ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal 4 (2015), no.2, 121–125
ISSN 1857-8365 UDC: 517.521:517.987.1

SOME FURTHER RESULTS ON THE BORWEIN-DITOR THEOREM

HARRY I. MILLER AND LEILA MILLER-VAN WIEREN[1]

Abstract. In 1978 D. Borwein and S. Z. Ditor published a paper answering a question of P. Erdős. Since then several authors including N. H. Bingham, A. J. Ostaszewski, H. I. Miller and L. Miller-Van Wieren have further extended this result by examining different gauges of size for sets (including Lebesgue measure and Baire category) and translates by null sequences. In this paper, we offer some further insights into related results using translates of sets by an additive function.

1. Introduction

There are several gauges of size of sets $A \subseteq \mathbb{R}$. For example, if $C \subseteq [0,1]$ denotes the classical Cantor middle third set, then C has cardinality c (continuum), i.e. C is large , but $m(C)$, the Lebesgue measure of C, is zero, i.e. C is small. However, $D(C) = \{x - x' : x, x' \in C\} = [-1,1]$, i.e. C is large. But C is nowhere dense, i.e. C is small. For more details on classical results comparing the largeness and smallness of A, using different gauges, see [14], [4], [11], [5], [3], [6] and [7]. Also the classical textbook by J.C. Oxtoby [12] and its extensive list of references provide many results in this area. Along the lines of these comparisons, D. Borwein and S.Z. Ditor [2] proved the following result.

Theorem 1.1 (Borwein, Ditor 1978). *If A is a measurable set in $\mathbb{R}$ with $m(A) > 0$, and (d_n) is a sequence of reals converging to 0, then for almost all $x \in A$, $x + d_n \in A$ for infinitely many n. There exists a measurable set in $\mathbb{R}$ with $m(A) > 0$, and a (decreasing) sequence (d_n) converging to 0, such that, for each x, $x + d_n \notin A$ for infinitely many n.*

In [8] H. I. Miller extended the Borwein-Ditor theorem by using a general function $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ instead of addition. In [1] N. H. Bingham and A. J. Ostaszewski considered homotopy and its relation to the Borwein-Ditor theorem. In [10] H. I. Miller and A. J. Ostaszewski considered general spaces, group action and shift-compactness and their relations to the Borwein-Ditor theorem. In [9], we proved the following analog of the first part of this theorem about translates of sets that are large in the sense of Baire category.

[1] *corresponding author*

2010 *Mathematics Subject Classification.* 40D25, 40G99, 28A12.

Key words and phrases. Lebesgue measure, Baire category, translates of sets, concentrated, additive function.

Theorem 1.2. *Suppose $A \subseteq [a,b]$ is such that $[a,b]\backslash A$ is of first category and suppose (d_n) is a sequence converging to 0. Then, there exists $x \in A$ and $n_0 \in \mathbb{N}$ such that $x + d_n \in A$ for all $n \geq n_0$. Furthermore, there exists $n_0 \in \mathbb{N}$ and a set $X \subseteq A$ that is contained in some subinterval of $[a,b]$ and has a complement of first category in that interval, such that $x + d_n \in A$ for all $x \in X$, $n \geq n_0$.*

Remark 1.1. *It is clear that the restriction of boundedness on the set A can be removed from the statement of Theorem 1.2, i.e. that we can simply assume the set A has a complement of first category in $\mathbb{R}$.*

2. First result

We are now ready to prove a generalization of Theorem 1.2 for a more general additive function of two variables.

Theorem 2.1. *Suppose that $f : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies the following conditions on a rectangle $[a,b]\times[c,d]$:*

(1) *There exists $e \in (a,b)$ such that $f(x,e) = x$ for all $x \in [a,b]$*
(2) *Partial derivatives f_x, f_y are continuous on $[a,b]\times[c,d]$*
(3) *There exist $m, M > 0$ such that $m < f_x < M$ and $m < f_y < M$ on $[a,b]\times[c,d]$.*

Suppose $A \subseteq [a,b]$ such that $[a,b]\backslash A$ is of first category and suppose (d_n) is a sequence converging to 0. Then, there exists $x \in A$ and $n_0 \in N$ such that:

$$f(x, e + d_n) \in A$$

for all $n \geq n_0$. Furthermore, there exists $n_0 \in \mathbb{N}$ and a set $X \subseteq A$ that is contained in some subinterval of $[a,b]$ and has a complement of first category in that interval, such that

$$f(x, e + d_n) \in A$$

for all $x \in X$, $n \geq n_0$.

