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ON THE EIGENVALUES OF A MATRIX REPRESENTING EQUIVALENCE RELATION

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ABSTRACT. We consider a matrix that represents an equivalence relation defined on a finite set. We find the explicit, factorized, form of its characteristic polynomial, which reveals very interesting and meaningful values of its eigenvalues. We give practical examples in which our theoretical results can be applied.

1. INTRODUCTION

Given the set $A = \{x_1, x_2, \ldots, x_n\}$. Any subset α from the set $A \times A$ is called *relation on A*. A relation α is called ([4]):

- *reflexive*, if $(x_i, x_i) \in \alpha$ for every $x_i \in A$;
- *symmetric*, if $(x_i, x_j) \in \alpha$, whenever $(x_j, x_i) \in \alpha$;
- *transitive*, if whenever $(x_i, x_j) \in \alpha$ and $(x_j, x_k) \in \alpha$, then $(x_i, x_k) \in \alpha$, and
- *equivalence relation*, if it is reflexive, symmetric and transitive.

When an equivalence relation α is defined on a set, there is a natural grouping of the set's elements into, what are called, equivalence classes of α. Formally, the equivalence class of an element $x_i \in A$, is the set

$$[x_i]_\alpha = \{x \in A \,|\, (x_i, x) \in \alpha\}.$$

Note that when there is no ambiguity, the index α in the notation for equivalence class can be omitted. The equivalence classes of two different elements of A are either identical or disjoint, i.e. they form a partition on A,

$$A = [x_{i_1}] \cup [x_{i_2}] \cup \ldots \cup [x_{i_k}], \quad k \leq n.$$

Any relation α can be represented by a matrix, so called adjacency matrix. The adjacency matrix $\mathbf{A}_\alpha = [a_{ij}]_{n \times n}$ of a relation α on a set A, is defined as follows,

$$a_{ij} = \begin{cases} 1, & (x_i, x_j) \in \alpha \\ 0, & (x_i, x_j) \notin \alpha \end{cases}, \quad i, j = 1, 2, \ldots, n.$$

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The adjacency matrix of an equivalence relation α, is a 0-1 matrix with a specific form: it contains ones on the principal diagonal (due to the reflexivity property) and it is symmetric (due to the symmetricity property). Also, due to the existence of the equivalence classes, which are k in total, the matrix $\mathbf{A}_\alpha$ contains k different rows (columns).

The scientific community works few decades on the problem of the eigenvalues of graphs, which actually reduces to the problem of eigenvalues of the graphs' adjacency matrices. For example, Hong in [1] states the bounds of the eigenvalues of graphs, while Rojo in [3] states the improved bounds of the largest eigenvalue of trees. Walker in [6] works on the bounds of the largest eigenvalue of a symmetric matrix. We also work with symmetric matrix, but of special type: a matrix that represents an equivalence relation. Exploring the properties of such matrix was initiated in [5], then continued in this paper.

2. The eigenvalues of the adjacency matrix of an equivalence relation

Let α be an equivalence relation defined on the set $A = \{x_1, x_2, \ldots, x_n\}$ and $\mathbf{A}_\alpha$ be its adjacency matrix. Let c_j be the cardinality of the class $[x_{i_j}]$, i.e., $c_j = |[x_{i_j}]|$, $j = 1, 2, \cdots, k$. Note that $c_1 + c_2 + \cdots + c_k = n$. We will denote by $\mathbf{1}_m$ the $m \times m$ unity matrix (the matrix whose all entries are ones), and by $diag\{\mathbf{D}_1, \mathbf{D}_2, \ldots, \mathbf{D}_l\}$ the block diagonal matrix, whose nonzero blocks $\mathbf{D}_1, \mathbf{D}_2, \ldots, \mathbf{D}_l$ are placed on the principal diagonal. The set of eigenvalues of the matrix $\mathbf{A}_\alpha$, i.e. the spectrum of $\mathbf{A}_\alpha$ will be denoted by $\sigma(\mathbf{A}_\alpha)$. The eigenvalue with the highest absolute value, i. e. the spectral radius, will be denoted by $\rho(\mathbf{A}_\alpha)$.

