the following equalities:*

(i) *for* $n = 1$*:* $\quad r_0 = \dfrac{p}{(p+1) + \sqrt{1+2p}}$,

(ii) *for* $p = 1$*:* $\quad r_0 = \left(\dfrac{1}{(1+n) + \sqrt{n^2+2n}}\right)^{\frac{1}{n}}$,

(iii) *for* $n = 1$, $p = 1$*:* $\quad r_0 = \dfrac{1}{2+\sqrt{3}}$.

Theorem 2.2. *Let* $\phi(z)$ *be an element of* $\mathcal{K}(p,n)$*, then*

$$\frac{pr^{p-1}(1-r^n)}{(1+r^n)^{\frac{2p}{n}+1}} \le |\phi'(z)| \le \frac{pr^{p-1}(1+r^n)}{(1-r^n)^{\frac{2p}{n}+1}} \quad (|z| = r < 1). \tag{2.12}$$

This inequality is sharp.

Proof. Using Lemma 2.1. and Lemma 2.2. we obtain

$$p(z) = p\frac{1+\psi(z)}{1-\psi(z)} \Leftrightarrow \psi(z) = \frac{p(z)-p}{p(z)+p} = \frac{(1-z^n)^{\frac{2p}{n}}\frac{\phi'(z)}{z^{p-1}} - p}{(1-z^n)^{\frac{2p}{n}}\frac{\phi'(z)}{z^{p-1}} + p}.$$

Since $\psi(z)$ satisfies the conditions of Schwarz lemma, then we have

$$\left|\frac{(1-z^n)^{\frac{2p}{n}}\frac{\phi'(z)}{z^{p-1}} - p}{(1-z^n)^{\frac{2p}{n}}\frac{\phi'(z)}{z^{p-1}} + p}\right| \le r \Leftrightarrow \left|(1-z^n)^{\frac{2p}{n}}\frac{\phi'(z)}{z^{p-1}} - p\right| \le r\left|(1-z^n)^{\frac{2p}{n}}\frac{\phi'(z)}{z^{p-1}} + p\right|$$

or

$$\left|(1-z^n)^{\frac{2p}{n}}\frac{\phi'(z)}{z^{p-1}} - \frac{p(1+r^{2n})}{1-r^{2n}}\right| \le \frac{2pr^n}{1-r^{2n}}. \tag{2.13}$$

After the simple calculations from (2.13) we get (2.12). This inequality is sharp because the extremal function can be obtained in the following manner:

$$(1-z^n)^{\frac{2p}{n}}\frac{\phi'(z)}{z^{p-1}} = p(z) \Rightarrow (1-z^n)^{\frac{2p}{n}}\frac{\phi'(z)}{z^{p-1}} = p\frac{1+z^n}{1-z^n}.$$

$\square$

3. An Application To The Harmonic Functions

Theorem 3.1. *Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}_H^0(p,n)$, then*

$$\frac{pr^{p-1}(1-r^n)}{(1+r^n)^{\frac{2p}{n}+2}} \leq |f_z| \leq \frac{pr^{p-1}(1+r^n)}{(1-r^n)^{\frac{2p}{n}+2}}, \tag{3.1}$$

and

$$\frac{|w(z)|pr^{p-1}(1-r^n)}{(1+r^n)^{\frac{2p}{n}+2}} \leq |f_{\overline{z}}| \leq \frac{|w(z)|pr^{p-1}(1+r^n)}{(1-r^n)^{\frac{2p}{n}+2}} \tag{3.2}$$

for $|z| = r < 1$. These distortions are sharp.

