ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **4** (2015), no.2, 183–194
ISSN 1857-8365 UDC: 517.546.3:517.983

NOTES ON GENERALIZED INTEGRAL OPERATOR INCLUDES PRODUCT OF p-VALENT MEROMORPHIC FUNCTIONS

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Presented at the 11th International Symposium
GEOMETRIC FUNCTION THEORY AND APPLICATIONS
24-27 August 2015, Ohrid, Republic of Macedonia

ABSTRACT. We define new general integral operator and new function of product of p-valent meromorphic functions. By studying the function of product of p-valent meromorphic functions on defined subclasses of p-valent meromorphic functions , we deduce the properties of the general integral operator. Various known and new results are derived as special cases.

1. INTRODUCTION

Let Σ_p denote the class of meromorphic functions of the form

$$f(z)=\frac{1}{z^p}+\sum_{n=p+1}^{\infty}a_nz^n \quad (p\in\mathbb{N}=\{1,2,\cdots\}),$$

which are analytic and p -valent in the punctured unit disc:

$$\mathbb{U}^*=\{z\in\mathbb{C}:0<|z|<1\}=\mathbb{U}-\{0\}.$$

A function $f\in\Sigma_p$ is said to be in the class $\Sigma_p^{\star}(\delta)$, of meromorphic p -valent starlike of order δ $(0\leq\delta<p)$ if it satisfies the following inequality:

$$-\Re\left(\frac{zf'(z)}{f(z)}\right)>\delta.$$

A function $f\in\Sigma_p$ is the meromorphic p -valent convex function of order δ $(0\leq\delta<p)$, if f satisfies the following inequality:

$$-\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\delta,$$

and we denote this class by $\Sigma\mathcal{K}_p(\delta)$.

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2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Analytic function, p-valent meromorphic function, differential operator, integral operator.

Define a linear operator $\mathcal{D}_\lambda^k$, as following (cf., e.g., [1], [2]):

$$\mathcal{D}_\lambda f(z) = (1+p\lambda)f(z) + \lambda z f'(z), \quad \lambda \geq -p, f \in \Sigma_p,$$

$$\mathcal{D}_\lambda^0 f(z) = f(z)$$

$$\mathcal{D}_\lambda^1 f(z) = \mathcal{D}_\lambda f(z) \tag{1.1}$$

$$\mathcal{D}_\lambda^2 f(z) = \mathcal{D}_\lambda(\mathcal{D}_\lambda^1 f(z)),$$

and in general for $k = 0, 1, 2, ...$, we can write

$$\mathcal{D}_\lambda^k f(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} (1+p\lambda+n\lambda)^k a_n z^n, \quad (k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, p \in \mathbb{N}).$$

It is easy to see that for $f \in \Sigma_p$, we have

$$\lambda z \left(\mathcal{D}_\lambda^k f(z)\right)' = \mathcal{D}_\lambda^{k+1} f(z) - (1+p\lambda)\mathcal{D}_\lambda^k f(z), \quad (k \in \mathbb{N}_0, p \in \mathbb{N}). \tag{1.2}$$

Meromorphically multivalent functions have been extensively studied by several authors, see for example, Uralegaddi and Somanatha ([10] and [11]), Liu and Srivastava ([13] and [14]), Mogra ([15]and [16]), Srivastava et a1.[17], Aouf et al. ([19] and [20]), Joshi and Srivastava [21], Owa et a1. [22] and Kulkarni et al. [23].

Now, for $f \in \Sigma_p$, and by using the linear operator $\mathcal{D}_\lambda^k$, we define the following new subclasses.

Definition 1.1. *Let a function $f \in \Sigma_p$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma_p^\star(\delta, b, \lambda)$ if, and only if, f satisfies*

$$\Re\left\{p - \frac{1}{b}\left(\frac{z\left(\mathcal{D}_\lambda^{k+1} f(z)\right)'}{\mathcal{D}_\lambda^{k+1} f(z)} + p\right)\right\} > \delta,$$

where $\delta \in [0, p), b \in \mathbb{C}\backslash\{0\}, \lambda \geq -p, k \in \mathbb{N}_0$.

Putting $\lambda = k = 0$, in Definition 1.1 and by using (1.2), we have,

Definition 1.2. *Let a function $f \in \Sigma_p$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma_p^\star(\delta, b)$if, and only if, f satisfies*

$$\Re\left\{p - \frac{1}{b}\left(\frac{zf'(z)}{f(z)} + p\right)\right\} > \delta,$$

where $\delta \in [0, p), b \in \mathbb{C}\backslash\{0\}$.