Lemma 1. *$\mathbf{A}_\alpha$ is similar to the block – matrix $\overline{\mathbf{A}}_\alpha = diag\{\mathbf{1}_{c_1}, \mathbf{1}_{c_2}, \ldots, \mathbf{1}_{c_k}\}$.*

Proof. As said in the introductory part, $\mathbf{A}_\alpha$ contains k groups of identical rows (columns). The identical rows and columns can be rearranged to be placed one next to another, forming blocks of ones on the principal diagonal. Such rearrangement of the matrix $\mathbf{A}_\alpha$ can be performed by multiplication with a matrix $\mathbf{P}^{-1}$ from left and with $\mathbf{P}$ from right, where $\mathbf{P}$ is a product of appropriate permutation matrices. That is,

$$\overline{\mathbf{A}}_\alpha = \mathbf{P}^{-1}\mathbf{A}_\alpha\mathbf{P}.$$

□

Remark 1. The block $\mathbf{1}_{c_j}$ corresponds to the equivalence class $[x_{i_j}]$. The matrix $\overline{\mathbf{A}}_\alpha = diag\{\mathbf{1}_{c_1}, \mathbf{1}_{c_2}, \ldots, \mathbf{1}_{c_k}\}$ is an adjacency matrix of the relation α defined on the rearranged set A, obtained when elements of the class $[x_{i_1}]$ are first listed, followed by the elements of the class $[x_{i_2}]$, and so on, ending with the elements of the class $[x_{i_k}]$.

Example 1. Let $A = \{x_1, x_2, x_3, x_4, x_5\}$ and $\alpha \subset A \times A$ be an equivalence relation defined on A such that $[x_1] = \{x_1, x_4\}$, $[x_2] = \{x_2, x_3, x_5\}$. Then

$$\mathbf{A}_\alpha = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Also,

$$\overline{\mathbf{A}}_\alpha = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} = \mathbf{P}_{2,4}^{-1}\mathbf{A}_\alpha \mathbf{P}_{2,4}.$$

In the last equation, the matrix $\mathbf{P}_{2,4}$ is the permutation matrix obtained when the second and fourth rows in the 5×5 identity matrix are interchanged. In other words, the matrix $\overline{\mathbf{A}}_\alpha$ represents the same relation α, but on the rearranged set, $\{x_1, x_4, x_3, x_2, x_5\}$.

The results in the following lines will help us in finding the characteristic polynomial of the matrix representing equivalence relation.

For $m \in \mathbb{N}$ and $s \in \mathbb{R}$ we will define define two functional determinants of order m, $B_m(s)$ and $D_m(s)$, by

$$B_m(s) = \begin{vmatrix} s & -1 & -1 & -1 & \dots & -1 \\ -1 & s & -1 & -1 & \dots & -1 \\ -1 & -1 & s & -1 & \dots & -1 \\ & & & & \ddots & \\ -1 & -1 & -1 & -1 & \dots & s \end{vmatrix}$$

and

$$D_m(s) = \begin{vmatrix} -1 & -1 & -1 & -1 & \dots & -1 \\ -1 & s & -1 & -1 & \dots & -1 \\ -1 & -1 & s & -1 & \dots & -1 \\ & & & & \ddots & \\ -1 & -1 & -1 & -1 & \dots & s \end{vmatrix}.$$

Using the Laplace expansion of both determinants along their first columns, we obtain

$$B_m(s) = sB_{m-1}(s) + (m-1)D_{m-1}(s),$$

$$D_m(s) = -B_{m-1}(s) + (m-1)D_{m-1}(s).$$

Using the results $B_1(s) = s$ and $D_1(s) = -1$ and applying the principle of mathematical induction we will obtain the following values of the determinants

(2.1) $$B_m(s) = (s+1)^{m-1}(s-m+1)$$

and

$$D_m(s) = -(s+1)^{m-1}.$$

Proposition 1. *The characteristic polynomial of the matrix $\mathbf{A}_\alpha$ is*

$$c_{\mathbf{A}_\alpha}(\lambda) = \lambda^{n-k}(\lambda - c_1)(\lambda - c_2)\dots(\lambda - c_k).$$

Proof. We will first find the characteristic polynomial of $\overline{\mathbf{A}}_\alpha$. The equation (2.1) will be useful here.

$$\begin{aligned}
&c_{\overline{\mathbf{A}}_\alpha}(\lambda) = \det(\lambda \mathbf{I} - \overline{\mathbf{A}}_\alpha) \\
&= B_{c_1}(\lambda-1) \cdot B_{c_2}(\lambda-1) \cdot \ldots \cdot B_{c_k}(\lambda-1) \\
&= \lambda^{c_1-1}(\lambda-c_1) \cdot \lambda^{c_2-1}(\lambda-c_2) \cdot \ldots \cdot \lambda^{c_k-1}(\lambda-c_k) \\
&= \lambda^{n-k}(\lambda-c_1)(\lambda-c_2)\ldots(\lambda-c_k).
\end{aligned}$$

Knowing that two similar matrices have identical characteristic polynomials and eigenvalues (see [2]) and using the lemma 1, we finally obtain the characteristic polynomial of the matrix that represents the equivalence relation α:

$$c_{\mathbf{A}_\alpha}(\lambda) = \lambda^{n-k}(\lambda-c_1)(\lambda-c_2)\ldots(\lambda-c_k).$$

□

In the following corollary, we will just highlight what obviously results from the proposition.

