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GENERALIZATION OF JENSEN'S AND JENSEN-STEFFENSEN'S INEQUALITIES AND THEIR CONVERSES BY HERMITE'S POLYNOMIAL AND MAJORIZATION THEOREM

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Abstract. In this paper, using majorization theorems and Hermite's interpolating polynomials we obtain results concerning Jensen's and Jensen-Steffensen's inequalities and their converses in both the integral and the discrete case. We give bounds for identities related to these inequalities by using Čebyšev functionals. We also give Grüss type inequalities and Ostrowsky type inequalities for these functionals.

1. Introduction

Majorization makes precise the vague notion that the components of a vector $\mathbf{x}$ are "less spread out" or "more nearly equal" than the components of a vector $\mathbf{y}$. For fixed $m \geq 2$ let

$$\mathbf{x} = (x_1, ..., x_m), \ \mathbf{y} = (y_1, ..., y_m)$$

denote two m-tuples. Let

$$x_{[1]} \geq x_{[2]} \geq ... \geq x_{[m]}, \ y_{[1]} \geq y_{[2]} \geq ... \geq y_{[m]},$$

$$x_{(1)} \leq x_{(2)} \leq ... \leq x_{(m)}, \ y_{(1)} \leq y_{(2)} \leq ... \leq y_{(m)}$$

be their ordered components.

Majorization: (see [14, p. 319]) $\mathbf{x}$ is said to majorize $\mathbf{y}$ (or $\mathbf{y}$ is said to be majorized by $\mathbf{x}$), in symbol, $\mathbf{x} \succ \mathbf{y}$, if

$$\sum_{i=1}^{l} y_{[i]} \leq \sum_{i=1}^{l} x_{[i]} \tag{1.1}$$

holds for $l = 1, 2, ..., m-1$ and

$$\sum_{i=1}^{m} x_{[i]} = \sum_{i=1}^{m} y_{[i]}.$$

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Note that (1.1) is equivalent to

$$\sum_{i=m-l+1}^{m} y_{(i)} \leq \sum_{i=m-l+1}^{m} x_{(i)}$$

holds for $l = 1, 2, ..., m-1$.

There are several equivalent characterizations of the majorization relation $\mathbf{x} \succ \mathbf{y}$ in addition to the conditions given in the definition of majorization. One is actually the answer of the question posed and answered in 1929 by Hardy, Littlewood and Polya in [8] and [9]: $\mathbf{x}$ majorizes $\mathbf{y}$ if

$$\sum_{i=1}^{m} F(y_i) \leq \sum_{i=1}^{m} F(x_i)$$

for every continuous convex function F. Another interesting characterization of $\mathbf{x} \succ \mathbf{y}$, also by Hardy, Littlewood and Polya in [8] and [9], is that $\mathbf{y} = \mathbf{P}\mathbf{x}$ for some double stochastic matrix $\mathbf{P}$. In fact, the previous characterization implies that the set of vectors $\mathbf{x}$ that satisfy $\mathbf{x} \succ \mathbf{y}$ is the convex hull spanned by the $n!$ points formed from the permutations of the elements of $\mathbf{x}$.

The following theorem is well-known as the majorization theorem and a convenient reference for its proof is given by Marshall and Olkin in [12, p. 14] (see also [14, p. 320]):

Theorem 1. *Let $\mathbf{x} = (x_1, ..., x_m)$, $\mathbf{y} = (y_1, ..., y_m)$ be two m-tuples such that $x_i, y_i \in [a, b]$, $i = 1, ..., m$. Then*

$$\sum_{i=1}^{m} F(y_i) \leq \sum_{i=1}^{m} F(x_i)$$

holds for every continuous convex function $F : [a, b] \to \mathbb{R}$ iff $\mathbf{x} \succ \mathbf{y}$ holds.

The following theorem can be regarded as a generalization of Theorem 1 known as Weighted Majorization Theorem and is proved by Fuchs in [7] (see also [12, p. 580] and [14, p. 323]).

Theorem 2. *Let $\mathbf{x} = (x_1, ..., x_m)$, $\mathbf{y} = (y_1, ..., y_m)$ be two decreasing real m-tuples with $x_i, y_i \in [a, b]$, $i = 1, ..., m$, let $\mathbf{w} = (w_1, ..., w_m)$ be a real m-tuple such that*

$$\sum_{i=1}^{l} w_i y_i \leq \sum_{i=1}^{l} w_i x_i, \;\; for\ l = 1, ..., m-1 \tag{1.2}$$

and

$$\sum_{i=1}^{m} w_i y_i = \sum_{i=1}^{m} w_i x_i. \tag{1.3}$$

Then for every continuous convex function $F : [a, b] \to \mathbb{R}$, we have

$$\sum_{i=1}^{m} w_i F(y_i) \leq \sum_{i=1}^{m} w_i F(x_i). \tag{1.4}$$

The following integral version of Theorem 2 is a simple consequence of Theorem 12.14. in [13] (see also [14, p. 328]).

Theorem 3. *Let $x, y : [a, b] \to [\alpha, \beta]$ be decreasing and $w : [a, b] \to \mathbb{R}$ be continuous functions. If*

$$\int_a^{\nu} w(t)y(t)dt \leq \int_a^{\nu} w(t)x(t)dt, \;\; \nu \in [a, b] \tag{1.5}$$

and

$$\int_a^b w(t)y(t)dt = \int_a^b w(t)x(t)dt \tag{1.6}$$

hold, then for every continuous convex function $F : [\alpha, \beta] \to \mathbb{R}$*, we have*

$$\int_a^b w(t)F(y(t))\,dt \le \int_a^b w(t)F(x(t))\,dt.$$

Consider the Greens's function G defined on $[a,b] \times [a,b]$ by

$$G(t,s) = \begin{cases} \frac{(t-b)(s-a)}{b-a}, & a \le s \le t, \\ \frac{(s-b)(t-a)}{b-a,}, & t \le s \le b. \end{cases} \tag{1.7}$$

The function G is convex in s, it is symetric, so it is also convex in t. The function G is continuous in s and continuous in t.

For any function $F : [a,b] \to \mathbb{R}$, $F \in C^2[a,b]$, we can easily show by integrating by parts that the following is valid

$$F(t) = \frac{b-t}{b-a}F(a) + \frac{t-a}{b-a}F(b) + \int_a^b G(t,s)F''(s)ds, \tag{1.8}$$

where the function G is defined as above in (1.7).

We follow here notations and terminology about **Hermite interpolating polynomial** from [2, p. 62]:

Let $-\infty < a < b < \infty$ and $a \le a_1 < a_2 ... < a_r \le b$, $r \ge 2$ be given. For $F \in C^n[a,b]$ a unique polynomial $P_H(t)$ of degree $(n-1)$, exists, fulfilling one of the following conditions:

Hermite conditions:

$$P_H^{(i)}(a_j) = F^{(i)}(a_j);\ 0 \le i \le k_j,\ 1 \le j \le r,\ \sum_{j=1}^r k_j + r = n,$$

in particular:

Simple Hermite or Osculatory conditions:
($n = 2m$, $r = m$, $k_j = 1$ for all j)

$$P_O(a_j) = F(a_j),\ P_O'(a_j) = F'(a_j),\ 1 \le j \le m,$$

Lagrange conditions: ($r = n$, $k_j = 0$ for all j)

$$P_L(a_j) = F(a_j),\ 1 \le j \le n,$$

Type $(m, n-m)$ **conditions:** ($r = 2$, $1 \le m \le n-1$, $k_1 = m-1$, $k_2 = n-m-1$)

$$P_{mn}^{(i)}(a) = F^{(i)}(a),\ 0 \le i \le m-1,$$
$$P_{mn}^{(i)}(b) = F^{(i)}(b),\ 0 \le i \le n-m-1,$$