Putting $\lambda = -\frac{1}{p}$, $k = 0$ in Definition 1.1, we have,

Definition 1.3. *Let a function $f \in \Sigma_p$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma\mathcal{K}_p(\delta, b)$ if, and only if, f satisfies*

$$\Re\left\{p - \frac{1}{b}\left(\frac{zf''(z)}{f'(z)} + p + 1\right)\right\} > \delta,$$

where $\delta \in [0, p), b \in \mathbb{C}\backslash\{0\}$.

We note that $f \in \Sigma\mathcal{K}_p(\delta, b)$ if, and only if, $-\frac{zf'(z)}{p} \in \Sigma_p^\star(\delta, b)$.

Putting $\lambda = 1$, $k = 0$ in Definition 1.1, we have,

Definition 1.4. *Let a function $f \in \Sigma_p$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma_p\mathcal{F}_1(\delta, b)$ if, and only if, f satisfies*

$$\Re\left\{p - \frac{1}{b}\left(\frac{z\left(zf''(z) + (p+2)f'(z)\right)}{zf'(z) + (p+1)f(z)} + p\right)\right\} > \delta,$$

where $\delta \in [0, p), b \in \mathbb{C}\backslash\{0\}$.

We note that $f \in \Sigma_p\mathcal{F}_1(\delta, b)$ if, and only if, $zf'(z) + (p+1)f(z) \in \Sigma_p^\star(\delta, b)$.

Remark 1.5. For $p = 1$ in Definitions 1.2, 1.3 and 1.4, respectively, we get the classes $\Sigma_b^\star(\delta)$, $\Sigma\mathcal{K}_b(\delta)$, and $\Sigma\mathcal{F}_1(\delta, b)$ [[6], Definitions 1.1, 1.2 and 1.7, respectively].

Definition 1.6. *Let a function $f \in \Sigma_p$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma_pUS_k(\alpha, \delta, b, \lambda)$ if, and only if, f satisfies*

$$\Re\left\{p - \frac{1}{b}\left(\frac{z\left(\mathcal{D}_\lambda^{k+1}f(z)\right)'}{\mathcal{D}_\lambda^{k+1}f(z)} + p\right)\right\} > \alpha\left|\frac{1}{b}\left(\frac{z\left(\mathcal{D}_\lambda^{k+1}f(z)\right)'}{\mathcal{D}_\lambda^{k+1}f(z)} + p\right)\right| + \delta, \tag{1.3}$$

where $\alpha \geq 0$, $\delta \in [-1, p)$, $\alpha + \delta \geq 0$, $b \in \mathbb{C}\backslash\{0\}$, $\lambda \geq -p$, $k \in \mathbb{N}_0$.

Putting $\lambda = k = 0$, in Definition 1.6, we have,

Definition 1.7. *Let a function $f \in \Sigma_p$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma_p^\star U(\alpha, \delta, b)$ if, and only if, f satisfies*

$$\Re\left\{p - \frac{1}{b}\left(\frac{zf'(z)}{f(z)} + p\right)\right\} > \alpha\left|\frac{1}{b}\left(\frac{zf'(z)}{f(z)} + p\right)\right| + \delta,$$

where $\alpha \geq 0, \delta \in [-1, p), \alpha + \delta \geq 0$, $b \in \mathbb{C}\backslash\{0\}$.

Putting $\lambda = -\frac{1}{p}$, $k = 0$ in Definition 1.6, we have,

Definition 1.8. *Let a function $f \in \Sigma_p$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma_p\mathcal{K}U(\alpha, \delta, b)$ if, and only if, f satisfies*

$$\Re\left\{p - \frac{1}{b}\left(\frac{zf''(z)}{f'(z)} + p + 1\right)\right\} > \alpha\left|\frac{1}{b}\left(\frac{zf''(z)}{f'(z)} + p + 1\right)\right| + \delta,$$

where $\alpha \geq 0, \delta \in [-1, p), \alpha + \delta \geq 0, b \in \mathbb{C}\backslash\{0\}$.

We note that $f \in \Sigma_p\mathcal{K}U(\alpha, \delta, b)$ if, and only if, $-\frac{zf'(z)}{p} \in \Sigma_p^\star U(\alpha, \delta, b)$.

Putting $\lambda = 1$, $k = 0$ in Definition 1.6, we have,

Definition 1.9. *Let a function $f \in \Sigma_p$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma_p\mathcal{K}\mathcal{F}_2(\alpha, \delta, b)$ if, and only if, f satisfies*

$$\Re\left\{p - \frac{1}{b}\left(\frac{z\left(zf''(z) + (p+2)f'(z)\right)}{zf'(z) + (p+1)f(z)} + p\right)\right\} > \alpha\left|\frac{1}{b}\left(\frac{z\left(zf''(z) + (p+2)f'(z)\right)}{zf'(z) + (p+1)f(z)} + p\right)\right| + \delta,$$

where $\alpha \geq 0, \delta \in [-1,p), \alpha+\delta \geq 0, b \in \mathbb{C}\backslash\{0\}$.