Proof. $A = [a,b] \setminus \bigcup_{k=1}^{\infty} D_k$ where $D_k \subseteq [a,b]$ are nowhere dense for $k \in \mathbb{N}$. Let

$$\begin{aligned} f_e(x) &= f(x,e) = x \\ f_{e+d_n}(x) &= f(x, e+d_n) \end{aligned}$$

for $x \in [a,b]$, $n \in \mathbb{N}$. Then f_{e+d_n} maps $[a,b]$ onto an interval $[a_n, b_n]$ continuously and is increasing. Hence

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} f_{e+d_n}(a) = f_e(a) = a \\ \lim_{n\to\infty} b_n &= \lim_{n\to\infty} f_{e+d_n}(b) = f_e(b) = b. \end{aligned}$$

Therefore there exists $n_1 \in \mathbb{N}$ and $[x_0, x_1] \subseteq [a,b]$ so that $[x_0, x_1] \subseteq [a_n, b_n]$ for $n \geq n_1$. Let $\alpha_n = f_{e+d_n}^{-1}(x_0)$, $\beta_n = f_{e+d_n}^{-1}(x_1)$ for $n \geq n_1$. It is easy to see that $\lim_{n\to\infty} \alpha_n = x_0$ and $\lim_{n\to\infty} \beta_n = x_1$. Hence there exists $\epsilon > 0$, $n_0 \in \mathbb{N}$, $n_0 \geq n_1$ such that

$$[x_0 + \epsilon, x_1 - \epsilon] \subseteq [\alpha_n, \beta_n]$$

for $n \geq n_0$. Now for $k \in \mathbb{N}$, D_k is nowhere dense in $[a, b]$ and consequently in $[x_0, x_1]$, so therefore $f^{-1}_{e+d_n}(D_k)$ is nowhere dense in $[\alpha_n, \beta_n]$ and consequently in $[x_0 + \epsilon, x_1 - \epsilon]$ for $n \geq n_0$. Therefore $f^{-1}_{e+d_n}(\bigcup_{k=1}^{\infty} D_k) = \bigcup_{k=1}^{\infty} f^{-1}_{e+d_n}(D_k)$ is of first category in $[x_0 + \epsilon, x_1 - \epsilon]$ for $n \geq n_0$. Since $A = [a, b] \setminus \bigcup_{k=1}^{\infty} D_k$ it follows that

$$f^{-1}_{e+d_n}(A) \bigcap [x_0 + \epsilon, x_1 - \epsilon]$$

has a complement of first category in $[x_0 + \epsilon, x_1 - \epsilon]$, for $n \geq n_0$. Therefore

$$\left(\bigcap_{n=n_0}^{\infty} f^{-1}_{e+d_n}(A)\right) \bigcap [x_0 + \epsilon, x_1 - \epsilon]$$

has a complement of first category in $[x_0 + \epsilon, x_1 - \epsilon]$ and we conclude that

$$A \bigcap \left(\bigcap_{n=n_0}^{\infty} f^{-1}_{e+d_n}(A)\right) \bigcap [x_0 + \epsilon, x_1 - \epsilon]$$

has a complement of first category in $[x_0 + \epsilon, x_1 - \epsilon]$. This completes the proof of the theorem. □

The following corollary is easy to obtain.