Two-point Taylor conditions: ($n = 2m$, $r = 2$, $k_1 = k_2 = m-1$)

$$P_{2T}^{(i)}(a) = F^{(i)}(a),\ P_{2T}^{(i)}(b) = F^{(i)}(b),\ 0 \le i \le m-1.$$

Theorem 4. *Let* $F \in C^n[a,b]$*, and let* P_H *be its Hermite interpolating polynomial. Then*

$$\begin{aligned} F(t) &= P_H(t) + e_H(t) \\ &= \sum_{j=1}^r \sum_{i=0}^{k_j} H_{ij}(t)F^{(i)}(a_j) + \int_a^b G_{H,n}(t,s)F^{(n)}(s)ds, \end{aligned}$$

where H_{ij} are fundamental polynomials of the Hermite basis defined by

$$H_{ij}(t) = \frac{1}{i!}\frac{\omega(t)}{(t-a_j)^{k_j+1-i}} \sum_{k=0}^{k_j-i} \frac{1}{k_j!}\left[\frac{(t-a_j)^{k_j+1}}{\omega(t)}\right]^{(k)}_{t=a_j} (t-a_j)^k, \tag{1.9}$$

where

$$\omega(t) = \prod_{j=1}^{r} (t-a_j)^{k_j+1}, \tag{1.10}$$

and $G_{H,n}$ is the Green's function defined by

$$G_{H,n}(t,s) = \begin{cases} \sum_{j=1}^{l}\sum_{i=0}^{k_j} \frac{(a_j-s)^{n-i-1}}{(n-i-1)!} H_{ij}(t), & s \le t \\ -\sum_{j=l+1}^{r}\sum_{i=0}^{k_j} \frac{(a_j-s)^{n-i-1}}{(n-i-1)!} H_{ij}(t), & s \ge t, \end{cases} \tag{1.11}$$

for all $a_l \le s \le a_{l+1}$, $l = 0, ..., r$ with $a_0 = a$ and $a_{r+1} = b$.

The following Lemma describes the positivity of Green's function (1.11) (see [5] and [11]).

Lemma 1. *The Green's function $G_{H,n}(t,s)$ has the following properties:*

(1) $\frac{G_{H,n}(t,s)}{\omega(t)} > 0$, $a_1 \le t \le a_r$, $a_1 \le s \le a_r$;
(2) $G_{H,n}(t,s) \le \frac{1}{(n-1)!(b-a)}|\omega(t)|$;
(3) $\int_a^b G_{H,n}(t,s)ds = \frac{\omega(t)}{n!}$

In order to recall the definition of n-convex function, first we write the definition of divided difference.

Definition 1. *Let f be a real-valued function defined on the segment $[a,b]$. The* **divided difference** *of order n of the function f at distinct points $x_0, ..., x_n \in [a,b]$, is defined recursively (see* [4], [14]*) by*

$$f[x_i] = f(x_i), \; (i = 0, \ldots, n)$$

and

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}.$$

The value $f[x_0, \ldots, x_n]$ is independent of the order of the points $x_0, \ldots, x_n$.
The definition may be extended to include the case that some (or all) of the points coincide. Assuming that $f^{(j-1)}(x)$ exists, we define

$$f[\underbrace{x, \ldots, x}_{j-times}] = \frac{f^{(j-1)}(x)}{(j-1)!}.$$

The notion of n-**convexity** goes back to Popoviciu [15]. We follow the definition given by Karlin [10]:

Definition 2. *A function $f : [a,b] \to \mathbb{R}$ is said to be n-convex on $[a,b]$, $n \ge 0$, if for all choices of $(n+1)$ distinct points in $[a,b]$, n-th order divided difference of f satisfies*

$$f[x_0, ..., x_n] \ge 0.$$

In fact, Popoviciu proved that each continuous n-convex function on $[0,1]$ is the uniform limit of the sequence of corresponding Bernstein's polynomials (see for example [14, p. 293]). Also, Bernstein's polynomials of continuous n-convex function are also n-convex functions. Therefore, when stating our results for a continuous n-convex function f, without any loss in generality we assume that $f^{(n)}$ exists and is continuous.

In [1] the authors proved the following Fuch's majorization theorems for n-convex function:

Theorem 5. *Let $-\infty < a = a_1 < a_2 \cdots < a_r = b < \infty$, $r \geq 2$ be the given points, $\mathbf{x} = (x_1, ..., x_m)$ and $\mathbf{y} = (y_1, ..., y_m)$ be decreasing m-tuples and $\mathbf{w} = (w_1, ..., w_m)$ be any m-tuple with $x_i, y_i \in [a,b]$, $w_i \in \mathbb{R}$, $i = 1, ..., m$ which satisfy* (1.2) *and* (1.3). *Let H_{lj} be as defined in* (1.9) *and $F : [a,b] \to \mathbb{R}$ be n-convex, then*

$$\begin{aligned}
(1.12) \qquad & \sum_{i=1}^{m} w_i F(x_i) - \sum_{i=1}^{m} w_i F(y_i) \\
\geq & \int_a^b \left[\sum_{i=1}^{m} w_i \left(G(x_i, t) - G(y_i, t) \right) \right] \sum_{j=1}^{r} \sum_{l=0}^{k_j} F^{(l+2)}(a_j) H_{lj}(t) dt.
\end{aligned}$$

(1) *If k_j is odd for every $j = 2, \ldots, r$, then the inequality* (1.12) *holds.*
(2) *If k_j is odd for every $j = 2, \ldots, r-1$ and k_r is even, then the reverse inequality in* (1.12) *holds.*

If the inequality (reverse inequality) in (1.12) *holds and the function $\phi(.) = \sum\limits_{j=1}^{r} \sum\limits_{l=0}^{k_j} F^{(l+2)}(a_j) H_{lj}(.)$ is non negative (non positive), then the right hand side of* (1.12) *will be non negative (non positive), that is the inequality (reverse inequality) in* (1.4) *will hold.*

In [3] using Hermite's interpolating polynomials and conditions on Green's functions, the authors present results for Jensen's inequality and converse of Jensen's inequality for signed measure. In this paper we give generalized results of Jensen's and Jensen-Steffensen's inequalities and their converses by using majorization theorem and Hermite's polynomial for n-convex functions. Then we give bounds for identities related to these inequalities by using Čebyšev functionals. We give Grüss type inequalities and Ostrowsky type inequalities for these functionals.

2. Generalization of Jensen's inequality

Theorem 6. *Let $-\infty < a \leq a_1 < a_2 \cdots < a_r \leq b < \infty$, $r \geq 2$ be the given points, let $\mathbf{x} = (x_1, ..., x_m)$ and $\mathbf{w} = (w_1, ..., w_m)$ be m-tuples such that $x_i \in [a,b]$, $w_i \in \mathbb{R}$, $i = 1, ..., m$, $W_m = \sum_{i=1}^{m} w_i$, $\overline{x} = \frac{1}{W_m} \sum_{i=1}^{m} w_i x_i$ and $F \in C^n[a,b]$. Also let H_{lj}, $G_{H,n}$*

and G *be as defined in* (1.9), (1.11) *and* (1.7) *respectively. Then*

$$\begin{aligned}
(2.1)\qquad & \frac{1}{W_m}\sum_{i=1}^{m} w_i F(x_i) - F(\overline{x}) \\
&= \int_a^b \left[\frac{1}{W_m}\sum_{i=1}^{m} w_i G(x_i,s) - G(\overline{x},s)\right] \sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j) H_{lj}(s) ds \\
&+ \int_a^b \int_a^b \left[\frac{1}{W_m}\sum_{i=1}^{m} w_i G(x_i,s) - G(\overline{x},s)\right] G_{H,n-2}(s,t) F^{(n)}(t)\, dt ds.
\end{aligned}$$

Proof. Consider $\frac{1}{W_m}\sum_{i=1}^{m} w_i F(x_i) - F(\overline{x})$. Using (1.8), we have

$$\begin{aligned}
(2.2)\qquad & \frac{1}{W_m}\sum_{i=1}^{m} w_i F(x_i) - F(\overline{x}) \\
&= \int_a^b \left[\frac{1}{W_m}\sum_{i=1}^{m} w_i G(x_i,s) - G(\overline{x},s)\right] F''(s) ds.
\end{aligned}$$

By Theorem 4, $F''(s)$ can be expressed as

$$(2.3)\qquad F''(s) = \sum_{j=1}^{r}\sum_{l=0}^{k_j} H_{lj}(s) F^{(l+2)}(a_j) + \int_a^b G_{H,n-2}(s,t) F^{(n)}(t) dt.$$

Using (2.2) and (2.3) we get (2.1). □

Using previous result and Theorem 5, here we give generalization of Jensen's inequality for n-convex function.