We note that $f \in \Sigma_p\mathcal{KF}_2(\alpha,\delta,b)$ if, and only if, $zf'(z)+(p+1)f(z) \in \Sigma_p^{\star}U(\alpha,\delta,b)$.

Remark 1.10. For $p=1$ in Definitions 1.7, 1.8 and 1.9, respectively, we get the classes $\Sigma^{\star}U(\alpha,\delta,b)$, $\Sigma\mathcal{K}U(\alpha,\delta,b)$ and $\Sigma\mathcal{F}_2(\alpha,\delta,b)$ [[6], Definitions 1.3, 1.4 and 1.8, respectively].

By using the differential operator given by (1.1), we introduce the following integral operator and another function of product p-valent meromorphic functions.

Definition 1.11. Let $\gamma_j > 0$, $1 \leq j \leq n$ and $\lambda > -p$. One defines the integral operator $\mathcal{J}_p : \Sigma_p^n \to \Sigma_p$,

$$\mathcal{J}_p(z) = \frac{1}{z^{p+1}} \int_0^z \prod_{j=1}^n \left(u^p \mathcal{D}_\lambda^{k+1} f_j(u)\right)^{\gamma_j} du, \tag{1.4}$$

where $f_1,\ldots,f_n \in \Sigma_p$ and $\mathcal{D}_\lambda^{k+1}$ is define as (1.2).

Remark 1.12. The integral operator $\mathcal{J}_p$ generalizes many operators which were introduced and studied recently.
(i) For $\lambda = k = 0$ and for $\lambda = -\frac{1}{p}$, $k=0$ respectively, we have the integral operators

$$F_{p,\gamma_1,\cdots\gamma_n}(z) = \frac{1}{z^{p+1}} \int_0^z \prod_{j=1}^n \left(u^p f_j(u)\right)^{\gamma_j} du \tag{1.5}$$

and

$$G_{p,\gamma_1,\cdots\gamma_n}(z) = \frac{1}{z^{p+1}} \int_0^z \prod_{j=1}^n \left(-\frac{u^{p+1} f_j'(u)}{p}\right)^{\gamma_j} du, \tag{1.6}$$

introduced and studied by Mohammed and Darus ([7]), (see also [8]).
(ii) For $p=1$, in (1.14) and (1.15) respectively, we have the integral operators

$$H_n(z) = \frac{1}{z^2} \int_0^z \prod_{j=1}^n \left(u f_j(u)\right)^{\gamma_j} du \tag{1.7}$$

and

$$H_{\gamma_1,\cdots,\gamma_n}(z) = \frac{1}{z^2} \int_0^z \prod_{j=1}^n \left(-u^2 f_j'(u)\right)^{\gamma_j} du, \tag{1.8}$$

introduced and studied by Mohammed and Darus ([3], [5], respectively), (see also [6]).

Definition 1.13. Let $\gamma_j > 0$, $1 \leq j \leq n$ and $\lambda > -p$. One defines the function of product p-valent meromorphic functions $\mathcal{L}_p : \Sigma_p^n \to \Sigma_p$,

$$\mathcal{L}_p(z) = \frac{1}{z^p} \prod_{j=1}^n \left(z^p \mathcal{D}_\lambda^{k+1} f_j(z)\right)^{\gamma_j}, \tag{1.9}$$

where $f_1,\ldots,f_n \in \Sigma_p$ and $\mathcal{D}_\lambda^{k+1}$ is define as (1.2).

Remark 1.14. The function $\mathcal{L}_p(z)$ generalizes many known and new functions.
(i)For $\lambda = k = 0$ and for $\lambda = \frac{-1}{p}$, $k = 0$ respectively, we obtain the functions,

$$\phi(z) = \frac{1}{z^p} \prod_{j=1}^{n} (z^p f_j(z))^{\gamma_j} \tag{1.10}$$

and

$$\Upsilon(z) = \frac{1}{z^p} \prod_{j=1}^{n} \left(-\frac{z^{p+1} f_j'(z)}{p} \right)^{\gamma_j}, \tag{1.11}$$

introduced and studied by Mohammed and Darus [7].

(ii) For $\gamma_1 = \cdots = \gamma_n = 1$, in (1.10) and (1.11), we get the following two functions,

$$F_p(z) = \frac{1}{z^p} \prod_{j=1}^{n} (z^p f_j(z)) \tag{1.12}$$

and

$$G_p(z) = \frac{1}{z^p} \prod_{j=1}^{n} \left(-\frac{z^{p+1} f_j'(z)}{p} \right), \tag{1.13}$$

studied by Srivastava et al. [18].

