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ON APPROXIMATION PROPERTIES OF NON-CONVOLUTION TYPE OF INTEGRAL OPERATORS FOR NON-INTEGRABLE FUNCTION

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Abstract. In this paper, we study the problem of pointwise convergence of non-convolution type of integral operators family at Lebesgue point when f is non-integrable functions. In this study third theorems are proved on the operator convergence to $f(x)$. It is examined separately for finite and infinite intervals.

1. Introduction

In [5] Mamedov studied various results pointwise convergence and on the order of convergence at generalized lebesgue points and lebesgue points by a family of non-convolution type singular integrals operators of the form

$$T_\lambda(f,x)=\int_{-\infty}^{\infty} f(t)\Phi_\lambda(x,t)dt\,,$$

in L_p space. After, Mamedov in [4] give proofs of many on the rate of convergence of the form

$$\int_a^b f(t)K_\lambda(x,t)dt,\ x\in(a,b)\,,$$

as $\lambda\to\infty$ for $f\in L_1(a,b)$, $f\in L_1(a,\infty)$ and $f\in L_p(a,b)$. And he shown similar results for functions of several variables. Similarly, Gadjiev [3] investigated that pointwise convergence and on the order of above the operator for f is bounded and differentiable.

In [1] Bardaro and Vinti studied convergence of integral operators $T_n(f)$ defined by

$$T_n(f)(s)=\int_\Omega f(t)K(n,s,t)dt\,,$$

where Ω is an open subset of R^m concerning certain variational functionals.
In [2] the authors given some approximation theorems with respect to pointwise convergence and the rate of pointwise convergence for non-convolution type lınear operators of

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the form

$$T_\lambda(f,x)=\int_0^\infty f(z)L_\lambda(x,z)\frac{dz}{z},\quad x>0\,,$$

with kernel satisfying some general homogeneity assumptions. In [6] the authors obtained pointwise convergence and the rates of certain non-integrable functions f by double singular integral operators with radial kernel on $D=\langle-\pi,\pi\rangle\times\langle-\pi,\pi\rangle$, at generalized Lebesgue point.

In this paper, we investigate the pointwise convergence of $L_\lambda(f,x)$ to $f(x)$ in $L_1(a.b)$ at lebesgue points by family of non-convolution type singular integral operators depending on two parameters of the form:

$$L_\lambda(f;x)=\int_a^b f(t)K_\lambda(t;x)dt, x\in(a,b)\,. \tag{1.1}$$

here f is non-integrable, $f\notin L_1(a.b)$

First, we give the following definition.

Definition 1 (Class A). *We take a family $F=(K_\lambda)_{\lambda\in\Lambda}$ of functions $K_\lambda(t;x):R\times\Lambda\to R$, with K_λ is non-negative. We will say that the function $K_\lambda(t;x)$ belongs to class A, if the following condition are satisfied.*

a) *$K_\lambda(t;x)$ is a function defined for all $x,t\in(a,b)$ and $\lambda\in\Lambda$.*

b) $\lim\limits_{\lambda\to\infty}\int_a^b K_\lambda(t;x)dt=1,\ \lambda>0,\ a\le x\le b$

c) *As functions of t, $K_\lambda(t;x)$ is non-decreasing on $[a,x]$ and non-increasing on $[x,b]$.*

d) $\lim\limits_{\lambda\to\infty}K_\lambda(x\pm\delta;x)dt=0$, *for chossen $x\in[a,b]$ and $\delta>0$.*

2. Approximation

In this section some results on the approximation for non-convolution of integral operators for non-integrable functions is obtained. Let $\rho\in L_1(a,b)$ and E_ρ is Lebesgue points set of ρ. Then, the operator

$$L_\lambda(\rho;x)=\int_a^b \rho(t)K_\lambda(t;x)dt,\ a\le x\le b,\ \lambda>0\,,$$

is convergence to $\rho(x)$ as $\lambda\to\infty$, with $x\in E_\rho$, if the functions $K_\lambda(t;x)$ belong to class A, see [5]. Let $E_{\frac{f}{\rho}}$ be the set of Lebesgue points of $\frac{f}{\rho}$. Here $E=E_\rho\cap E_{\frac{f}{\rho}}$ is the set of Lebesgue points both f and ρ. Now we will give the main theorems .