Theorem 7. *Let* $-\infty < a = a_1 < a_2 \cdots < a_r = b < \infty$, $r \geq 2$ *be the given points, let* $\mathbf{x} = (x_1, ..., x_m)$ *be decreasing real* m*-tuple with* $x_i \in [a,b]$, $i = 1, \ldots, m$, *let* $\mathbf{w} = (w_1, ..., w_m)$ *be positive* m*-tuple such that* $w_i \in \mathbb{R}$, $i = 1, ..., m$, $W_m = \sum_{i=1}^{m} w_i$, $\overline{x} = \frac{1}{W_m}\sum_{i=1}^{m} w_i x_i$ *and* H_{lj} *be as defined in* (1.9). *Let* $F : [a,b] \to \mathbb{R}$ *be* n*-convex function. Consider the inequality*

$$\begin{aligned}
(2.4)\qquad & \frac{1}{W_m}\sum_{i=1}^{m} w_i F(x_i) - F(\overline{x}) \\
&\geq \int_a^b \left[\frac{1}{W_m}\sum_{i=1}^{m} w_i G(x_i,s) - G(\overline{x},s)\right] \sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j) H_{lj}(s) ds.
\end{aligned}$$

(1) *If* k_j *is odd for every* $j = 2, \ldots, r$, *then the inequality* (2.4) *holds.*
(2) *If* k_j *is odd for every* $j = 2, \ldots, r-1$ *and* k_r *is even, then the reverse inequality in* (2.4) *holds.*

If the inequality (reverse inequality) in (2.4) *holds and the function* $\phi(.) = \sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j) H_{lj}(.)$ *is non negative (non positive), then the right hand side of* (2.4) *will be non negative (non positive), that is the inequality (reverse inequality)*

$$(2.5)\qquad \frac{1}{W_m}\sum_{i=1}^{m} w_i F(x_i) - F(\overline{x}) \geq 0$$

holds.

Proof. For $l = 1, ..., k$, such that $x_k \geq \overline{x}$ we get

$$\sum_{i=1}^{l} w_i \overline{x} \leq \sum_{i=1}^{l} w_i x_i.$$

If $l = k+1, ..., m-1$, such that $x_{k+1} < \overline{x}$ we have

$$\sum_{i=1}^{l} w_i x_i = \sum_{i=1}^{m} w_i x_i - \sum_{i=l+1}^{m} w_i x_i > \sum_{i=1}^{m} w_i \overline{x} - \sum_{i=l+1}^{m} w_i \overline{x} = \sum_{i=1}^{l} w_i \overline{x}.$$

So,

$$\sum_{i=1}^{l} w_i \overline{x} \leq \sum_{i=1}^{l} w_i x_i \text{ for all } l = 1, \ldots, m-1$$

and obviously

$$\sum_{i=1}^{m} w_i \overline{x} = \sum_{i=1}^{m} w_i x_i.$$

Now, we put $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (\bar{x}, \ldots, \bar{x})$ in Theorem 5 to get inequalities (2.4) and (2.5). □

Using $(p, n-p)$ type conditions, we get the following corollary:

Corollary 1. *Let $[a, b]$ be the given interval, $\mathbf{x} = (x_1, ..., x_m)$ be decreasing real m-tuple with $x_i \in [a, b]$, $i = 1, \ldots, m$, let $\mathbf{w} = (w_1, ..., w_m)$ be positive m-tuple such that $w_i \in \mathbb{R}$, $i = 1, ..., m$, $W_m = \sum_{i=1}^{m} w_i$ and $\overline{x} = \frac{1}{W_m} \sum_{i=1}^{m} w_i x_i$. Let $F : [a, b] \to \mathbb{R}$ be n-convex function. Consider the inequality*

$$\frac{1}{W_m} \sum_{i=1}^{m} w_i F(x_i) - F(\overline{x}) \tag{2.6}$$

$$\geq \int_a^b \left[\frac{1}{W_m} \sum_{i=1}^{m} w_i G(x_i, s) - G(\overline{x}, s) \right] \left[\sum_{l=0}^{p-1} F^{(l+2)}(a) H_{l1}(s) + \sum_{l=0}^{n-p-1} F^{(l+2)}(b) H_{l2}(s) \right] ds,$$

where

$$H_{l1}(s) = \frac{1}{l!}(s-a)^l \left(\frac{s-b}{a-b} \right)^{n-p} \sum_{k=0}^{p-1-l} \binom{n-p+k-1}{k} \left(\frac{s-a}{b-a} \right)^k$$

and

$$H_{l2}(s) = \frac{1}{l!}(s-b)^l \left(\frac{s-a}{b-a} \right)^{p} \sum_{k=0}^{n-p-1-l} \binom{p+k-1}{k} \left(\frac{s-b}{a-b} \right)^k.$$

(1) *If $n - p$ is even, then the inequality* (2.6) *holds.*
(2) *If $n - p$ is odd, then the reverse inequality in* (2.6) *holds.*

If the inequality (reverse inequality) in (2.6) *holds and the function $\phi(.) = \sum_{l=0}^{p-1} F^{(l+2)}(a) H_{l1}(.) + \sum_{l=0}^{n-p-1} F^{(l+2)}(b) H_{l2}(.)$ is non negative (non positive), then the right hand side of* (2.6) *will be non negative (non positive), that is the inequality (reverse inequality)* (2.5) *holds.*

Using Two-point Taylor conditions, we get the following corollary:

Corollary 2. *Let $[a,b]$ be the given interval, $\mathbf{x} = (x_1, ..., x_m)$ be decreasing real m-tuple with $x_i \in [a,b]$, $i = 1, \ldots, m$, let $\mathbf{w} = (w_1, ..., w_m)$ be positive m-tuple such that $w_i \in \mathbb{R}$, $i = 1, ..., m$, $W_m = \sum_{i=1}^m w_i$ and $\overline{x} = \frac{1}{W_m} \sum_{i=1}^m w_i x_i$. Let $F : [a,b] \to \mathbb{R}$ be n-convex function. Consider the inequality*

$$\begin{aligned} (2.7) \qquad & \frac{1}{W_m} \sum_{i=1}^m w_i F(x_i) - F(\overline{x}) \\ & \geq \int_a^b \left[\frac{1}{W_m} \sum_{i=1}^m w_i G(x_i, s) - G(\overline{x}, s) \right] \sum_{l=0}^{p-1} \sum_{k=0}^{p-1-l} \binom{p+k-1}{k} \cdot \\ & \left[\frac{(s-a)^l}{l!} \left(\frac{s-b}{a-b} \right)^p \left(\frac{s-a}{b-a} \right)^k F^{(l+2)}(a) + \frac{(s-b)^l}{l!} \left(\frac{s-a}{b-a} \right)^p \left(\frac{s-b}{a-b} \right)^k F^{(l+2)}(b) \right] ds \end{aligned}$$