For $p = 1$ in (1.12) and (1.13), respectively, we get the functions,

$$F(z) = \frac{1}{z} \prod_{j=1}^{n} (z f_j(z))^{\gamma_j}, \tag{1.14}$$

and

$$G(z) = \frac{1}{z} \prod_{j=1}^{n} \left(-z^2 f_j'(z) \right)^{\gamma_j}, \tag{1.15}$$

It is clear that from Definitions 1.11 and 1.13, the relation between $\mathcal{J}_p$ and $\mathcal{L}_p$ is given by

$$\mathcal{L}_p(z) = (p+1)\mathcal{J}_p + z\mathcal{J}_p'(z),$$

where $\mathcal{J}_p$ and $\mathcal{L}_p$ define in (1.4) and (1.9), respectively.

In addition, from Remarks 1.12 and 1.14, we deduce the following identities,

$$\begin{aligned}
\phi(z) &= (p+1)F_{p,\gamma_1,\cdots\gamma_n}(z) + zF'_{p,\gamma_1,\cdots\gamma_n}(z),\\
\Upsilon(z) &= (p+1)G_{p,\gamma_1,\cdots\gamma_n}(z) + zG'_{p,\gamma_1,\cdots\gamma_n}(z),\\
F(z) &= (p+1)H_n + z\mathcal{H}_n'(z),\\
G(z) &= (p+1)H_{\gamma_1,\cdots,\gamma_n}(z) + zH'_{\gamma_1,\cdots,\gamma_n}(z).
\end{aligned}$$

The last identities play important rules in our investigation.

In this paper we study some sufficient conditions for the function $\mathcal{L}_p(z)$ which define in (1.9) and by using the above identities we state some properties of the integral operator defined in (1.4). In addition some corollaries as special cases for the integral operators and functions mentioned in Remarks 1.12 and 1.14 are also presented.

2. Main results

Our first theorem is the following:

Theorem 2.1. *For $j \in \{1, \dots, n\}$, let $\gamma_j > 0$ and $f_j \in \Sigma_p^{\star}(\delta_j, b, \lambda)$ $(0 \le \delta_j < 1)$. If*

$$0 < \sum_{j=1}^{n} \gamma_i \left(p - \delta_i\right) \le p,$$

then the function $\mathcal{L}_p(z)$ define by (1.9) is in the class $\Sigma_p^{\star}(\mu, b)$, $\mu = p - \sum\limits_{j=1}^{n} \gamma_j \left(p - \delta_j\right)$.

Proof. By differentiating (1.9) logarithmically, with respect to z we get,

$$\frac{\mathcal{L}_p'(z)}{\mathcal{L}_p(z)} + \frac{p}{z} = \sum_{j=1}^{n} \gamma_j \left(\frac{(\mathcal{D}_\lambda^{k+1} f_j(z))'}{\mathcal{D}_\lambda^{k+1} f_j(z)} + \frac{p}{z} \right). \tag{2.1}$$

By multiplying (2.1) with z yield

$$\frac{z\mathcal{L}_p'(z)}{\mathcal{L}_p(z)} + p = \sum_{j=1}^{n} \gamma_j \left(\frac{z(\mathcal{D}_\lambda^{k+1} f_j(z))'}{\mathcal{D}_\lambda^{k+1} f_j(z)} + p \right).$$

This is equivalent to

$$p - \frac{1}{b}\left(\frac{z\mathcal{L}_p'(z)}{\mathcal{L}_p(z)} + p \right) = \sum_{j=1}^{n} \gamma_j \left\{ p - \frac{1}{b}\left(\frac{z(\mathcal{D}_\lambda^{k+1} f_j(z))'}{\mathcal{D}_\lambda^{k+1} f_j(z)} + p \right) \right\} + p\left(1 - \sum_{j=1}^{n} \gamma_j\right). \tag{2.2}$$

Taking real parts of both sides of (2.2), we obtain

$$\Re\left\{ p - \frac{1}{b}\left(\frac{z\mathcal{L}_p'(z)}{\mathcal{L}_p(z)} + p \right) \right\} = \sum_{j=1}^{n} \gamma_j \Re \left\{ p - \frac{1}{b}\left(\frac{z(\mathcal{D}_\lambda^{k+1} f_j(z))'}{\mathcal{D}_\lambda^{k+1} f_j(z)} + p \right) \right\} + p\left(1 - \sum_{j=1}^{n} \gamma_j\right). \tag{2.3}$$

Since $f_j \in \Sigma_p^{\star}(\delta_j, b, \lambda)$, for $j \in \{1, \dots, n\}$, we receive

$$\Re\left\{ p - \frac{1}{b}\left(\frac{z\mathcal{L}_p'(z)}{\mathcal{L}_p(z)} + p \right) \right\} > p - \sum_{j=1}^{n} \gamma_j \left(p - \delta_j\right).$$

That is $\mathcal{L}_p(z)$ define by (1.9) is in the class $\Sigma_p^{\star}(\mu, b)$, $\mu = p - \sum\limits_{j=1}^{n} \gamma_j \left(p - \delta_j\right)$. $\square$

Corollary 2.2. *For $j \in \{1, \dots, n\}$, let $\gamma_j > 0$ and $f_j \in \Sigma_b^{\star}(\delta_j, b, \lambda)$ $(0 \le \delta_j < 1)$. If*

$$0 < \sum_{j=1}^{n} \gamma_i \left(p - \delta_i\right) \le p,$$

then the integral operator $\mathcal{J}_p(z)$ define by (1.4) is in the class $\Sigma_p \mathcal{F}_1(\mu, b)$, $\mu = p - \sum\limits_{j=1}^{n} \gamma_j \left(p - \delta_j\right)$.