Theorem 1. *Suppose that $\dfrac{f}{\rho}\in L_1(a,b)$ for $\rho>0$, $f\notin L_1(a,b)$ and that the non-negative function K_λ be the element of the class A. Moreover, if ρ and K_λ be almost everywhere differentiable and for every λ and x, the condition*

$$\rho'(t)\frac{\partial}{\partial t}K_\lambda(t;x)>0\,, \tag{2.1}$$

holds. Then, the operators $L_\lambda(f;x)$, which are defined in (1.1),

$$\lim_{\lambda\to\infty}L_\lambda(f;x)=f(x)\,,$$

holds for at every $x-$ Lebesgue point.

Note that when $\rho(t) = 1$ for $f \in L_1$, i.e., f belongs space L_1, the condition (2.1) is not needed.

Proof. If the statement (1.1) is multiplied and divided by $\rho(t)$, we have:

$$L_\lambda(f;x) = \int_a^b \frac{f(t)}{\rho(t)} \rho(t) K_\lambda(t;x)dt\,,$$

and we can write

$$\begin{aligned} L_\lambda(f;x) - f(x) &= \int_a^b \left\{ \frac{f(t)}{\rho(t)} - \frac{f(x)}{\rho(x)} \right\} (\rho(t)K_\lambda(t;x))dt \\ &\quad + \frac{f(x)}{\rho(x)} \left\{ \int_a^b \rho(t)K_\lambda(t;x)dt - \rho(x) \right\} . \end{aligned} \tag{2.2}$$

On the order hand, since $\dfrac{f}{\rho} \in L_1(a,b)$ and x is a Lebesgue point of $\dfrac{f}{\rho}$, the statements

$$\lim_{\lambda\to\infty} \frac{1}{h} \int_0^h \left| \frac{f(x+t)}{\rho(x+t)} - \frac{f(x)}{\rho(x)} \right| dt = 0$$

and

$$\lim_{\lambda\to\infty} \frac{1}{h} \int_0^h \left| \frac{f(x-t)}{\rho(x-t)} - \frac{f(x)}{\rho(x)} \right| dt = 0$$

hold. Then, for every $\varepsilon > 0$, $\exists \rho > 0$, $\forall h \le \delta$, we have

$$\frac{1}{h} \int_0^h \left| \frac{f(x+t)}{\rho(x+t)} - \frac{f(x)}{\rho(x)} \right| dt < \epsilon h\,, \tag{2.3}$$

and

$$\frac{1}{h} \int_0^h \left| \frac{f(x-t)}{\rho(x-t)} - \frac{f(x)}{\rho(x)} \right| dt < \epsilon h\,. \tag{2.4}$$

For a chosen δ and since $K_\lambda(t,x)$ is positive, we can write the equation (2.2) as follows:

$$\begin{aligned} |L_\lambda(f;x) - f(x)| &\le \left\{ \int_a^{x-\delta} + \int_{x-\delta}^{x} + \int_x^{x+\delta} + \int_{x+\delta}^{b} \right\} \left| \frac{f(t)}{\rho(t)} - \frac{f(x)}{\rho(x)} \right| \rho(t)K_\lambda(t;x)dt \\ &\quad + \left| \frac{f(x)}{\rho(x)} \right| \left| \int_a^b \rho(t)K_\lambda(t;x)dt - \rho(x) \right| \\ &= A_{1,\lambda} + A_{2,\lambda} + A_{3,\lambda} + A_{4,\lambda} + A_{5,\lambda} \end{aligned}$$

It is sufficent to show that the terms on the right hand side of the last inequality tends to zero as $\lambda \to \infty$. Now, we will calculate the integrals $A_{1,\lambda}$, $A_{2,\lambda}$, $A_{3,\lambda}$, $A_{4,\lambda}$ and $A_{5,\lambda}$.

