ADV MATH SCI JOURNAL

Advances in Mathematics: Scientific Journal **5** (2016), no.2, 91–96
ISSN 1857-8365 printed version
ISSN 1857-8438 electronic version,

THIRD HANKEL DETERMINANT FOR BAZILEVIČ FUNCTIONS

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ABSTRACT. In this paper we study the class B-the so-called class of Bazilevič functions. It is known that B is a subclass of S, the class of univalent functions in $U = \{z : |z| < 1\}$. The objective of this paper is to obtain an upper bound to the third Hankel for Bazilevič functions.

1. INTRODUCTION

Let A denote the class of analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

in the open unit disk

$$U = \{z : |z| < 1\}.$$

We denote by S be the class of all functions $f \in A$ which are univalent in U.

Denote by S^* the subclass of S of starlike functions, so that $f \in S^*$ if, and only if, for $z \in U$

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0.$$

The Fekete-Szegö functional $\left|a_3 - \mu a_2^2\right|$ for normalized univalent functions

$$f(z) = z + a_2 z^2 + \cdots$$

is well known for its rich history in the theory of geometric functions. Its origin was in the disproof by Fekete and Szegö of the 1933 conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity (see [4]). The functional has since received great attention, particularly in many subclasses of the family of univalent functions. For that reason Fekete-Szegö functional was studied by many authors and some estimates were found in a many subclasses of normalized univalent functions (see [2], [6], [7], [9]).

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2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Analytic functions, univalent functions, Bazilevič functions, Hankel determinant.

In 1976, Noonan and Thomas [10] defined the q^{th} Hankel determinant of f for $n \geq 0$ and $q \geq 1$ is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \qquad (a_1 = 1).$$

This determinant has also been considered by several authors. For example, Noor [11] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions f given by (1.1) with bounded boundary. In particular, sharp upper bounds on $H_2(2)$ were obtained by the authors of articles ([11], [12]) for different classes of functions.

Note that

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2$$

and

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

The Hankel determinant $H_2(1) = a_3 - a_2^2$ is well-known as Fekete-Szegö functional.

For our discussion in this paper, we consider the Hankel determinant in the case $q = 3$ and $n = 1$, denoted by $H_3(1)$, given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

For $f \in A$, $a_1 = 1$, so that, we have

$$H_3(1) = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2)$$

and by applying triangle inequality, we get

$$|H_3(1)| \leq |a_3| \left| a_2 a_4 - a_3^2 \right| + |a_4| \left| a_4 - a_2 a_3 \right| + |a_5| \left| a_3 - a_2^2 \right|.$$

Definition 1.1. *(see [14]) For $0 \leq \beta < 1$ and $f \in A$, let $B(\beta)$ denote the class of Bazilevič functions if and only if*

$$\operatorname{Re}\left(\left(\frac{z}{f(z)} \right)^{1-\beta} f'(z) \right) > 0, \qquad z \in U.$$

2. Preliminary Results

Let P denote the class of functions consisting of p, such that

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

which are regular in the open unit disc U and satisfy $\operatorname{Re} p(z) > 0$ for any $z \in U$. Here, $p(z)$ is called Caratheodory function [3].

Lemma 2.1. [13] *If $p \in P$, then*

$$|p_n| \leq 2 \qquad (n \in \mathbb{N} = \{1, 2, \ldots\})$$

and

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2}.$$

Lemma 2.2. [5] *If the function $p \in P$, then*

$$\begin{aligned} 2p_2 &= p_1^2 + x(4 - p_1^2) \\ 4p_3 &= p_1^3 + 2(4 - p_1^2)p_1 x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z \end{aligned}$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

3. Main results

Theorem 3.1. *Let f given by (1.1) be in the class $B(\beta)$,and $0 \leq \beta < 1$. Then*

$$|a_2| \leq \frac{2}{\beta + 1},$$

$$|a_3| \leq \frac{2}{\beta + 2} + \frac{2(1 - \beta)}{(\beta + 1)^2},$$

and

$$|a_4| \leq \tfrac{2}{\beta+3} + \tfrac{4(1-\beta)}{(\beta+1)(\beta+2)} + \tfrac{4(1-\beta)|2\beta-1|}{3(\beta+1)^3}.$$

Proof. Let $f \in B(\beta)$. Then there exists a $p \in P$ such that

$$\left(\frac{z}{f(z)}\right)^{1-\beta} f'(z) = \left(\frac{f(z)}{z}\right)^{\beta} \left(\frac{zf'(z)}{f(z)}\right) = p(z) \tag{3.1}$$

for some $z \in U$. From relation (3.1):

$$a_2 = \frac{p_1}{\beta + 1}, \tag{3.2}$$

$$a_3 = \frac{p_2}{\beta + 2} - \frac{(\beta - 1)p_1^2}{2(\beta + 1)^2}, \tag{3.3}$$

and

$$a_4 = \tfrac{p_3}{\beta+3} - \tfrac{(\beta-1)p_1 p_2}{(\beta+1)(\beta+2)} + \tfrac{(\beta-1)(2\beta-1)p_1^3}{6(\beta+1)^3}, \tag{3.4}$$

and the results follow by triangle inequality and using Lemma 2.1. □

Theorem 3.2. *Let f given by (1.1) be in the class $B(\beta)$. Then*

$$|a_2 a_3 - a_4| \leq \begin{cases} 2 & if \quad \beta = 0 \\ \\ \frac{2(\beta^2+3\beta+6)}{3(\beta+1)(\beta+2)(\beta+3)} \sqrt{\frac{2(\beta^3+4\beta^2+9\beta+6)}{\beta^3+4\beta^2+\beta+18}} & if \quad 0 < \beta < 1 \end{cases}.$$