Proof. From the fact that

$$f \in \Sigma_p \mathcal{F}_1 (\delta_j, b) \Longleftrightarrow z f'(z) + (p+1) f(z) \in \Sigma_p^\star (\delta_j, b),$$

and since,

$$\mathcal{L}_p(z) = z\mathcal{J}_p'(z) + (p+1)\mathcal{J}_p(z),$$

by replacing $\mathcal{L}_p(z)$ by $z\mathcal{J}_p'(z)+(p+1)\mathcal{J}_p(z)$ in Theorem 2.1, we get the desired result. □

Now, we prove a sufficient condition for the function $\mathcal{L}_p(z)$ defined by (1.9) to belong to the class $\Sigma_p^\star U(\alpha, \delta, b)$.

Theorem 2.3. Let $\alpha \geq 0$, $\delta \in [-1, p)$, $\alpha + \delta \geq 0$ and $b \in \mathbb{C}\backslash\{0\}$, $\lambda \geq -p$. Suppose that

$$\sum_{j=1}^{n} \gamma_j \leq 1, \quad 1 \leq j \leq n.$$

If $f_j \in \Sigma_p U S_k (\alpha, \delta_j, b, \lambda)$ $(1 \leq j \leq n))$, then the function $\mathcal{L}_p(z)$ define by (1.9) is in the class $\Sigma_p^\star U(\alpha, \delta, b)$.

Proof. Since $f_j \in \Sigma_p U S_k (\alpha, \delta, b, \lambda)$ $(1 \leq j \leq n))$, by (1.3) we have

$$\Re\left\{p - \frac{1}{b}\left(\frac{z\left(\mathcal{D}_\lambda^{k+1} f_j(z)\right)'}{\mathcal{D}_\lambda^{k+1} f_j(z)} + p\right)\right\} > \alpha\left|\frac{1}{b}\left(\frac{z\left(\mathcal{D}_\lambda^{K+1} f_j(z)\right)'}{\mathcal{D}_\lambda^{K+1} f_j(z)} + p\right)\right| + \delta.$$

Considering Definition 1.7 and with the help of (2.3) we obtain

$$\begin{aligned}
&\Re\left\{p - \frac{1}{b}\left(\frac{z\mathcal{L}_p'(z)}{\mathcal{L}_p(z)} + p\right)\right\} - \alpha\left|\frac{1}{b}\left(\frac{z\mathcal{L}_p'(z)}{\mathcal{L}_p(z)} + p\right)\right| - \delta \\
&= p - p\sum_{j=1}^{n}\gamma_j + \sum_{j=1}^{n}\gamma_j \Re\left\{p - \frac{1}{b}\left(\frac{z\left(\mathcal{D}_\lambda^{k+1} f_j(z)\right)'}{\mathcal{D}_\lambda^{k+1} f_j(z)} + p\right)\right\} \\
&- \alpha\left|\sum_{j=1}^{n}\gamma_j \frac{1}{b}\left(\frac{z\left(\mathcal{D}_\lambda^{k+1} f_j(z)\right)'}{\mathcal{D}_\lambda^{k+1} f_j(z)} + p\right)\right| - \delta \\
&\geq p - p\sum_{j=1}^{n}\gamma_j + \sum_{j=1}^{n}\gamma_j \Re\left\{p - \frac{1}{b}\left(\frac{z\left(\mathcal{D}_\lambda^{k+1} f_j(z)\right)'}{\mathcal{D}_\lambda^{k+1} f_j(z)} + p\right)\right\} \\
&- \alpha\sum_{j=1}^{n}\gamma_j\left|\frac{1}{b}\left(\frac{z\left(\mathcal{D}_\lambda^{k+1} f_j(z)\right)'}{\mathcal{D}_\lambda^{k+1} f_j(z)} + p\right)\right| - \delta
\end{aligned}$$

$$> p - p\sum_{j=1}^{n}\gamma_j + \sum_{j=1}^{n}\gamma_j \left\{ \alpha \left| \frac{1}{b}\left(\frac{z\left(\mathcal{D}_{\lambda}^{k+1} f_j(z)\right)'}{\mathcal{D}_{\lambda}^{k+1} f_j(z)} + p \right) \right| + \delta \right\}$$