Corollary 2.1. *Suppose that $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies conditions (1), (2) and (3) from Theorem 2.1 on a rectangle $[a, b] \times [c, d]$. There exists a set A of real numbers with $m(A) = 0$ such that for every sequence (d_n) converging to 0, there exists $x \in A$ and $n_0 \in \mathbb{N}$ such that $f(x, e + d_n) \in A$ for $n \geq n_0$.*

Proof. It is well known that we can construct subsets of $[0, 1]$ of any positive measure L, $0 < L < 1$ that are nowhere dense. Therefore, we can obtain a sequence of nowhere dense sets X_n, $n \in \mathbb{N}$, such that $X_n \subseteq [0, 1]$ with $m(X_n) = 1 - \dfrac{1}{n}$. If we set $A = [0, 1] \setminus \bigcup_{n=1}^{\infty} X_n$, then clearly $m(A) = 0$ and the conclusion follows from Theorem 2.1. □

3. A result on sets concentrated on $\mathbb{Q}$

Next we examine a different type of small set and prove a result about translates by null sequences of such a set. Let $\mathbb{Q}$ denote the set of rational numbers. The following definition can be found in the classical textbook by C. A. Rogers [13].

Definition 3.1. *A set of real numbers A is said to be concentrated on $\mathbb{Q}$ if every open set containing $\mathbb{Q}$ contains all but at most countably many elements of A.*

Assuming the Continuum hypothesis, we prove the following analog of Corollary 2.1.

Theorem 3.1. *Suppose the Continuum hypothesis holds. Suppose that $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies conditions (1), (2) and (3) from Theorem 2.1 on a rectangle $[a, b] \times [c, d]$. There exists a set A concentrated on $\mathbb{Q}$ such that for every sequence (d_n) converging to 0, there exists $x \in A$ and $n_0 \in \mathbb{N}$ such that $f(x, e + d_n) \in A$ for $n \geq n_0$.*

Proof. The set of all open sets that contain $\mathbb{Q}$ has cardinality c (continuum) and hence these sets can be arranged into a transfinite sequence G_α, $\alpha < \omega$ where ω is the least ordinal of cardinality c. Likewise the set of all null sequences has cardinality c and can also be arranged into a transfinite sequence (d_n^α), $\alpha < \omega$. Since $G_1 \bigcap (a,b)$ is an open set and $f(x, e + d_n^1) \longrightarrow x$ in (a,b) , we can find $x_1 \in G_1 \bigcap (a,b)$ and n_1 such that $f(x_1, e + d_n^1) \in G_1 \bigcap (a,b)$ for $n \geq n_1$. Now suppose $\alpha < \omega$ is a fixed ordinal. Then because of the continuum hypothesis, α has countable cardinality and hence the set $(\bigcap_{\beta \leq \alpha} G_\beta) \bigcap (a,b)$ has a complement of first category in $[a,b]$ (as each G_β has a nowhere dense complement in $\mathbb{R}$). Therefore by Theorem 2.1, we can find $x_\alpha \in \bigcap_{\beta \leq \alpha} G_\beta \bigcap (a,b)$ and n_α such that $f(x_\alpha, e + d_n^\alpha) \in \bigcap_{\beta \leq \alpha} G_\beta \bigcap (a,b)$ for $n \geq n_\alpha$. Now let:

$$A = \{x_\alpha : \alpha < \omega\} \bigcup_{\alpha < \omega} \{f(x_\alpha, e + d_n^\alpha) : n \geq n_\alpha\}.$$

It is easy to see that the Continuum hypothesis implies that A is concentrated on $\mathbb{Q}$ and it satisfies the desired conclusion. □

The special case when $f(x,y) = x + y$ has been examined earlier through oral communication with A. J. Ostaszewski.

4. Non-measurable sets

Here we prove the existence of a non-measurable set that satisfies the conclusion of Theorem 2.1.