(1) *If p is even then the inequality* (2.7) *holds.*
(2) *If p is odd then the reverse inequality in* (2.7) *holds.*

If the inequality (reverse inequality) in (2.7) *holds and the function* $\phi(s) = \sum_{l=0}^{p-1} \sum_{k=0}^{p-1-l} \binom{p+k-1}{k} \left[\frac{(s-a)^l}{l!} \left(\frac{s-b}{a-b} \right)^p \left(\frac{s-a}{b-a} \right)^k F^{(l+2)}(a) + \frac{(s-b)^l}{l!} \left(\frac{s-a}{b-a} \right)^p \left(\frac{s-b}{a-b} \right)^k F^{(l+2)}(b) \right]$ *is non negative (non positive), then the right hand side of* (2.7) *will be non negative (non positive), that is the inequality (reverse inequality)* (2.5) *holds.*

Using Simple Hermite or Osculatory conditions, we get the following corollary:

Corollary 3. *Let $-\infty < a = a_1 < a_2 \cdots < a_r = b < \infty$, $r \geq 2$ be the given points, let $\mathbf{x} = (x_1, ..., x_m)$ be decreasing real m-tuple with $x_i \in [a,b]$, $i = 1, \ldots, m$, let $\mathbf{w} = (w_1, ..., w_m)$ be positive m-tuple such that $w_i \in \mathbb{R}$, $i = 1, ..., m$, $W_m = \sum_{i=1}^m w_i$ and $\overline{x} = \frac{1}{W_m} \sum_{i=1}^m w_i x_i$. Let $F : [a,b] \to \mathbb{R}$ be $(2r)$-convex function. Then we have*

$$\begin{aligned} & \frac{1}{W_m} \sum_{i=1}^m w_i F(x_i) - F(\overline{x}) \\ \geq \; & \int_a^b \left[\frac{1}{W_m} \sum_{i=1}^m w_i G(x_i, s) - G(\overline{x}, s) \right] \sum_{j=1}^r \left[F''(a_j) H_{0j}(s) + F'''(a_j) H_{1j}(s) \right] ds, \end{aligned}$$

where

$$H_{0j}(s) = \frac{P_r^2(s)}{(s-a_j)^2 \left[P_r'(a_j) \right]^2} \left(1 - \frac{P_r''(a_j)}{P_r'(a_j)} (s - a_j) \right)$$

$$H_{1j}(s) = \frac{P_r^2(s)}{(s-a_j) \left[P_r'(a_j) \right]^2},$$

and

$$P_r(s) = \prod_{j=1}^r (s - a_j).$$

Proof. We put $k_j = 1$ for $j = 1, ..., r$ in Theorem 7. $\square$

In the following remark we give the integral version of the Theorem 7.

Remark 1. For the given points $-\infty < \alpha = a_1 < a_2 \cdots < a_r = \beta < \infty$, $r \geq 2$, $x : [a,b] \to \mathbb{R}$ continuous decreasing function, such that $x([a,b]) \subseteq [\alpha, \beta]$, $\lambda : [a,b] \to \mathbb{R}$ increasing, bounded function with $\lambda(a) \neq \lambda(b)$ and $\overline{x} = \frac{\int_a^b x(t)\, d\lambda(t)}{\int_a^b d\lambda(t)}$, for $x(c) \geq \overline{x}$, we have:

$$\int_a^c x(t)\, d\lambda(t) \geq \int_a^c x(c)\, d\lambda(t) \geq \int_a^c \overline{x}\, d\lambda(t), \;\; c \in [a,b]\,.$$

If $x(c) < \overline{x}$ we have

$$\begin{aligned} \int_a^c x(t)\, d\lambda(t) &= \int_a^b x(t)\, d\lambda(t) - \int_c^b x(t)\, d\lambda(t) \\ > \;\; & \int_a^b \overline{x}\, d\lambda(t) - \int_c^b \overline{x}\, d\lambda(t) = \int_a^c \overline{x}\, d\lambda(t), \;\; c \in [a,b]\,. \end{aligned}$$

Equality

$$\int_a^b x(t)\, d\lambda(t) = \int_a^b \overline{x} d\lambda(t)$$

obviously holds, so majorization conditions (1.5) and (1.6) are satisfied.
Consider the inequality:

$$\begin{aligned} &\frac{\int_a^b F\left(x(t)\right)\, d\lambda(t)}{\int_a^b d\lambda(t)} - F(\overline{x}) \\ \geq\;\; &\int_\alpha^\beta \left[\frac{\int_a^b G(x(t),s)\, d\lambda(t)}{\int_a^b d\lambda(t)} - G(\overline{x}, s)\right] \sum_{j=1}^{r} \sum_{l=0}^{k_j} F^{(l+2)}(a_j) H_{lj}(s) ds, \end{aligned} \tag{2.8}$$

where H_{lj} is as defined in (1.9) and $F : [\alpha, \beta] \to \mathbb{R}$ is n-convex function.

(1) If k_j is odd for every $j = 2, \ldots, r$, then the inequality (2.8) holds.
(2) If k_j is odd for every $j = 2, \ldots, r-1$ and k_r is even, then the reverse inequality in (2.8) holds.

If the inequality (reverse inequality) in (2.8) holds and the function
$\phi(.) = \sum\limits_{j=1}^{r} \sum\limits_{l=0}^{k_j} F^{(l+2)}(a_j) H_{lj}(.)$ is non negative (non positive), then the right hand side of (2.8) will be non negative (non positive), that is the inequality (reverse inequality)

$$\frac{\int_a^b F\left(x(t)\right)\, d\lambda(t)}{\int_a^b d\lambda(t)} - F(\overline{x}) \geq 0 \tag{2.9}$$

holds.

Remark 2. Motivated by the inequalities (2.4) and (2.8), we define functionals $\Theta_1(F)$ and $\Theta_2(F)$, by

$$\begin{aligned} \Theta_1(F) \;\;=\;\; & \frac{1}{W_m} \sum_{i=1}^{m} w_i F(x_i) - F(\overline{x}) \\ - \;\; & \int_a^b \left[\frac{1}{W_m} \sum_{i=1}^{m} w_i G(x_i, s) - G(\overline{x}, s)\right] \sum_{j=1}^{r} \sum_{l=0}^{k_j} F^{(l+2)}(a_j) H_{lj}(s) ds, \end{aligned}$$

$$\begin{aligned}\Theta_2(F) &= \frac{\int_a^b F(x(t))\,d\lambda(t)}{\int_a^b d\lambda(t)} - F(\overline{x}) \\ &- \int_\alpha^\beta \left[\frac{\int_a^b G(x(t),s)\,d\lambda(t)}{\int_a^b d\lambda(t)} - G(\overline{x},s)\right]\sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j)H_{lj}(s)ds,\end{aligned}$$

Similarly as in [3] we can construct new families of exponentially convex function and Cauchy type means by looking at these linear functionals. The monotonicity property of the generalized Cauchy means obtained via these functionals can be proved by using the properties of the linear functionals associated with this error representation, such as n-exponential and logarithmic convexity.

3. Generalization of Jensen Steffensen's inequality

Theorem 8. *Let $-\infty < a = a_1 < a_2 \cdots < a_r = b < \infty$, $r \geq 2$ be the given points, let $\mathbf{x} = (x_1, ..., x_m)$ be decreasing real m-tuple with $x_i \in [a,b]$, $i = 1, ..., m$, let $\mathbf{w} = (w_1, ..., w_m)$ be real m-tuple such that $0 \leq W_k \leq W_m$, $k = 1, \cdots, m$, $W_m > 0$, where $W_k = \sum_{i=1}^k w_i$, $\overline{x} = \frac{1}{W_m}\sum_{i=1}^m w_i x_i$ and H_{lj} be as defined in (1.9). Let $F : [a,b] \to \mathbb{R}$ be n-convex function.*

1. *If k_j is odd for every $j = 2, \ldots, r$, then the inequality (2.4) holds.*
2. *If k_j is odd for every $j = 2, \ldots, r-1$ and k_r is even, then the reverse inequality in (2.4) holds.*

If the inequality (reverse inequality) in (2.4) holds and the function $\phi(.) = \sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j)H_{lj}(.)$ is non negative (non positive), then the right hand side of (2.4) will be non negative (non positive), that is the inequality (reverse inequality) (2.5) holds.