$$- \alpha \sum_{j=1}^{n}\gamma_j \left| \frac{1}{b}\left(\frac{z\left(\mathcal{D}_{\lambda}^{k+1} f_j(z)\right)'}{\mathcal{D}_{\lambda}^{k+1} f_j(z)} + p \right) \right| - \delta$$

$$= (p-\delta)\left(1 - \sum_{j=1}^{n}\gamma_j\right) \geq 0.$$

This completes the proof. □

Next, by using the relation

$$f \in \Sigma_p\mathcal{KF}_2\left(\alpha,\delta,b\right) \Longleftrightarrow zf'(z) + (p+1)f(z) \in \Sigma_p^{\star}U\left(\alpha,\delta,b\right),$$

and by setting

$$\mathcal{L}_p(z) = z\mathcal{J}_p'(z) + (p+1)\mathcal{J}_p(z),$$

in Theorem 2.3, we have the following corollary.

Corollary 2.4. Let $\alpha \geq 0$, $\delta \in [-1,p), \alpha+\delta \geq 0$ and $b \in \mathbb{C}\backslash\{0\}, \lambda \geq -p$. Suppose that

$$\sum_{j=1}^{n}\gamma_j \leq 1, \quad 1 \leq j \leq n.$$

If $f_j \in \Sigma_p US_k\left(\alpha,\delta,b,\lambda\right)$ $(1 \leq j \leq n))$, then the integral operator $\mathcal{J}_p(z)$ define by (1.4) is in the class $\Sigma_p\mathcal{KF}_2\left(\alpha,\delta,b\right)$.

By adopting the same method and technique used in the proof of Theorems 2.1 and 2.3 and Corollaries 2.2 and 2.4, we receive the following corollaries.

Corollary 2.5. *For $j \in \{1,\ldots,n\}$, let $\gamma_j > 0$ and $\left(0 \leq \delta_j < 1\right)$. If*

$$0 < \sum_{j=1}^{n}\gamma_j\left(p-\delta_j\right) \leq p.$$

Also let

$$\mu = p - \sum_{j=1}^{n}\gamma_j\left(p-\delta_j\right).$$

Then each of the following assertion holds true:

(i) If $f_j \in \Sigma_p^{\star}(\delta_j,b)$, then the function $\phi(z)$ defined by (1.10) is in the class $\Sigma_p^{\star}(\mu,b)$.

(ii) If $f_j \in \Sigma\mathcal{K}_p\left(\delta_j,b\right)$, then the the function $\Upsilon(z)$ defined by (1.11) is in the class $\Sigma_p^{\star}(\mu,b)$.

(iii) If $f_j \in \Sigma_p^{\star}(\delta_j,b)$, then the integral operator $F_{p,\gamma_1,\cdots\gamma_n}(z)$ defined by (1.5) is in the class $\Sigma_p\mathcal{F}_1\left(\mu,b\right)$.

(iv) If $f_j \in \Sigma\mathcal{K}_p(\delta_j, b)$, then the integral operator $G_{p,\gamma_1,\cdots\gamma_n}(z)$ defined by (1.6) is in the class $\Sigma_p\mathcal{F}_1(\mu, b)$.

Corollary 2.6. *Let $\alpha \geq 0$, $\delta \in [-1, p), \alpha + \delta \geq 0$ and $b \in \mathbb{C}\backslash\{0\}$. Suppose that*

$$\sum_{j=1}^{n} \gamma_j \leq 1, \quad 1 \leq j \leq n.$$

Then each of the following assertion holds true:

(i) If $f_j \in \Sigma_p^{\star}U(\alpha, \delta, b)$, then the function $\phi(z)$ defined by (1.10) is in the class $\Sigma_p^{\star}U(\alpha, \delta, b)$.

(ii) If $f_j \in \Sigma_p\mathcal{K}U(\alpha, \delta, b)$, then the the function $\Upsilon(z)$ defined by (1.11) is in the class $\Sigma_p^{\star}U(\alpha, \delta, b)$.

(iii) If $f_j \in \Sigma_p^{\star}U(\alpha, \delta, b)$, then the integral operator $F_{p,\gamma_1,\cdots\gamma_n}(z)$ defined by (1.5) is in the class $\Sigma_p\mathcal{K}\mathcal{F}_2(\alpha, \delta, b)$.

(iv) If $f_j \in \Sigma_p\mathcal{K}U(\alpha, \delta, b)$, then the integral operator $G_{p,\gamma_1,\cdots\gamma_n}(z)$ defined by (1.6) is in the class $\Sigma_p\mathcal{K}\mathcal{F}_2(\alpha, \delta, b)$.