Theorem 4.1. *Suppose that $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies conditions (1), (2) and (3) from Theorem 2.1 on a rectangle $[a,b] \times [c,d]$. There exists a non-measurable set A in $[a,b]$ such that for every sequence (d_n) converging to 0, there exists $x \in A$ and $n_0 \in \mathbb{N}$ such that $f(x, e + d_n) \in A$ for $n \geq n_0$.*

Proof. Let F_α, $\alpha < \omega$ be the set of all closed subsets of $[a,b]$ of positive Lebesgue measure and (d_n^α), $\alpha < \omega$ denote the transfinite sequence of all null sequences. Fix an arbitrary x_1 in F_1 . Put x_1 and $f(x_1, e + d_n^1)$, $n \geq 1$ into A. Fix an arbitrary y_1 from $F_1 \setminus \{x_1, f(x_1, e + d_n^1) : n \geq 1\}$ and put y_1 into B. Suppose that for all $\beta < \alpha$ (some $\alpha < \omega$) we have already put $x_\beta \in F_\beta$, $f(x_\beta, e + d_n^\beta)$ for $n \geq 1$ into A and $y_\beta \in F_\beta$ into B, so that A and B are disjoint. The set F_α is closed and has positive measure, and hence has cardinality c. Therefore we can now fix an element $x_\alpha \in F_\alpha \setminus \bigcup_{\beta < \alpha} \{y_\beta, f_{e+d_n^\alpha}^{-1}(y_\beta) : n \in \mathbb{N}\}$ (using notation from the proof of Theorem 2.1). Put x_α and $f(x_\alpha, e + d_n^\alpha)$ for $n \geq 1$ into A. Now we can fix $y_\alpha \in F_\alpha \setminus \bigcup_{\beta \leq \alpha} \{x_\beta, f(x_\beta, e + d_n^\beta), n \geq 1\}$. Put y_α into B. It is clear that after step α, A and B remain disjoint. Therefore by transfinite induction we have constructed disjoint sets A and B. Since both A and B intersect each closed set of positive measure, the set A is non-measurable. Also from our construction, it is clear that the set A satisfies the conclusion of the theorem. □

Again the special case when $f(x,y) = x + y$ has been looked into before together with A. J. Ostazewski.

References

[1] N.H. Bingham, A.J. Ostazewski: *Homotopy of the Kestelman-Borwein-Ditor Theorem*, Canad.Math. Bulletin, **54**(1) (2011), 12–20.

[2] D. Borwein, S.Z. Ditor: *Translates of sequences in sets of positive measure*, Canad.Math.Bull., **21** (1978), 497–498.

[3] R.O. Davies: *Some remarks on the Kakeya problem*, Proc.Cambridge. Phil. Soc., **69** (1971), 417–421.

[4] J.R. Kinney: *A thin set of cicles*, Amer.Math.Monthly, **75** (1968), 1077–1081.

[5] J.R. Kinney: *A thin set of lines*, Israel. J. Math., **8** (1970), 97–102.

[6] H.I. Miller: *Generalization of a classical theorem of measure theory*, Radovi ANUBiH, **XLV** (1973), 45–48.

[7] H.I. Miller: *Relationships between various gauges of the size of sets of real numbers*, Glasnik Matematički, **9**(29) (1974), 59–64.

[8] H.I. Miller: *Generalization of a result of Borwein and Ditor*, Proc. Amer.Math.Soc., **105**(4) (1989), 889–893.

[9] H.I. Miller, L. Miller-Van Wieren: *Translates of sequences for some small sets*, Sarajevo Journal of Mathematics, **7**(20) (2011), 201–205.

[10] H.I. Miller, A.J. Ostazewski: *Group action, shift compactness and the KBD theorem*, Journal of Math.Anal. and Applications, (2012), 22–39.

[11] T. Neubrunn, T. Šalát: *Distance stes, ratio sets and certain transformations of sets of real numbers*, Časopis. Pest.Mat., **94** (1969), 381–393.

[12] J.C. Oxtoby: *Measure and Category*, Springer-Verlag, Vol. 2, 1971.

[13] C.A. Rogers: *Hausdorf Measures*, Cambridge University Press, 1970.

[14] H. Steinhaus: *Sur les distances des points des ensembles de measure positive*, Fund.Math., **1** (1920), 93–104.

Faculty of Engineering and Natural Sciences
International University of Sarajevo
Sarajevo, 71000
Bosnia and Herzegovina
E-mail address: himiller@hotmail.com

Faculty of Engineering and Natural Sciences
International University of Sarajevo
Sarajevo, 71000
Bosnia and Herzegovina
E-mail address: lmiller@ius.edu.ba