Proof. Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}_H^0(p,n)$, and let $\phi(z) \in \mathcal{K}(p,n)$, if we take

$$\phi(z) = h(z) - g(z),$$

then we have

$$h'(z) = \frac{\phi'(z)}{1-w(z)} \text{ and } g'(z) = w(z)\frac{\phi'(z)}{1-w(z)},$$

where $w(z)$ is the second dilatation of f and satisfies the conditions of Schwarz's lemma. Therefore we have

$$\frac{|\phi'(z)|}{1+|w(z)|} \leq |f_z| = |h'(z)| \leq \frac{|\phi'(z)|}{1-|w(z)|} \tag{3.3}$$

and

$$\frac{|w(z)||\phi'(z)|}{1+|w(z)|} \leq |f_{\overline{z}}| = |g'(z)| \leq \frac{|w(z)||\phi'(z)|}{1-|w(z)|}. \tag{3.4}$$

Using Theorem 2.2 in the inequalities (3.3) and (3.4) we get (3.1) and (3.2). We also note that these distortions are sharp because the extremal function can be found in the following manner:

$$(1-z^n)^{\frac{2p}{n}}\frac{\phi'(z)}{z^{p-1}} = p\frac{1+z^n}{1-z^n}, \quad (1-z^n)^{\frac{2p}{n}}\frac{\phi'(z)}{z^{p-1}} = p\frac{1-z^n}{1+z^n}$$

(i.e, $p(z) \in \mathcal{P}(p,n)$, then $1/p(z) \in \mathcal{P}(p,n)$), so if we take a sıitable second dilatation $w(z)$, then

$$h(z) = \int_o^z \frac{\phi'(\xi)}{1-w(\xi)}d\xi, \quad g(z) = \int_o^z \frac{\phi'(\xi)w(\xi)}{1-w(\xi)}d\xi.$$

Note that the solution of $h(z)$ and $g(z)$ must be found under the conditions $h(0) = g(0) = 0$. So we have

$$\begin{aligned} f = h(z) + \overline{g(z)} &= \int_o^z \frac{\phi'(\xi)}{1-w(\xi)}d\xi + \overline{\int_o^z \frac{\phi'(\xi)w(\xi)}{1-w(\xi)}d\xi} \\ &= \int_o^z \frac{\phi'(\xi)}{1-w(\xi)}d\xi + \overline{\int_o^z \frac{\phi'(\xi)}{1-w(\xi)}d\xi - \int_0^z \phi'(z)d\xi} \\ &= \operatorname{Re}\left(\int_o^z \frac{2\phi'(\xi)}{1-w(\xi)}d\xi\right) - \overline{\phi(z)}. \end{aligned}$$

□

Corollary 3.1. *Let* $f = h(z) + \overline{g(z)}$ *be an element of* $\mathcal{S}_H^0(p,n)$, *then*

$$|f| \leq \frac{n}{p}\Big[B\left(r^n, \frac{p}{n}, -1-\frac{2p}{n}\right) + 2B\left(r^n, \frac{n+p}{n}, -1-\frac{2p}{n}\right) + B\left(r^n+2+\frac{p}{n}, -1-\frac{2p}{n}\right)\Big], \tag{3.5}$$

where $B\left(r^n, \frac{p}{n}, -1-\frac{2p}{n}\right)$, $B\left(r^n, \frac{n+p}{n}, -1-\frac{2p}{n}\right)$, *and* $B\left(r^n+2+\frac{p}{n}, -1-\frac{2p}{n}\right)$ *are beta functions [1].*

Proof. Using the below formula which can be find in [3],

$$(|f_z| - |f_{\overline{z}}|)|dz| \leq |f| \leq (|f_z| + |f_{\overline{z}}|)dz$$

for all $z \in \mathbb{D}$ we obtain

$$|f| \leq \int_o^r \frac{p\varrho^{p-1}(1+\varrho^n)}{(1-\varrho^n)^{\frac{2p}{n}+2}}d\varrho + \int_o^r \frac{p\varrho^{p+n-1}(1+\varrho^n)}{(1-\varrho^n)^{\frac{2p}{n}+2}}d\varrho.$$

After the simple calculations we get (3.5). □

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