Proof. For $l = 1, ..., k$, such that $x_k \geq \overline{x}$ we have

$$\sum_{i=1}^{l} w_i x_i - W_l x_l = \sum_{i=1}^{l-1}(x_i - x_{i+1})W_i \geq 0$$

and so we get

$$\sum_{i=1}^{l} w_i \overline{x} = W_l \overline{x} \leq W_l x_l \leq \sum_{i=1}^{l} w_i x_i.$$

For $l = k+1, ..., m-1$, such that $x_{k+1} < \overline{x}$ we have

$$x_l\,(W_m - W_l) - \sum_{i=l+1}^{m} w_i x_i = \sum_{i=l+1}^{m}(x_{i-1} - x_i)(W_m - W_{i-1}) \geq 0$$

and now

$$\sum_{i=l+1}^{m} w_i \overline{x} = (W_m - W_l)\,\overline{x} > (W_m - W_l)\,x_l \geq \sum_{i=l+1}^{m} w_i x_i.$$

So, similarly as in Theorem 7, we get that conditions (1.2) and (1.3) for majorization are satisfied, so inequalities (2.4) and (2.5) are valid. □

Remark 3. For the given points $-\infty < \alpha = a_1 < a_2 \cdots < a_r = \beta < \infty$, $r \geq 2$, $x : [a,b] \to \mathbb{R}$ continuous, decreasing function, such that $x([a,b]) \subseteq [\alpha,\beta]$ and $\lambda : [a,b] \to \mathbb{R}$ is either continuous or of bounded variation satisfying $\lambda(a) \leq \lambda(t) \leq \lambda(b)$ for all $t \in [a,b]$, $\overline{x} = \frac{\int_a^b x(t)\,d\lambda(t)}{\int_a^b d\lambda(t)}$ and $F : [\alpha,\beta] \to \mathbb{R}$ n-convex function, for $x(c) \geq \overline{x}$, we have:

$$\int_a^c x(t)d\lambda(t) - x(c)\int_a^c d\lambda(t) = -\int_a^c x'(t)\left(\int_a^t d\lambda(x)\right)dt \geq 0$$

and so

$$\overline{x}\int_a^c d\lambda(t) \leq x(c)\int_a^c d\lambda(t) \leq \int_a^c x(t)d\lambda(t).$$

If $x(c) < \overline{x}$ we have

$$x(c)\int_c^b d\lambda(t) - \int_c^b x(t)d\lambda(t) = -\int_c^b x'(t)\left(\int_t^b d\lambda(x)\right)dt \geq 0$$

and now

$$\overline{x}\int_c^b d\lambda(t) > x(c)\int_c^b d\lambda(t) \geq \int_c^b x(t)d\lambda(t).$$

Similarly as in the Remark 1 we get that conditions for majorization are satisfied, so inequalities (2.8) and (2.9) are valid.

4. Generalization of converse of Jensen's inequality

Theorem 9. *Let $-\infty < a = a_1 < a_2 \cdots < a_r = b < \infty$, $r \geq 2$ be the given points, let $\mathbf{x} = (x_1, ..., x_p)$ be real p-tuple with $x_i \in [m,M] \subseteq [a,b]$, $i = 1,\dots,p$, let $\mathbf{w} = (w_1, ..., w_p)$ be positive p-tuple such that $w_i \in \mathbb{R}$, $i = 1, ..., p$, $W_p = \sum_{i=1}^p w_i$, $\overline{x} = \frac{1}{W_p}\sum_{i=1}^p w_i x_i$ and H_{lj} be as defined in* (1.9). *Let $F : [a,b] \to \mathbb{R}$ be n-convex function. Consider the inequality*

$$\begin{aligned}\frac{1}{W_p}\sum_{i=1}^p w_i F(x_i) &\leq \frac{\overline{x}-m}{M-m}F(M) + \frac{M-\overline{x}}{M-m}F(m)\\ &- \int_a^b\left[\frac{\overline{x}-m}{M-m}G(M,s) + \frac{M-\overline{x}}{M-m}G(m,s) - \frac{1}{W_p}\sum_{i=1}^p w_i G(x_i,s)\right]\\ &\times\sum_{j=1}^r\sum_{l=0}^{k_j} F^{(l+2)}(a_j)H_{lj}(s)ds.\end{aligned} \tag{4.1}$$

(1) *If k_j is odd for every $j = 2,\dots,r$, then the inequality* (4.1) *holds.*
(2) *If k_j is odd for every $j = 2,\dots,r-1$, and k_r is even, then the reverse inequality in* (4.1) *holds.*

Moreover, if the inequality (reverse inequality) in (4.1) *holds and the function $\phi(.) = \sum_{j=1}^r\sum_{l=0}^{k_j} F^{(l+2)}(a_j)H_{lj}(.)$ is non negative (non positive), then the right hand side of* (4.1) *will be non positive (non negative), that is the inequality (reverse inequality)*

$$\frac{1}{W_p}\sum_{i=1}^p w_i F(x_i) \leq \frac{\overline{x}-m}{M-m}F(M) + \frac{M-\overline{x}}{M-m}F(m) \tag{4.2}$$

holds.

Proof. Using inequality (2.4) we have

$$\begin{aligned}
&\frac{1}{W_p}\sum_{i=1}^{p} w_i F(x_i) = \frac{1}{W_p}\sum_{i=1}^{p} w_i F\left(\frac{x_i - m}{M - m}M + \frac{M - x_i}{M - m}m\right)\\
\leq\ & \frac{\overline{x} - m}{M - m}F(M) + \frac{M - \overline{x}}{M - m}F(m)\\
-\ & \int_a^b \left[\frac{\overline{x} - m}{M - m}G(M, s) + \frac{M - \overline{x}}{M - m}G(m, s) - \frac{1}{W_p}\sum_{i=1}^{p} w_i G(x_i, s)\right]\\
&\times \sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j) H_{lj}(s) ds.
\end{aligned}$$

For the inequality (4.2) we use the fact that for every convex function φ we have

$$\frac{1}{W_p}\sum_{i=1}^{p} w_i \varphi(x_i) \leq \frac{\overline{x} - m}{M - m}\varphi(M) + \frac{M - \overline{x}}{M - m}\varphi(m).$$

□

Corollary 4. *Let $-\infty < m < a_2 \cdots < a_{r-1} < M < \infty$, $r \geq 2$ be the given points, let $\mathbf{x} = (x_1, ..., x_p)$ be real p-tuple with $x_i \in [m, M]$, $i = 1, \ldots, p$, let $\mathbf{w} = (w_1, ..., w_p)$ be positive p-tuple such that $w_i \in \mathbb{R}$, $i = 1, ..., p$, $W_p = \sum_{i=1}^{p} w_i$, $\overline{x} = \frac{1}{W_p}\sum_{i=1}^{p} w_i x_i$ and H_{lj} be as defined in* (1.9)*. Let $F : [m, M] \to \mathbb{R}$ be n-convex function. Consider the inequality*

$$\begin{aligned}
&\frac{1}{W_p}\sum_{i=1}^{p} w_i F(x_i) \leq \frac{\overline{x} - m}{M - m}F(M) + \frac{M - \overline{x}}{M - m}F(m)\\
&+ \sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j)\frac{1}{W_p}\sum_{i=1}^{p} w_i \int_m^M G(x_i, s) H_{lj}(s) ds. \qquad (4.3)
\end{aligned}$$