Since $\rho \in L_1(a,b)$, K_λ is an element of class A and $x \in E_\rho$, we know (see [5]):

$$\lim_{\lambda\to\infty}\int_a^b \rho(t)K_\lambda(t;x)dt = \rho(x).$$

It is easy seen that

$$\lim_{\lambda\to\infty} A_{5,\lambda} = 0\,.$$

Next, we consider $A_{1,\lambda}$ and $A_{4,\lambda}$:

$$A_{1,\lambda} = \int_a^{x-\delta} \left|\frac{f(t)}{\rho(t)} - \frac{f(x)}{\rho(x)}\right| \rho(t)K_\lambda(t;x)dt\,.$$

From the triangle inequality, we can write:

$$A_{1,\lambda} \le \int_a^{x-\delta} \left|\frac{f(t)}{\rho(t)}\right|\rho(t)K_\lambda(t;x)dt + \left|\frac{f(x)}{\rho(x)}\right| \int_a^{x-\delta} \rho(t)K_\lambda(t;x)dt\,.$$

From condition c) and (2.1), we get:

$$\begin{aligned} A_{1,\lambda} &< \rho(x-\rho)K_\lambda(x-\rho;x)\left\{\int_a^{x-\delta}\left|\frac{f(t)}{\rho(t)}\right|dt + \left|\frac{f(x)}{\rho(x)}\right|\int_a^{x-\delta} dt\right\} \\ &\le \rho(x-\rho)K_\lambda(x-\rho;x)\left\{\int_a^{b}\left|\frac{f(t)}{\rho(t)}\right|dt + \left|\frac{f(x)}{\rho(x)}\right|\int_a^{b} dt\right\} \\ &\le \rho(x-\rho)K_\lambda x-\rho;x)\left\{\left\|\frac{f}{\rho}\right\|_{L_1} + \left|\frac{f(x)}{\rho(x)}\right|(b-a\right\}. \end{aligned}$$

In the same way, we find that

$$\begin{aligned} A_{4,\lambda} &< \rho(x+\rho)K_\lambda x+\rho;x)\left\{\int_{x+\delta}^{b}\left|\frac{f(t)}{\rho(t)}\right|dt + \left|\frac{f(x)}{\rho(x)}\right|\int_{x+\delta}^{b} dt\right\} \\ &\le \rho(x+\rho)K_\lambda(x+\rho;x)\left\{\left\|\frac{f}{\rho}\right\|_{L_1} + \left|\frac{f(x)}{\rho(x)}\right|(b-a\right\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} A_{1,\lambda} + A_{4,\lambda} &< \left\{\left\|\frac{f}{\rho}\right\|_{L_1} + \left|\frac{f(x)}{\rho(x)}\right|(b-a\right\}\rho(x-\rho)K_\lambda(x-\rho;x) \\ &\quad + \left\{\left\|\frac{f}{\rho}\right\|_{L_1} + \left|\frac{f(x)}{\rho(x)}\right|(b-a\right\}\rho(x+\rho)K_\lambda(x+\rho;x). \end{aligned}$$

By using the condition d), we obtain that

$$\lim_{\lambda\to\infty}(A_{1,\lambda} + A_{4,\lambda}) = 0. \tag{2.5}$$

Next, we consider $A_{2,\lambda}$. Let us define the function

$$\Phi(t) = \int_0^t \left| \frac{f(x-u)}{\rho(x-u)} - \frac{f(x)}{\rho(x)} \right| du \, .$$

Then

$$d\Phi(t) = \left| \frac{f(x-t)}{\rho(x-t)} - \frac{f(x)}{\rho(x)} \right| dt \, . \tag{2.6}$$

From (2.4) and $t \leq \delta$, the inequality

$$\Phi(t) < \epsilon t \, . \tag{2.7}$$

holds. Thus, for the integral $A_{2,\lambda}$ with an appropriate transformation ($t = x - u$ and then $u = t$), we get:

$$A_{2,\lambda} = \int_0^\delta \left| \frac{f(x-t)}{\rho(x-t)} - \frac{f(x)}{\rho(x)} \right| \rho(x-t) K_\lambda(x-t;x) dt \, .$$