Corollary 1. (a) $\sigma(\mathbf{A}_\alpha) = \{0, c_1, c_2, \ldots, c_k\}$, *i.e., all the eigenvalues of the matrix* $\mathbf{A}_\alpha$ *are the cardinalities of the classes of equivalence and zero;*
(b) $rank(\mathbf{A}_\alpha) = k$;
(c) $\rho(\mathbf{A}_\alpha) = \max\{c_1, c_2, \ldots, c_k\}$, *i.e. the spectral radius of* $\mathbf{A}_\alpha$ *is equal to the highest cardinality of the classes of equivalence.*

Remark 2. Note that the cardinalities $c_1, c_2, \ldots, c_k$ need not to be different. In case that c_i is repeated j times, it means that the algebraic multiplicity of the eigenvalue c_i is j. The algebraic multiplicity of the eigenvalue 0 is the difference between the cardinality of the set A and the number of classes of the equivalence relation $\alpha \subseteq A \times A$, i.e. $|A| - k$.

3. Application of the results

The following example is taken from [4] and adjusted.

Example 2. Let the set A contains all the bit-strings of length 4, i.e.

$$\begin{aligned}
A &= \{1111, 0111, 1011, 1101, 1110, 1100, 1010, 1001, \\
&\quad 0110, 0101, 0011, 1000, 0100, 0010, 0001, 0000\} \\
&= \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}\}.
\end{aligned}$$

The relation α defined on A by: $(x, y) \in \alpha$ iff x and y agree in their first two bits, is an equivalence relation. The matrix that represents the relation α has the form

$$\mathbf{A}_\alpha = \begin{bmatrix} 1&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&1&1&0&0&1&0&0&0 \\ 0&0&1&0&0&0&1&1&0&0&0&1&0&0&0&0 \\ 1&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&1&1&0&0&0&1&0&0&0&0 \\ 0&0&1&0&0&0&1&1&0&0&0&1&0&0&0&0 \\ 0&1&0&0&0&0&0&0&1&1&0&0&1&0&0&0 \\ 0&1&0&0&0&0&0&0&1&1&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&1 \\ 0&0&1&0&0&0&1&1&0&0&0&1&0&0&0&0 \\ 0&1&0&0&0&0&0&0&1&1&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&1 \end{bmatrix}.$$

Its corresponding matrix $\overline{\mathbf{A}}_\alpha$ can be obtained by rearrangement of the rows and columns of the matrix $\mathbf{A}_\alpha$ (or by appropriately rearranging the set A and then defining the relation α on that set):

$$\overline{\mathbf{A}}_\alpha = \begin{bmatrix} 1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{bmatrix}.$$

The classes of equivalence of the relation $\alpha \subseteq A \times A$ are:

$$[1111] = \{1111, 1101, 1110, 1100\} = \{x_1, x_4, x_5, x_6\} = [x_1],$$

$$[0111] = \{0111, 0110, 0101, 0100\} = \{x_2, x_9, x_{10}, x_{13}\} = [x_2],$$

$$[1011] = \{1011, 1010, 1001, 1000\} = \{x_3, x_7, x_8, x_{12}\} = [x_3],$$

$$[0011] = \{0011, 0010, 0001, 0000\} = \{x_{11}, x_{14}, x_{15}, x_{16}\} = [x_{11}].$$

Without even computing the characteristic polynomial, we know that there are two eigenvalues of the matrix $\mathbf{A}_\alpha : \lambda_1 = 4$ with algebaric multiplicity 4 (since there are 4 classes of equivalence, each with cardinality 4) and $\lambda_2 = 0$ with algebraic multiplicity 8 (obtained when subtracting the number of different classes from the dimension of the matrix, that is 12-4). However, we are stating its characteristic polynomial in a factorized form

$$c_{\mathbf{A}_\alpha}(\lambda) = \lambda^{12}(\lambda - 4)^4.$$

Example 3. Furthermore, we will apply our results on the Voronoi diagram. Instead of partitioning the real plane, we will partition the finite set $X = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{R}^2$. Let the set of points (generators) be $P = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} \subset X$. We will use the Euclidean metric d_E to define the corresponding Voronoi cells $R[x_{i_l}]$, for $l = 1, 2, \ldots, k$,

$$x_j \in R[x_{i_l}] \quad \Leftrightarrow \quad d_E(x_j, x_{i_l}) = \min\{d_E(x_j, x_{i_m}) : m = 1, 2, ..., k\}.$$