Corollary 2.7. *For $j \in \{1, \ldots, n\}$, let $\gamma_j > 0$ and $(0 \leq \delta_j < 1)$. If*

$$0 < \sum_{j=1}^{n} \gamma_i (1 - \delta_i) \leq 1.$$

Also let

$$\mu = 1 - \sum_{j=1}^{n} \gamma_j (1 - \delta_j).$$

Then each of the following assertion holds true:

(i) If $f_j \in \Sigma_b^{\star}(\delta_j)$, then the function $F(z)$ defined by (1.14) is in the class $\Sigma_b^{\star}(\mu)$.

(ii) If $f_j \in \Sigma\mathcal{K}_b(\delta_j)$, then the the function $G(z)$ defined by (1.15) is in the class $\Sigma_b^{\star}(\mu)$.

(iii) If $f_j \in \Sigma_b^{\star}(\delta_j)$, then the integral operator $H_n(z)$ defined by (1.7) is in the class $\Sigma\mathcal{F}_1(\mu, b)$.

(iv) If $f_j \in \Sigma\mathcal{K}_b(\delta_j)$, then the integral operator $H_{\gamma_1,\cdots\gamma_n}(z)$ defined by (1.8) is in the class $\Sigma\mathcal{F}_1(\mu, b)$.

Corollary 2.8. *Let $\alpha \geq 0$, $\delta \in [-1, p), \alpha + \delta \geq 0$ and $b \in \mathbb{C}\backslash\{0\}$. Suppose that*

$$\sum_{j=1}^{n} \gamma_j \leq 1, \quad 1 \leq j \leq n.$$

Then each of the following assertion holds true:

(i) If $f_j \in \Sigma^{\star}U(\alpha, \delta, b)$, then the function $F(z)$ defined by (1.14) is in the class $\Sigma^{\star}U(\alpha, \delta, b)$.

(ii) If $f_j \in \Sigma\mathcal{K}U(\alpha, \delta, b)$, then the the function $G(z)$ defined by (1.15) is in the class $\Sigma^{\star}U(\alpha, \delta, b)$.

(iii) If $f_j \in \Sigma^{\star}U(\alpha,\delta,b)$, then the integral operator $H_n(z)$ defined by (1.7) is in the class $\Sigma\mathcal{F}_2(\alpha,\delta,b)$.

(iv) If $f_j \in \Sigma\mathcal{K}U(\alpha,\delta,b)$, then the integral operator $H_{\gamma_1,\cdots\gamma_n}(z)$ defined by (1.8) is in the class $\Sigma\mathcal{F}_2(\alpha,\delta,b)$.

Corollary 2.9. *Let $0 \le \delta_j < 1, \ \ 1 \le j \le n$. If*

$$0 < np - \sum_{j=1}^{n} \delta_j \le 1.$$

Also let

$$\mu = p(1-n) + \sum_{j=1}^{n} \delta_j.$$

Then the following assertion holds true:

(i) If $f_j \in \Sigma_b^{\star}(\delta_j)$, then the function $F_p(z)$ defined by (1.12) is in the class $\Sigma_b^{\star}(\mu)$.

(ii) If $f_j \in \Sigma\mathcal{K}_b(\delta_j)$, then the the function $G_p(z)$ defined by (1.13) is in the class $\Sigma_b^{\star}(\mu)$.

Corollary 2.10. *Let $\alpha \ge 0, \ \delta \in [-1,p), \alpha+\delta \ge 0$ and $b \in \mathbb{C}\backslash\{0\}$. Suppose that*

$$1 - n \ge 0.$$

Then the following assertion holds true:

(i) If $f_j \in \Sigma_p^{\star}U(\alpha,\delta,b)$, then the function $F_p(z)$ defined by (1.12) is in the class $\Sigma_p^{\star}U(\alpha,\delta,b)$.

(ii) If $f_j \in \Sigma_p\mathcal{K}U(\alpha,\delta,b)$, then the the function $G_p(z)$ defined by (1.13) is in the class $\Sigma_p^{\star}U(\alpha,\delta,b)$.

Remark 2.11. (i) Some of the above result presented in [5], [6] and [7], by using different techniques.

(ii) We can obtain many other consequences for relatively more familiar subclasses of meromorphically $p-$ valent functions from the above result.

Other work that we can look at regarding differential and integral operators see ([4], [9], [12], [24], [25], [26], [27]).

Acknowledgement: The work here is fully supported by MOHE grant: FRGSTOP-DOWN/2013/STG06/UKM/01/1.

References

[1] S. P. Goyal and J. K. Prajapat: *A new class of meromorphic multivalent functions involving certain linear operator*, Tamsui Oxford Journal of Mathematical Sciences, **25**(2)(2009), 167–176.

[2] A. Saif and A. Kiliçman: *On Certain Subclasses of Meromorphically $p-$ Valent Functions Associated by the Linear Operator $\mathcal{D}_{\lambda}^{n}$*, Journal of Inequalities and Applications, *2011*(2011), Article ID 401913, pp.16.