By using (2.6) we can write:

$$A_{2,\lambda} = \int_0^\delta \rho(x-t) K_\lambda(x-t;x) d\Phi(t) \, .$$

By partial integration and by using the fact that $K_\lambda(t,x)$ is positive, we obtain the following inequality:

$$A_{2,\lambda} \leq \Phi(\delta)\rho(x-\delta)K_\lambda x - \delta; x) + \int_0^\delta \Phi(t) d_t \left(\rho(x-t) K_\lambda x - t; x) \right) .$$

Then, since the derivative of $\rho(x-t)K_\lambda(x-t;x)$ is positive, the function is increasing. Therefore, from (2.7) the inequality:

$$A_{2,\lambda} \leq \epsilon\delta\rho(x-\delta)K_\lambda(x-\delta;x) + \epsilon \int_0^\delta t d_t \left(\rho(x-t)K_\lambda(x-t;x) \right)$$

is satisfied. By partial integration again, we obtain the inequality

$$A_{2,\lambda} \leq \epsilon \int_0^\delta \rho(x-t) K_\lambda(x-t;x) dt$$

and we get

$$A_{2,\lambda} \leq \epsilon \int_{x-\delta}^{x} \rho(t) K_\lambda(t;x) dt$$

($t = x - u$ and then $u = t$). When $\rho(t)$ and $K_\lambda(t;x)$ are positive, we have:

$$A_{2,\lambda} \leq \epsilon \int_a^b \rho(t) K_\lambda(t;x) dt \, .$$

We can use similar method for evaluating $A_{3,\lambda}$. Let

$$\Psi(t)=\int\limits_0^t\left|\frac{f(x+u)}{\rho(x+u)}-\frac{f(x)}{\rho(x)}\right|du\,,$$

then, it follows:

$$d\Psi(t)=\left|\frac{f(x+t)}{\rho(x+t)}-\frac{f(x)}{\rho(x)}\right|dt\,.$$

From (2.3) and by using $t\leq\delta$ we have:

$$\Psi(t)<\epsilon t\,. \tag{2.8}$$

After the transformations, first $t=x+u$ later $t=u$, we obtain:

$$A_{3,\lambda}=\int\limits_0^\delta\left|\frac{f(x+t)}{\rho(x+t)}-\frac{f(x)}{\rho(x)}\right|\rho(x+t)K_\lambda(x+t;x)dt\,.$$

From differentation of $\Psi(t)$, the equality

$$A_{3,\lambda}=\int\limits_0^\delta\rho(x+t)K_\lambda(x+t;x)d\Psi(t)\,,$$

is obtained. By using partial integration, we get:

$$A_{3,\lambda}\leq\Psi(\delta)\rho(x+\delta)K_\lambda(x+\delta;x)+\int\limits_0^\delta\Psi(t)d_t\left(\rho(x+t)K_\lambda(x+t;x)\right)\,.$$

From the conditions c) and (2.1), $\rho(x+t)K_\lambda(x+t;x)$ is decreasing on $[x,b]$. Thus $-\rho(x+t)K_\lambda(x+t;x)$ is increasing and its derivative is positive. Then, by using (2.8), it is obtained:

$$A_{3,\lambda}\leq\epsilon\delta\rho(x+\delta)K_\lambda(x+\delta;x)+\epsilon\int\limits_0^\delta td_t\left(-\rho(x+t)K_\lambda(x+t;x)\right)\,.$$

Again by applying the partial integration,

$$A_{3,\lambda}\leq\epsilon\int\limits_0^\delta\left(\rho(x+t)K_\lambda(x+t;x)\right)dt\,,$$

and it follows that:

$$A_{3,\lambda}\leq\epsilon\int\limits_x^{x-\delta}\left(\rho(t)K_\lambda t;x)\right)dt\,.$$

Since $\rho(t)$ and $K_\lambda(t;x)$ are positive, we find that:

$$A_{3,\lambda}\leq\epsilon\int\limits_a^b\rho(t)K_\lambda(t;x)dt\,,$$

and it is clear that:

$$A_{2,\lambda}+A_{3,\lambda}\leq2\epsilon\int\limits_a^b\rho(t)K_\lambda(t;x)dt\,.$$