Proof. Using equations (3.2) to (3.4) we find that

$$|a_2a_3 - a_4| = \left|\frac{\beta p_1 p_2}{(\beta+1)(\beta+2)} - \frac{p_3}{\beta+3} - \frac{(\beta-1)p_1^3}{3(\beta+1)^2}\right|.$$

Substituting for p_2 and p_3 from Lemma 2.1. and letting $p_1 = p$ we get

$$\begin{aligned}|a_2a_3 - a_4| &= \left|\frac{\beta p\left[p^2+x(4-p^2)\right]}{2(\beta+1)(\beta+2)} - \frac{(\beta-1)p^3}{3(\beta+1)^2}\right. \\ &\quad \left. - \frac{p^3+2(4-p^2)px-p(4-p^2)x^2+2(4-p^2)(1-|x|^2)z}{4(\beta+3)}\right|\end{aligned}$$

which gives

$$\begin{aligned}|a_2a_3 - a_4| &= \left|-\frac{\left(\beta^3+4\beta^2+\beta-18\right)p^3}{12(\beta+1)^2(\beta+2)(\beta+3)} + \frac{px^2(4-p^2)}{4(\beta+3)}\right. \\ &\quad \left. - \frac{px(4-p^2)}{(\beta+1)(\beta+2)(\beta+3)} - \frac{(4-p^2)(1-|x|^2)z}{2(\beta+3)}\right|.\end{aligned}$$

Since $p \in P$, so $|p_1| \le 2$. We may assume without restriction that $p \in [0,2]$.
For $\eta = |x| \le 1$, we get

$$\begin{aligned}|a_2a_3 - a_4| &\le \frac{(\beta^3+4\beta^2+\beta-18)p^3}{12(\beta+1)^2(\beta+2)(\beta+3)} + \frac{(2-p)\eta^2(4-p^2)}{4(\beta+3)} \\ &\quad + \frac{p\eta(4-p^2)}{(\beta+1)(\beta+2)(\beta+3)} + \frac{4-p^2}{2(\beta+3)} \\ &= G(\eta).\end{aligned}$$

Then

$$G'(\eta) = \frac{p(4-p^2)}{(\beta+1)(\beta+2)(\beta+3)} + \frac{(2-p)\eta(4-p^2)}{2(\beta+3)}$$

Note also that $G'(\eta) \ge G'(1) > 0$. Then there exists $p^* \in [0,2]$ such that $G'(\eta) > 0$ for $p^* \in (0,2]$ and $G'(\eta) \le 0$ otherwise. Then for $p \in (p^*,2]$, $G(\eta) \le G(1)$, that is:

$$\begin{aligned}|a_2a_3 - a_4| &\le \frac{\left(\beta^2+3\beta+6\right)p}{(\beta+1)(\beta+2)(\beta+3)} - \frac{(\beta^2+2\beta+9)p^3}{6(\beta+1)^2(\beta+3)} \\ &= G_1(p).\end{aligned}$$

If $\beta = 0$, we have $G_1(p) = p - \frac{p^3}{2} \le 2$. Otherwise, by elementary calculation $G_1(p)$ is maximum at

$$p_{01} = \sqrt{\frac{2(\beta^3+4\beta^2+9\beta+6)}{\beta^3+4\beta^2+\beta+18}}$$

and is given by

$$\max G_1(p) = G_1(p_{01}) = \frac{2\left(\beta^2+3\beta+6\right)}{3(\beta+1)(\beta+2)(\beta+3)}\sqrt{\frac{2(\beta^3+4\beta^2+9\beta+6)}{\beta^3+4\beta^2+\beta+18}}.$$

Now suppose $p \in [0,p^*]$, then $G(\eta) \le G(0)$, that is:

$$\begin{aligned}|a_2a_3 - a_4| &\le \frac{\left(\beta^3+4\beta^2+\beta-18\right)p^3}{12(\beta+1)^2(\beta+2)(\beta+3)} + \frac{4-p^2}{2(\beta+3)} \\ &= G_2(p)\end{aligned}$$

which implies that $G_2(p)$ turns at $p_{02} = 0$ and

$$p_{03} = \frac{4(\beta+1)^2(\beta+2)}{\beta^3+4\beta^2+\beta-18}$$

with its maximum at $p_{02} = 0$. That is

$$\max G_2(p) = G_2(p_{02}) = \frac{2}{\beta + 3}.$$

$\square$

Remark 3.1. *For $\beta = 0$, Theorem 3.2 readily yields the following coefficient estimates for starlike functions.*

Corollary 3.1. *Let f given by (1.1) be in the class S^*. Then*

$$|a_2 a_3 - a_4| \leq 2.$$

This result is sharp and agree with those obtained by Babalola in [1].

Theorem 3.3. *Let f given by (1.1) be in the class $B(\beta)$, and $0 \leq \beta < 1$. Then we have the best possible bound*

$$\left|a_3 - a_2^2\right| \leq \frac{2}{\beta + 2}.$$

Proof. Since $f \in B(\beta)$, From (3.2) and (3.4) we find that

$$\begin{aligned} \left|a_3 - a_2^2\right| &\leq \frac{p^2}{2(\beta+1)(\beta+2)} - \frac{\left(4 - p^2\right)\eta}{2(\beta+2)} \\ &= F(\eta). \end{aligned}$$

The rest of the proof follows as in Theorem 3.2. $\square$

Remark 3.2. *Again for $\beta = 0$ in Theorem 3.3, we have*

$$\left|a_3 - a_2^2\right| \leq 1.$$

This sharp result also agree with those obtained by Keogh [8].

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