We will define a relation α on X with

$$(x, y) \in \alpha \quad \Leftrightarrow \quad x \text{ and } y \text{ are in the same Voronoi cell.} \tag{3.1}$$

The relation α is reflexive, symmetric and transitive, therefore equivalence relation. Its adjacency matrix has $k + 1$ eigenvalues. One of them is 0, the rest are $|R[x_{i_l}]|$, that is, the number of points of X belonging to the region $R[x_{i_l}]$, $l = 1, 2, \ldots, k$.

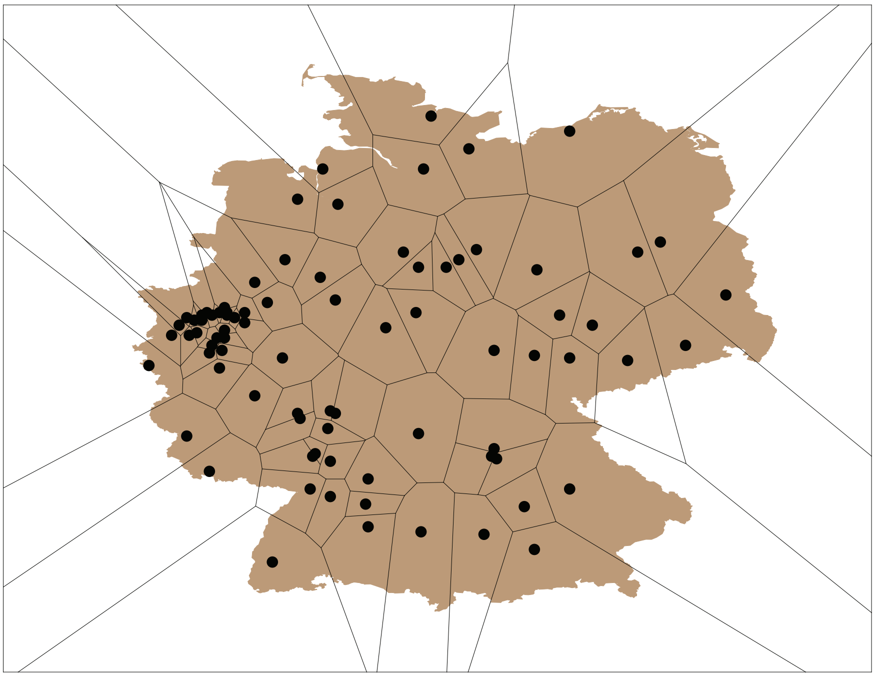

FIGURE 1. The black points and their Voronoi cells represent the network of the bigger cities in Germany and their closest surroundings.

Figure 1 shows possible division of a country (Germany) to regions, where the dots are locations of the generator cities, i.e. some representative cities in the regions. It is created using Wolfram Mathematica 10 Software. The eigenvalues of the adjacency matrix of the relation defined as in (3.1), are the numbers of cities in the regions and zero. The multiplicities of each eigenvalue depends on whether its corresponding region has a unique cardinality, as described in the Remark 2. This connection enables the well developed theory for eigenvalues to be applied in the practical divisions of regions in general, done by means of the Voronoi diagram.

4. Conclusion

The relation of equivalence α, defined on a set A, represents many different kinds of partitioning of a particular set. In this work we explicitly find the characteristic polynomial of an adjacency matrix of an equivalence relation. We thus show an elegant property of that matrix: its spectrum is composed of zero and the cardinalities of the equivalence classes. The algebraic multiplicity of the nonzero eigenvalues, i.e. the nonzero cardinality of a particular class, coincides with the number of classes with the same cardinality. The algebraic multiplicity of the zero eigenvalue can be got when the total number of classes of α is subtracted from the number of elements in the set A. We apply the results on few examples. They can successfully be applied in graph theory, for graphs representing equivalence relations.

The theory developed in this paper enables finding a matrix whose eigenvalues will be predefined nonnegative integers.

Our future work will be directed in finding the set of eigenvectors and extending the application of the results.

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