On the other hand, from conditions of the theorem, we see that

$$\begin{aligned}\int_a^b \rho(t)K_\lambda(t;x)dt &= \int_a^x \rho(t)K_\lambda(t;x)dt + \int_x^b \rho(t)K_\lambda(t;x)dt \\ &\leq \rho(x)\int_a^b K_\lambda(t;x)dt + \rho(x)\int_a^b K_\lambda(t;x)dt \\ &= 2\rho(x)\int_a^b K_\lambda(t;x)dt\,.\end{aligned}$$

From the condition

$$\lim_{\lambda\to\infty}\int_a^b K_\lambda(t;x)dt = 1$$

this integration is bounded. Then, the integration

$$\int_a^b \rho(t)K_\lambda(t;x)dt\,,$$

is bounded. Thus

$$A_{2,\lambda} + A_{3,\lambda} < \epsilon C\,, \tag{2.9}$$

with C is a fixed. Combining (2.9) and (2.5), we get:

$$|l_\lambda(f,x) - f(t)| < \left\{\begin{array}{c} \left\{\left\|\frac{f}{\rho}\right\|_{L_1} + \left|\frac{f(x)}{\rho(x)}\right|(b-a)\right\}\rho(x-\rho)K_\lambda(x-\rho;x) \\ +\left\{\left\|\frac{f}{\rho}\right\|_{L_1} + \left|\frac{f(x)}{\rho(x)}\right|(b-a)\right\}\rho(x+\rho)K_\lambda(x+\rho;x) \\ +\epsilon C + \left|\frac{f(x)}{\rho(x)}\right|\left|\int_a^b \rho(t)K_\lambda(t;x)dt - \rho(x)\right|\,. \end{array}\right\}$$

Therefore our theorem now follows as $\lambda\to\infty$

$$\lim_{\lambda\to\infty} L_\lambda(f,x) = f(x)\,.$$

This completes the proof of the theorem. □

In this theorem, specially it may be $a=\infty$ and $b=\infty$. In this case, we can give the following theorem.

Theorem 2. *Let* $\dfrac{f}{\rho}\in L_1(-\infty,\infty)$ *with* $\rho>0, \rho\in L_1(-\infty,\infty)$. *Suppose that non-negative function* K_λ *be the element of the Class-A. For the chosen* $\delta>0$, *let conditions*

$$\lim_{\lambda\to\infty}\int_{-\infty}^{x-\delta} K_\lambda(t;x)dt = 0$$

and

$$\lim_{\lambda\to\infty}\int_{x+\delta}^{\infty} K_\lambda(t;x)dt = 0$$

be satisfied. Also if ρ and K_λ be almost everywhere differenable and for every λ and x, the condition (2.1) *holds, then*

$$\lim_{\lambda\to\infty} L_\lambda(f,x) = f(x),$$

at $x \in E$.

Proof. As in the proof of Theorem 1, we can write:

$$\begin{aligned}
|L_\lambda(f;x) - f(x)| &\le \left\{ \int_{-\infty}^{x-\delta} + \int_{x-\delta}^{x} + \int_{x}^{x+\delta} + \int_{x+\delta}^{\infty} \right\} \left| \frac{f(t)}{\rho(t)} - \frac{f(x)}{\rho(x)} \right| \rho(t) K_\lambda(t;x)dt \\
&\quad + \left| \frac{f(x)}{\rho(x)} \right| \left| \int_{-\infty}^{\infty} \rho(t) K_\lambda(t;x)dt - \rho(x) \right| \\
&= B_{1,\lambda} + B_{2,\lambda} + B_{3,\lambda} + B_{4,\lambda} + B_{5,\lambda}.
\end{aligned}$$

By using (2.1) and the condition c), we have

$$\begin{aligned}
B_{1,\lambda} &= \int_{-\infty}^{x-\delta} \left| \frac{f(t)}{\rho(t)} - \frac{f(x)}{\rho(x)} \right| \rho(t) K_\lambda(t;x)dt \\
&\le \rho(x-\rho) K_\lambda x - \rho; x) \left\| \frac{f}{\rho} \right\|_{L_1} + \left| \frac{f(x)}{\rho(x)} \right| \rho(x-\rho) \int_{-\infty}^{x-\delta} K_\lambda(t;x)dt \qquad (2.10)
\end{aligned}$$