(1) *If k_j is odd for every $j = 2, \ldots, r$, then the inequality* (4.3) *holds.*
(2) *If k_j is odd for every $j = 2, \ldots, r - 1$, and k_r is even, then the reverse inequality in* (4.3) *holds.*

Proof. We use inequality (4.1) for $m = a = a_1$ and $M = b = a_r$. Therefore we get $G(m, s) = 0$ and $G(M, s) = 0$ and so obtain inequality (4.3). □

Remark 4. For the given points $-\infty < \alpha = a_1 < a_2 \cdots < a_r = \beta < \infty$, $r \geq 2$, $x : [a, b] \to \mathbb{R}$ continuous function, such that $x([a, b]) \subseteq [m, M] \subseteq [\alpha, \beta]$ and $\lambda : [a, b] \to \mathbb{R}$ increasing, bounded function with $\lambda(a) \neq \lambda(b)$, $\overline{x} = \frac{\int_a^b x(t)\, d\lambda(t)}{\int_a^b d\lambda(t)}$, H_{lj} as defined in (1.9) and $F : [\alpha, \beta] \to \mathbb{R}$ n-convex function, consider the inequality

$$\begin{aligned}
&\frac{\int_a^b F(x(t)) d\lambda(t)}{\int_a^b d\lambda(t)} \leq \frac{\overline{x} - m}{M - m}F(M) + \frac{M - \overline{x}}{M - m}F(m) \qquad (4.4)\\
-\ & \int_\alpha^\beta \left[\frac{\overline{x} - m}{M - m}G(M, s) + \frac{M - \overline{x}}{M - m}G(m, s) - \frac{\int_a^b G(x(t), s) d\lambda(t)}{\int_a^b d\lambda(t)}\right] \sum_{j=1}^{r}\\
&\times \sum_{l=0}^{k_j} F^{(l+2)}(a_j) H_{lj}(s) ds.
\end{aligned}$$

(1) If k_j is odd for every $j = 2, \ldots, r$, then the inequality (4.4) holds.
(2) If k_j is odd for every $j = 2, \ldots, r-1$, and k_r is even, then the reverse inequality in (4.4) holds.

Moreover, if the inequality (reverse inequality) in (4.4) holds and the function $\phi(.) = \sum_{j=1}^{r} \sum_{l=0}^{k_j} F^{(l+2)}(a_j) H_{lj}(.)$ is non negative (non positive), then the right hand side of (4.4) will be non positive (non negative), that is the inequality (reverse inequality)

$$\frac{\int_a^b F(x(t))d\lambda(t)}{\int_a^b d\lambda(t)} \leq \frac{\overline{x}-m}{M-m}F(M) + \frac{M-\overline{x}}{M-m}F(m)$$

holds.

Remark 5. Motivated by the inequalities (4.1) and (4.4), we define functionals $\Theta_3(F)$ and $\Theta_4(F)$ by

$$\begin{aligned}
\Theta_3(F) &= \frac{1}{W_p}\sum_{i=1}^{p} w_i F(x_i) - \frac{\overline{x}-m}{M-m}F(M) - \frac{M-\overline{x}}{M-m}F(m) \\
+ &\int_a^b \left[\frac{\overline{x}-m}{M-m}G(M,s) + \frac{M-\overline{x}}{M-m}G(m,s) - \frac{1}{W_p}\sum_{i=1}^{p} w_i G(x_i,s)\right] \\
&\times \sum_{j=1}^{r} \sum_{l=0}^{k_j} F^{(l+2)}(a_j) H_{lj}(s) ds
\end{aligned}$$

and

$$\begin{aligned}
\Theta_4(F) &= \frac{\int_a^b F(x(t))d\lambda(t)}{\int_a^b d\lambda(t)} - \frac{\overline{x}-m}{M-m}F(M) - \frac{M-\overline{x}}{M-m}F(m) \\
+ &\int_\alpha^\beta \left[\frac{\overline{x}-m}{M-m}G(M,s) + \frac{M-\overline{x}}{M-m}G(m,s) - \frac{\int_a^b G(x(t),s)d\lambda(t)}{\int_a^b d\lambda(t)}\right] \\
&\times \sum_{j=1}^{r} \sum_{l=0}^{k_j} F^{(l+2)}(a_j) H_{lj}(s) ds.
\end{aligned}$$

Now, we can observe the same results which are mentioned in Remark 2.

5. Bounds for identities related to generalization of majorization inequality

For two Lebesgue integrable functions $f, h : [a,b] \to \mathbb{R}$ we consider Čebyšev functional

$$\Omega(f,h) = \frac{1}{b-a}\int_a^b f(t)h(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \frac{1}{b-a}\int_a^b h(t)dt. \tag{5.1}$$

In [6], the authors proved the following theorems:

Theorem 10. *Let $f : [a,b] \to \mathbb{R}$ be a Lebesgue integrable function and $h : [a,b] \to \mathbb{R}$ be an absolutely continuous function with $(. - a)(b - .)[h']^2 \in L[a,b]$. Then we have the* *inequality*

$$(5.2)\qquad |\,\Omega(f,h)\,| \leq \frac{1}{\sqrt{2}}\left[\Omega(f,f)\right]^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left(\int_a^b (x-a)(b-x)\left[h'(x)\right]^2 dx\right)^{\frac{1}{2}}.$$

The constant $\frac{1}{\sqrt{2}}$ *in* (5.2) *is the best possible.*

Theorem 11. *Assume that* $h:[a,b]\to\mathbb{R}$ *is monotonic nondecreasing on* $[a,b]$ *and* $f:[a,b]\to\mathbb{R}$ *is absolutely continuous with* $f'\in L_\infty[a,b]$. *Then we have the inequality*

$$(5.3)\qquad |\,\Omega(f,h)\,| \leq \frac{1}{2(b-a)}\parallel f'\parallel_\infty \int_a^b (x-a)(b-x)dh(x).$$

The constant $\frac{1}{2}$ *in* (5.3) *is the best possible.*

In the sequel we use the above theorems to obtain generalizations of the results proved in the previous sections.

For m-tuples $\mathbf{w}=(w_1,...,w_m)$, $\mathbf{x}=(x_1,...,x_m)$ with $x_i\in[a,b]$, $w_i\in\mathbb{R}$, $i=1,...,m$, $W_m=\sum_{i=1}^m w_i$, $\overline{x}=\frac{1}{W_m}\sum_{i=1}^m w_i x_i$ and the Green's functions G and $G_{H,n-2}$ as defined in (1.7) and (1.11), respectively, we denote

$$(5.4)\qquad \Upsilon(t)=\int_a^b\left[\frac{1}{W_m}\sum_{i=1}^m w_i G\left(x_i,s\right)-G\left(\overline{x},s\right)\right]G_{H,n-2}(s,t)ds,\ t\in[a,b].$$

Similarly for $x:[a,b]\to[\alpha,\beta]$ continuous function, $\lambda:[a,b]\to\mathbb{R}$ as defined in Remark 1 or in Remark 3, the Green's functions G and $G_{H,n-2}$ as defined in (1.7) and (1.11), respectively, and for all $s\in[\alpha,\beta]$ we denote

$$(5.5)\qquad \tilde{\Upsilon}(t)=\int_\alpha^\beta\left[\frac{\int_a^b G\left(x(p),s\right)d\lambda(p)}{\int_a^b d\lambda(p)}-G\left(\overline{x},s\right)\right]G_{H,n-2}(s,t)ds,\ t\in[\alpha,\beta].$$