[3] A. Mohammed and M. Darus: *A new integral operator for meromorphic functions* Acta Universitatis Apulensis, **24** (2010,) 231–238.

[4] A. Mohammed and M. Darus: *New properties for certain integral operators*, Int. Journal of Math. Analysis, **4**(42) (2010), 2101–2109.

[5] A. Mohammed and M. Darus: *Starlikeness properties for a new integral operator for meromorphic functions*, Journal of Applied Mathematics, **2011**(2011),Article ID 804150, 8 pages.

[6] A. Mohammed and M. Darus: *Integral operators on new families of meromorphic functions of complex order,* Journal of Inequalities and Applications, **2011**(2011), 12 pages.

[7] A. Mohammed and M. Darus: *The order of starlikeness of new p-valent meromorphic functions*, Int. Journal of Math. Analysis, **6**(27) (2012), pp. 1329–1340.

[8] A. Mohammed and M. Darus: *Some Properties of Certain Integral Operators on new subclasses of analytic functions with complex order,* Journal of Applied Mathematics, **2012**(2012), Article ID 161436, 9 pages.

[9] B.A. Frasin:*On an integral operator of meromorphic functions*, Matematicki Vesnik, **64**(2)(2012), 167–172.

[10] B.A.Uralegaddi and C.Somanatha: *New criteria for meromorphic starlike univalent functions*, Bull. Austral.Math. Soc., **43**(1991), 137–140.

[11] B.A.Uralegaddi and C.Somanatha: *Certain classes of meromorphic multivalent functions,* Tamkang J. Math., **23**(1992), 223–231.

[12] N. Breaz, D. Braez and M. Darus: *Convexity properties for some general integral operators on uniformly analytic functions classes*, Computers and Mathematics with Applications, **60** (2010), 3105–3107.

[13] J.-L. Liu and M. H. Srivastava: *A linear operator and associated families of meromorphicaly multivalent functions,* J.Math. Anal APPl., **259**(2001), 566–581.

[14] J.-L. Liu and M. H. Srivastava: *Some convolution conditions for starlikeness and convexity of meromorphically multivalent functions,* Appl.Math.Letter, **16**(2003),13–16.

[15] M. L. Mogra: *Meromorphic multivalent functions with positive coefficients I*, Math. Japonica, **35**(1990), 1–11.

[16] M. L. Mogra: *Meromorphic multivalent functions with positive coefficients II*, Math. Japonica, **35**(1990), 1089–1098.

[17] H. M. Srivastava, H.M.Hossen and M.K.Aouf: *A unified presentation of some classes of meromorphically multivalent functions,* Comput.Math Appl., **38** (11-12)(1999), 63–70.

[18] H. M. Srivastava, A. Y. Lashin and B. A. Frasin: *Starlikeness and Convexity of Certain Classes of Meromorphically Multivalent Functions,* Theory and Applications of Mathematics and Computer Science, **3**(2)(2013), 93–102.

[19] M.K.Aouf and H. M. Hossen: *New criteria for meromorphic $p-$ valent starlike functions,* Tsukuba J.Math., **17**(1993), 481–486.

[20] M. K. Aouf and M. H. Srivastava: *A new criterion for meromorphically $p-$ valent convex functions of order alpha,* Math.Sci.Res.Hot-Line, **1**(8) (1997), 7–12.

[21] S. B. Joshi and M. H. Srivastava: *A certain family of meromorphically multivalent functions,* Comput.Math. Appl., **38**(3-4)(1999), 201–211.

[22] S.Owa, H. E. Darwish and M.A.Aouf: *Meromorphically multivalent functions with positive and fixed second coefficients,* Math.Japon., **46**(1997), 231–236.

[23] S. R. Kulkarni, U. R. Naik and M. H. Srivastava: *A certain class of meromorphically $p-$ valent quasi-convex functions,* Pan.Amer.Math. J., **8**(1)(1998), 57–64.

[24] S. Bulut: *Some properties for an integral operator defined by Al-Oboudi differential operator,* Journal of Inequalities in Pure and Applied Mathematics, **9**(4) (2008), Article 115, 5 pages.

[25] S. Bulut, B. A. Frasin: *Starlikeness of a new general integral operator for meromorphic multivalent functions,* Journal of the Egyptian Mathematical Society, (2013), DOI: 10.1016/j.joems.2013.11.009.

[26] S. Bulut: *A new general integral operator defined by Al-Oboudi differential operator,* Journal of Inequalities and Applications, **2009**(2009), Article ID 158408, 13 pages.

[27] P. Goswami and S. Bulut: *Starlikeness of general integral operator for meromorphic multivalent functions,* Journal of Complex Analysis, **2013**(2013), Article ID 690584, 4 pages.

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