and the inequality

$$B_{4,\lambda} < \rho(x+\rho) K_\lambda x + \rho; x) \left\| \frac{f}{\rho} \right\|_{L_1} + \left| \frac{f(x)}{\rho(x)} \right| \rho(x+\rho) \int_{x+\delta}^{\infty} K_\lambda(t;x)dt \qquad (2.11)$$

is obtained. From the condition c), the relations (2.10) and (2.11), we get:

$$\lim_{\lambda\to\infty} (B_{1,\lambda} + B_{4,\lambda}) = 0\,.$$

In the other hand, $B_{2,\lambda}$ and $B_{3,\lambda}$ are calculated as Theorem 1, i. e.,

$$B_{2,\lambda} + B_{3,\lambda} < \epsilon C\,.$$

Thus, we obtain:

$$\begin{aligned}
&\left\| \frac{f}{\rho} \right\| \{ \rho(x-\rho) K_\lambda(x-\rho;x) + \rho(x+\rho) K_\lambda(x+\rho;x) \} \\
&+ \left| \frac{f(x)}{\rho(x)} \right| \left\{ \rho(x-\rho) \int_{-\infty}^{x-\delta} K_\lambda(t;x)dt + \rho(x+\rho) \int_{x+\delta}^{\infty} K_\lambda(t;x)dt \right\} \\
&+ \left| \frac{f(x)}{\rho(x)} \right| \left| \int_{-\infty}^{\infty} \rho(t) K_\lambda(t;x)dt - \rho(x) \right| \\
&+ \epsilon C\,.
\end{aligned}$$

By taking limit as $\lambda \to \infty$, from the hypothesis, since $K_\lambda(x \pm \delta; x) \to 0$, from

$$\int_{-\infty}^{x-\delta} K_\lambda(t;x)dt \to 0$$

and

$$\int_{x+\delta}^{\infty} K_\lambda(t;x)dt \to 0\,,$$

we have:

$$\lim_{\lambda\to\infty} L_\lambda(f,x) = f(x)\,.$$

This completes the proof. $\square$

Example 1. Let

$$\rho(t) = \frac{1}{(1+|t|)\sqrt{|t|}} = \left\{ \begin{array}{c} \frac{1}{(1+t)\sqrt{t}}, t>0 \\ 1, t=0 \\ \frac{1}{(1-t)\sqrt{-t}}, t<0 \end{array} \right\}.$$

Is it $\rho \in L_1(-\infty,\infty)$? Since ρ is even, we have: $\int\limits_{-\infty}^{\infty} \rho(t)dt = 2\int\limits_{0}^{\infty} \rho(t)dt$. It is sufficent show that the integration $\int\limits_{0}^{\infty} \rho(t)dt$ is bounded.

$$\begin{aligned} \int_0^\infty \frac{dt}{(1+t)\sqrt{t}} &= \int_0^1 \frac{dt}{(1+t)\sqrt{t}} + \int_1^\infty \frac{dt}{(1+t)\sqrt{t}} \\ &= I_1 + I_2. \end{aligned}$$

First, we will consider I_1.

i) Since $t<1$ on $0<t<1$, the inequality

$$\begin{aligned} \sqrt{t} &< 1 \\ \sqrt{t}(1+t) &> \sqrt{t} \\ \frac{1}{(1+t)\sqrt{t}} &< \frac{1}{\sqrt{t}} \end{aligned}$$

is satisfied. Hence,

$$\begin{aligned} I_1 &= \int_0^1 \frac{dt}{(1+t)\sqrt{t}} < \int_0^1 \frac{dt}{\sqrt{t}} \\ &= \lim_{\mu\to 0}\int_\mu^1 \frac{dt}{\sqrt{t}} = 2 < \infty. \end{aligned}$$

ii) Taking $1 < t < \infty$, we can write

$$\begin{aligned} I_2 &= \int_1^{\infty} \frac{dt}{(1+t)\sqrt{t}} < \int_1^{\infty} \frac{dt}{t^{\frac{3}{2}}} \\ &= \lim_{\beta\to\infty} \int_1^{\beta} \frac{dt}{t^{\frac{3}{2}}} = 2 , \end{aligned}$$

and I_2 is bounded.
Thus $\rho \in L_1(-\infty,\infty)$.