Theorem 12. *Let* $-\infty<a\leq a_1<a_2\cdots<a_r\leq b<\infty$, $r\geq 2$ *be the given points, let* $F:[a,b]\to\mathbb{R}$ *be such that* $F\in C^{n+1}[a,b]$ *for* $n\in\mathbb{N}$ *and* $\mathbf{x}=(x_1,...,x_m)$, $\mathbf{w}=(w_1,...,w_m)$ *be* m*-tuples such that* $x_i\in[a,b]$, $w_i\in\mathbb{R}$, $i=1,...,m$, $W_m=\sum_{i=1}^m w_i$, $\overline{x}=\frac{1}{W_m}\sum_{i=1}^m w_i x_i$ *and let the functions* H_{lj}, $l=0,...,k_j$, $j=1,...,r$, ω, G, Υ *and functional* Ω *be defined in* (1.9), (1.10), (1.7), (5.4) *and* (5.1), *respectively. Then we have:*

$$\begin{aligned}
&\frac{1}{W_m}\sum_{i=1}^m w_i F(x_i)-F(\overline{x})\\
=&\int_a^b\left[\frac{1}{W_m}\sum_{i=1}^m w_i G(x_i,s)-G(\overline{x},s)\right]\sum_{j=1}^r\sum_{l=0}^{k_j}F^{(l+2)}(a_j)H_{lj}(s)ds\\
&+\frac{F^{(n-1)}(b)-F^{(n-1)}(a)}{b-a}\int_a^b\left[\frac{1}{W_m}\sum_{i=1}^m w_i G(x_i,s)-G(\overline{x},s)\right]\frac{\omega(s)}{(n-2)!}ds\\
(5.6)\qquad&+H_n^1(F;a,b)
\end{aligned}$$

where the remainder $H_n^1(F;a,b)$ *satisfies the estimation*

$$(5.7)\qquad |\,H_n^1(F;a,b)\,|\leq\frac{\sqrt{b-a}}{\sqrt{2}}\left[\Omega(\Upsilon,\Upsilon)\right]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)\left[F^{(n+1)}(t)\right]^2 dt\right|^{\frac{1}{2}}.$$

Proof. If we apply Theorem 10 for $f \to \Upsilon$ and $h \to F^{(n)}$ we obtain

$$\left|\frac{1}{b-a}\int_a^b \Upsilon(t)F^{(n)}(t)dt - \frac{1}{b-a}\int_a^b \Upsilon(t)dt \cdot \frac{1}{b-a}\int_a^b F^{(n)}(t)dt\right|$$
$$\leq \frac{1}{\sqrt{2}}\left[\Omega(\Upsilon,\Upsilon)\right]^{\frac{1}{2}} \frac{1}{\sqrt{b-a}}\left|\int_a^b (t-a)(b-t)\left[F^{(n+1)}(t)\right]^2 dt\right|^{\frac{1}{2}}.$$

Therefore we have

$$\int_a^b \Upsilon(t)F^{(n)}(t)dt = \frac{F^{(n-1)}(b)-F^{(n-1)}(a)}{b-a}\int_a^b \Upsilon(t)dt + H_n^1(F;a,b),$$

where the remainder $H_n^1(F;a,b)$ satisfies the estimation (5.7). Now, from Lemma 1 we obtain (5.6). □

Integral case of the above theorem can be given:

Theorem 13. *Let* $-\infty < \alpha \leq a_1 < a_2 \cdots < a_r \leq \beta < \infty$, $r \geq 2$ *be the given points, let* $F : [\alpha,\beta] \to \mathbb{R}$ *be such that* $F \in C^{n+1}[\alpha,\beta]$ *for* $n \in \mathbb{N}$, *let* $x : [a,b] \to \mathbb{R}$ *continuous functions such that* $x([a,b]) \subseteq [\alpha,\beta]$, $\lambda : [a,b] \to \mathbb{R}$ *be as defined in Remark 1 or in Remark 3,* $\overline{x} = \frac{\int_a^b x(t)\,d\lambda(t)}{\int_a^b d\lambda(t)}$ *and let the functions* H_{lj}, $l = 0,...,k_j$, $j = 1,...,r$, ω, G, $\tilde{\Upsilon}$ *and functional* Ω *be defined in* (1.9), (1.10), (1.7), (5.5) *and* (5.1), *respectively. Then we have:*

$$\frac{\int_a^b F(x(t))d\lambda(t)}{\int_a^b d\lambda(t)} - F(\overline{x})$$
$$= \int_\alpha^\beta \left[\frac{\int_a^b G(x(t),s)\,d\lambda(t)}{\int_a^b d\lambda(t)} - G(\overline{x},s)\right]\sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j)H_{lj}(s)ds$$
$$+\frac{F^{(n-1)}(\beta)-F^{(n-1)}(\alpha)}{\beta-\alpha}\int_\alpha^\beta\left[\frac{\int_a^b G(x(t),s)d\lambda(t)}{\int_a^b d\lambda(t)} - G(\overline{x},s)\right]\frac{\omega(s)}{(n-2)!}ds$$
$$+\tilde{H}_n^1(F;\alpha,\beta) \tag{5.8}$$

where the remainder $\tilde{H}_n^1(F;\alpha,\beta)$ *satisfies the estimation*

$$|\,\tilde{H}_n^1(F;\alpha,\beta)\,| \leq \frac{\sqrt{\beta-\alpha}}{\sqrt{2}}\left[\Omega(\tilde{\Upsilon},\tilde{\Upsilon})\right]^{\frac{1}{2}}\left|\int_\alpha^\beta (s-\alpha)(\beta-s)\left[F^{(n+1)}(s)\right]^2 ds\right|^{\frac{1}{2}}.$$

Using Theorem 11 we also get the following Grüss type inequality.

Theorem 14. *Let* $-\infty < a \leq a_1 < a_2 \cdots < a_r \leq b < \infty$, $r \geq 2$ *be the given points, let* $F : [a,b] \to \mathbb{R}$ *be such that* $F \in C^{n+1}[a,b]$ *for* $n \in \mathbb{N}$, $F^{(n+1)} \geq 0$ *on* $[a,b]$ *and let* $\mathbf{x} = (x_1,...,x_m)$, $\mathbf{w} = (w_1,...,w_m)$ *be* m*-tuples such that* $x_i \in [a,b]$, $w_i \in \mathbb{R}$, $i = 1,...,m$, $W_m = \sum_{i=1}^m w_i$, $\overline{x} = \frac{1}{W_m}\sum_{i=1}^m w_i x_i$ *and let the function* Υ *be defined in* (5.4). *Then we* *have the representation* (5.6) *and the remainder* $H_n^1(F;a,b)$ *satisfies the bound*

$$| H_n^1(F;a,b) |\leq \frac{\|\Upsilon'\|_\infty}{2}\left\{(b-a)\left[F^{(n-1)}(b)+F^{(n-1)}(a)\right]-\left[F^{(n-2)}(b)-F^{(n-2)}(a)\right]\right\}. \tag{5.9}$$

Proof. Applying Theorem 11 for $f\to\Upsilon$ and $h\to F^{(n)}$ we obtain

$$\left|\frac{1}{b-a}\int_a^b \Upsilon(t)F^{(n)}(t)dt-\frac{1}{b-a}\int_a^b\Upsilon(t)dt\cdot\frac{1}{b-a}\int_a^b F^{(n)}(t)dt\right|$$

$$\leq \frac{1}{2(b-a)}\|\Upsilon'\|_\infty\int_a^b (t-a)(b-t)F^{(n+1)}(t)dt. \tag{5.10}$$

Since

$$\int_a^b (t-a)(b-t)F^{(n+1)}(t)dt=\int_a^b \left[2t-(a+b)\right]F^{(n)}(t)dt$$

$$= (b-a)\left[F^{(n-1)}(b)+F^{(n-1)}(a)\right]-2\left[F^{(n-2)}(b)-F^{(n-2)}(a)\right],$$

using the identity (2.1) and (5.10) we deduce (5.9). □

Integral version of the above theorem can be given as:

Theorem 15. *Let* $-\infty<\alpha\leq a_1<a_2\cdots<a_r\leq\beta<\infty$, $r\geq 2$ *be the given points, let* $F:[\alpha,\beta]\to\mathbb{R}$ *be such that* $F\in C^{n+1}[\alpha,\beta]$ *for* $n\in\mathbb{N}$ *and* $F^{(n+1)}\geq 0$ *on* $[\alpha,\beta]$, *let* $x:[a,b]\to\mathbb{R}$ *continuous functions such that* $x([a,b])\subseteq[\alpha,\beta]$, $\lambda:[a,b]\to\mathbb{R}$ *be as defined in Remark* 1 *or in Remark* 3, $\overline{x}=\frac{\int_a^b x(t)\,d\lambda(t)}{\int_a^b d\lambda(t)}$ *and let the function* $\tilde{\Upsilon}$ *be defined in* (5.5). *Then we have the representation* (5.8) *and the remainder* $\tilde{H}_n^1(F;\alpha,\beta)$ *satisfies the bound*

$$| \tilde{H}_n^1(F;\alpha,\beta) |\leq \frac{\|\Upsilon'\|_\infty}{2}\left\{(\beta-\alpha)\left[F^{(n-1)}(\beta)+F^{(n-1)}(\alpha)\right]-\left[F^{(n-2)}(\beta)-F^{(n-2)}(\alpha)\right]\right\}.$$

We also give the Ostrowsky type inequality related to the generalization of majorization inequality.

Theorem 16. *Let* $-\infty<a\leq a_1<a_2\cdots<a_r\leq b<\infty$ $r\geq 2$ *be the given points, let* $\mathbf{x}=(x_1,...,x_m)$ *and* $\mathbf{w}=(w_1,...,w_m)$ *be* m*-tuples such that* $x_i\in[a,b]$, $w_i\in\mathbb{R}$, $i=1,...,m$, $W_m=\sum_{i=1}^m w_i$, $\overline{x}=\frac{1}{W_m}\sum_{i=1}^m w_ix_i$. *Let* (p,q) *be a pair of conjugate exponents, that is* $1\leq p,q\leq\infty$ *and* $\frac{1}{p}+\frac{1}{q}=1$ *and let* $F\in C^n[a,b]$. *Also, let* H_{lj} *and* Υ *be as defined in* (1.9) *and* (5.4) *respectively.*
Then we have

$$\left|\frac{1}{W_m}\sum_{i=1}^m w_iF(x_i)-F(\overline{x})\right. \tag{5.11}$$

$$\left.-\int_a^b\left[\frac{1}{W_m}\sum_{i=1}^m w_iG(x_i,s)-G(\overline{x},s)\right]\sum_{j=1}^r\sum_{l=0}^{k_j}F^{(l+2)}(a_j)H_{lj}(s)ds\right|$$

$$\leq ||F^{(n)}||_p||\Upsilon||_q.$$

The constant on the right hand side of (5.11) *is sharp for* $1<p\leq\infty$ *and the best possible for* $p=1$.

Proof. Using the identity (2.1) and applying Hölder's inequality we obtain

$$\begin{aligned}&\left|\frac{1}{W_m}\sum_{i=1}^{m} w_i F(x_i) - F(\overline{x})\right.\\ - &\left.\int_a^b \left[\frac{1}{W_m}\sum_{i=1}^{m} w_i G(x_i,s) - G(\overline{x},s)\right]\sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j)H_{lj}(s)ds\right|\\ = &\left|\int_a^b \Upsilon(t)F^{(n)}(t)dt\right| \le ||F^{(n)}||_p||\Upsilon||_q.\end{aligned}$$

For the proof of the sharpness of the constant $||\Upsilon||_q$ let us find a function F for which the equality in (5.11) is obtained.
For $1 < p < \infty$ take F to be such that

$$F^{(n)}(t) = \operatorname{sgn}\Upsilon(t)\,|\Upsilon(t)|^{\frac{1}{p-1}}.$$

For $p = \infty$ take $F^{(n)}(t) = \operatorname{sgn}\Upsilon(t)$.
For $p = 1$ we prove that

$$\left|\int_a^b \Upsilon(t)F^{(n)}(t)dt\right| \le \max_{t\in[a,b]}|\Upsilon(t)|\left(\int_a^b \left|F^{(n)}(t)\right|dt\right) \tag{5.12}$$

is the best possible inequality. Suppose that $|\Upsilon(t)|$ attains its maximum at $t_0 \in [a,b]$. First we assume that $\Upsilon(t_0) > 0$. For ε small enough we define $F_\varepsilon(t)$ by

$$F_\varepsilon(t) = \begin{cases} 0, & a \le t \le t_0,\\ \frac{1}{\varepsilon\, n!}(t-t_0)^n, & t_0 \le t \le t_0+\varepsilon,\\ \frac{1}{(n-1)!}(t-t_0)^{n-1}, & t_0+\varepsilon \le t \le b.\end{cases}$$

Then for ε small enough

$$\left|\int_a^b \Upsilon(t)F^{(n)}(t)dt\right| = \left|\int_{t_0}^{t_0+\varepsilon}\Upsilon(t)\frac{1}{\varepsilon}dt\right| = \frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\Upsilon(t)dt.$$

Now from the inequality (5.12) we have

$$\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\Upsilon(t)dt \le \Upsilon(t_0)\int_{t_0}^{t_0+\varepsilon}\frac{1}{\varepsilon}dt = \Upsilon(t_0).$$

Since

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\Upsilon(t)dt = \Upsilon(t_0)$$

the statement follows. In the case $\Upsilon(t_0) < 0$, we define $F_\varepsilon(t)$ by

$$F_\varepsilon(t) = \begin{cases} \frac{1}{(n-1)!}(t-t_0-\varepsilon)^{n-1}, & a \le t \le t_0,\\ -\frac{1}{\varepsilon n!}(t-t_0-\varepsilon)^n, & t_0 \le t \le t_0+\varepsilon,\\ 0, & t_0+\varepsilon \le t \le b,\end{cases}$$

and the rest of the proof is the same as above.

□

Integral version of the above theorem can be stated as:

Theorem 17. *Let* $-\infty < \alpha \leq a_1 < a_2 \cdots < a_r \leq \beta < \infty$, $r \geq 2$ *be the given points, let* $x : [a,b] \to \mathbb{R}$ *continuous functions such that* $x([a,b]) \subseteq [\alpha,\beta]$, $\lambda : [a,b] \to \mathbb{R}$ *be as defined in Remark 1 or in Remark 3 and* $\overline{x} = \frac{\int_a^b x(t)\, d\lambda(t)}{\int_a^b d\lambda(t)}$. *Let* (p,q) *be a pair of conjugate exponents, that is* $1 \leq p, q \leq \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Let* $F \in C^n[\alpha,\beta]$ *and let the* H_{lj} *and* $\tilde{\Upsilon}$ *be defined in* (1.9) *and* (5.5).
Then we have

$$\begin{aligned} (5.13) \qquad & \left| \frac{\int_a^b F(x(t))\, d\lambda(t)}{\int_a^b d\lambda(t)} - F(\overline{x}) \right. \\ & - \left. \int_\alpha^\beta \left[\frac{\int_a^b G(x(t),s)\, d\lambda(t)}{\int_a^b d\lambda(t)} - G(\overline{x},s) \right] \sum_{j=1}^{r} \sum_{l=0}^{k_j} F^{(l+2)}(a_j) H_{lj}(s) ds \right| \\ & \leq ||F^{(n)}||_p ||\tilde{\Upsilon}||_q. \end{aligned}$$

The constant on the right hand side of (5.13) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

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