Now, we will find that f function with $f \notin L_1(-\infty,\infty)$ and $\frac{f}{\rho} \in L_1(-\infty,\infty)$. We we consider the function

$$f(t) = \frac{1}{|t|\,(1+t^2)} = \left\{ \begin{array}{c} \frac{1}{t(1+t^2)}, t > 0 \\ 1, t = 0 \\ \frac{1}{-t(1+t^2)}, t < 0 \end{array} \right\}.$$

Now, let us see $\frac{f}{\rho} \in L_1(-\infty,\infty)$ or $f \notin L_1(-\infty,\infty)$. For $0 < t < 1$ it is easy to see that

$$\begin{aligned} 1+t^2 &< 2 \\ \frac{1}{1+t^2} &> \frac{1}{2} \\ \frac{1}{t(1+t^2)} &> \frac{1}{2t}. \end{aligned}$$

Therefore, since: $\int_0^1 \frac{dt}{t(1+t^2)} > \frac{1}{2}\int_0^1 \frac{dt}{t} = \infty$, it follows that $f \notin L_1(-\infty,\infty)$.

Moreover, is $\frac{f}{\rho} \in L_1(-\infty,\infty)$? We write: $\int_{-\infty}^{\infty} \frac{f(t)}{\rho(t)}dt = \left(\int_{-\infty}^{0} + \int_0^{\infty} \right) \frac{f(t)}{\rho(t)}dt = I_1 + I_2$.

From $\frac{f}{\rho}$ is even function, we get $\int_{-\infty}^{\infty} \frac{f(t)}{\rho(t)}dt = 2\int_0^{\infty} \frac{f(t)}{\rho(t)}dt$. Hence, we obtain

$$\begin{aligned} I_2 &= \int_0^{\infty} \frac{\sqrt{t}(1+t)}{t(1+t^2)}dt = \int_0^{\infty} \frac{(1+t)}{\sqrt{t}(1+t^2)}dt \\ &= \left(\int_1^{\infty} + \int_0^{1} \right) \frac{(1+t)}{\sqrt{t}(1+t^2)}dt \\ &< \int_0^{1} \frac{2}{\sqrt{t}}dt + 2\int_1^{\infty} \frac{t}{\sqrt{t}(1+t^2)}dt \\ &< 2\int_0^{1} \frac{1}{\sqrt{t}}dt + 2\int_1^{\infty} \frac{1}{t^{\frac{3}{2}}}dt. \end{aligned}$$

Then, we see that these integration are bounded. That is $\dfrac{f}{\rho} \in L_1(-\infty,\infty)$.
From the condition of Class A, since $K_\lambda(t,x)$ is increasing on $(-\infty,x]$ and from (2.1), also $\rho(t)$ is increasing on $(-\infty,x]$ and $K_\lambda(t,x)$ is decreasing on $[x,\infty)$, hence $\rho(t)$ is decreasing. Let $-\infty < t < 0$ (for the special case: $x=0$). From the definition of $\rho(t)$, we have:

$$\rho'(t) = \frac{\frac{1}{2\sqrt{-t}}(1-t)+\sqrt{-t}}{(\sqrt{-t}(1-t))^2} > 0.$$

Thus, $\rho(t)$ is increasing on $(-\infty,x]$. For $0<t<\infty$, we find that

$$\rho'(t) = \frac{-\frac{1}{2\sqrt{t}}(1+t)+\sqrt{t}}{(\sqrt{t}(1+t))^2} < 0,$$

i. e, $\rho(t)$ is decreasing on $[x,\infty)$. Now, we will take $K_\lambda(t,x) = \frac{\lambda}{\sqrt{\pi}}e^{-\lambda^2(t-x)^2}$ (kernel of Gaus-Weierstrass). It is evident that this functions non-negative for every $t \in (-\infty,\infty)$. Is it

$$\lim_{\lambda\to\infty}\int_{-\infty}^{\infty} K_\lambda(t,x)dt = \lim_{\lambda\to\infty}\int_{-\infty}^{\infty}\frac{\lambda}{\sqrt{\pi}}e^{-\lambda^2(t-x)^2} = 1?$$

Next, we will calculate. By using

$$\int_{-\infty}^{\infty} K_\lambda(t,x)dt = \int_{-\infty}^{\infty}\frac{\lambda}{\sqrt{\pi}}e^{-\lambda^2(t-x)^2}dt = \frac{\lambda}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\lambda^2u^2}du,$$

if the transformation $\lambda t = w$ is made, we get for the last integral:

$$\int_{-\infty}^{\infty} e^{w^2}dw = 1.$$

Hence, the condition b) is satisfied.
For $-\infty < t < x$ (or let $x=0$), -$\infty < t < 0$, we see that

$$\frac{\partial}{\partial t}K_\lambda(t,x) = \frac{\partial}{\partial t}(\frac{\lambda}{\sqrt{\pi}}e^{-\lambda^2t^2}) = \frac{-2\lambda^3t}{\sqrt{\pi}}e^{-\lambda^2t^2} > 0.$$

Therefore, it is increasing.
Second, for $0<t<\infty$ we get

$$\frac{\partial}{\partial t}K_\lambda(t,x) = \frac{-2\lambda^3t}{\sqrt{\pi}}e^{-\lambda^2t^2} < 0.$$

Then, $K_\lambda(t,x)$ is decreasing as t on $[0,\infty)$ .Finally, we will show that the condition of the Theorem 2 is satisfied. For every $t\neq x$, we must show that,

$$\lim_{\lambda\to\infty}K_\lambda(t,x) = 0.$$

We took in particular, $x=0$.Then for every $t\neq 0$,we get

$$\lim_{\lambda\to\infty}K_\lambda(t,x) = \lim_{\lambda\to\infty}\frac{\lambda}{\sqrt{\pi}}e^{-\lambda^2t^2} = 0.$$

As a result, the integration

$$L_\lambda(f,x) = \int_{-\infty}^{\infty}\frac{1}{|t|\,(1+t^2)}e^{-\lambda^2(t-x)^2}dt$$

convergence to $f(0)$ as $\lambda \to \infty$ for $x = 0$.

If we take $\rho(t) = \frac{1}{\rho_1(x)}$ in Theorem 1, then, we can give the following theorem.

Theorem 3. *Let* $\dfrac{1}{\rho_1} \in L_1, f\rho_1 \in L_1(a,b)$ *and* $f \notin L_1(a,b)$. *Suppose that non-negative function* K_λ *be the element of Class A. Also if* $\dfrac{1}{\rho_1}$ *and* K_λ *be almost everywhere differentiable and for every* λ *and* x, *the condition,*

$$\left(\frac{1}{\rho_1(t)}\right)' \frac{\partial}{\partial t} K_\lambda(t,x) > 0$$

holds. Then

$$\lim_{\lambda\to\infty} L_\lambda(f,x) = f(x)$$

at $x-$ *Lebesgue point of* $\rho_1 \in L_1$.

Proof. The proof is done similar to that of Theorem 1. □

References

[1] C.Bardaro, G.Vinti: *On approximation properties of certain non-convolution integral operators*, J.Approx.Theory **62**(3) (1990), 358–371.

[2] C.Bardaro, G.Vinti, H.Karsli: *On pointwise convergence of linear integral operators with homogeneous kernels*, Integral Trans. Spec. Funct., **19**(5-6) (2008), 429–439.

[3] A.D.Gajiev: *On the order of convergence of some class of singular integrals*, Izv. Acad. Sc. of Azerbaijan SSR, **6** (1963), 27–31.

[4] R.G.Mamedov: *A study of order of convergence of one-dimensional and multidimensional singular integrals*, Studies in theory of differential equations and theory of functions (in Russian), Izdat. Akad. Nauk Azerbaĭdžan. SSR, Baku, **41**(50) (1965), 92–108.

[5] R.G.Mamedov: *A generalization of some results on the order of convergence of singular integrals*, (in Russian) Akad. Nauk Azerbaĭdžan, SSR Dokl., **18**(3) (1962), 3–7.

[6] G.Uysal, M.M.Yilmaz: *Some theorems on the approximation of non-integrable functions via singular integral operators*, Proc. Jangjeon Math. Soc., **18**(2) (2015), 241–251.

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