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RESULTS INVOLVING GAUSSIAN ERROR FUNCTION $\mathrm{erf}(|x|^{1/2})$ AND THE NEUTRIX CONVOLUTION

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ABSTRACT. The Gaussian error function $\mathrm{erf}(x)$ and its associated functions $\mathrm{erf}(x_+)$ and $\mathrm{erf}(x_-)$ are defined. Further, the generalized Gaussian error function $\mathrm{erf}_i(x)$ and the associated functions $\mathrm{erf}_i(x_+)$ and $\mathrm{erf}_i(x_-)$ are defined. Some neutrix convolutions of these functions and other functions are evaluated.

1. Introduction

The error function (also known as Gaussian error function) $\mathrm{erf}(x)$ [19] is defined for $x \in \mathbb{R}$ by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du = \frac{2}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(2i+1)} x^{2i+1}.$$

The error function is odd, convex on $(-\infty, 0]$, concave on $[0, \infty)$, and strictly increasing on $\mathbb{R}$. We refer to the reader ref. [2, 3] for other properties of the error function.

The Gaussian error function plays an important role in statistics, probability theories and in problems stemming from mathematical physics, especially in analytic solutions for problems of thermo mechanics and mass flow due to diffusion. Dirschmid and Fischer extended the classical Gaussian error function $\mathrm{erf}(x)$ to a family of infinite extended Gaussian error functions $\mathrm{erf}_i(x)$ (for $i \geq 1$) which can be easily programmed by current computational tools. The generalized Gaussian error function for $i \in \mathbb{N}$ are defined by

$$\mathrm{erf}_i(x) = \frac{2}{\sqrt{\pi}} \int_0^x u^i e^{-u^2}\, du,$$

see [5].

It can be easily noted that

$$\begin{aligned}\lim_{x\to 0} \mathrm{erf}_i(x) &= 0, \qquad \lim_{x\to\infty} \mathrm{erf}_i(x) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} u^i e^{-u^2}\, du = \frac{1}{\sqrt{\pi}}\Gamma\Big(\frac{i+1}{2}\Big),\\ \lim_{x\to -\infty} \mathrm{erf}_i(x) &= \frac{2}{\sqrt{\pi}} \int_0^{-\infty} u^i e^{-u^2}\, du = \frac{(-1)^{i+1}}{\sqrt{\pi}}\Gamma\Big(\frac{i+1}{2}\Big),\end{aligned}$$

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see [13] for the calculation.

The locally summable functions $\operatorname{erf}(x_+)$ and $\operatorname{erf}(x_-)$ are defined by

$$\operatorname{erf}(x_+) = H(x)\operatorname{erf}(x), \quad \operatorname{erf}(x_-) = H(-x)\operatorname{erf}(x),$$

where H denotes Heaviside's function and we define

$$\operatorname{erf}(-x_+) = -\operatorname{erf}(x_-), \quad \operatorname{erf}(-x_-) = -\operatorname{erf}(x_+).$$

The functions $\operatorname{erf}(|x|^{1/2})$, $\operatorname{erf}(x_+^{1/2})$ and $\operatorname{erf}(|x|^{1/2})$ are similarly defined by

$$\begin{aligned}\operatorname{erf}(|x|^{1/2}) &= \frac{2}{\sqrt{\pi}}\int_0^{|x|^{1/2}} \exp(-u^2)\,du,\\ \operatorname{erf}(x_+^{1/2}) &= H(x)\operatorname{erf}(|x|^{1/2}), \quad \operatorname{erf}(x_-^{1/2}) = H(-x)\operatorname{erf}(|x|^{1/2}).\end{aligned}$$

Similarly, we define the locally summable functions $\operatorname{erf}_i(x_+)$ and $\operatorname{erf}_i(x_-)$ by

$$\operatorname{erf}_i(x) = H(x)\operatorname{erf}_i(x), \quad \operatorname{erf}_i(x_-) = H(-x)\operatorname{erf}_i(x),$$

Before proving our results on the convolution, we need the following lemma, which is easily proved by induction:

Lemma 1.

$$\begin{aligned}\operatorname{erf}_{2i}(x) &= \frac{2}{\sqrt{\pi}}\int_0^x u^{2i}e^{-u^2}\,du\\ &= -\sum_{j=0}^{i-1}\frac{(2i)!(i-j)!}{\sqrt{\pi}2^{2j}i!(2i-2j)!}x^{2i-2j-1}\exp(-x^2) + \frac{(2i)!}{2^{2i}i!}\operatorname{erf}(x),\\ \operatorname{erf}_{2i+1}(x) &= \frac{2}{\sqrt{\pi}}\int_0^x u^{2i+1}e^{-u^2}\,du\\ &= -\sum_{j=0}^{i}\frac{i!}{\sqrt{\pi}(i-j)!}x^{2i-2j}\exp(-x^2) + \frac{i!}{\sqrt{\pi}}\end{aligned}$$

for $i = 0, 1, 2, \ldots$, where the sum in first relation is empty when $i = 0$.

The classical definition of the convolution of two locally summable functions f and g is as follows:

Definition 1. *Let f and g be functions. Then the convolution $f * g$ is defined by*

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt$$

for all points x for which the integral exist.

It follows easily from the definition that if $f * g$ exists then $g * f$ exists and

$$f * g = g * f$$

and if $(f*g)'$ and $f*g'$ (or $f'*g$) exists, then

$$(f*g)' = f*g' \quad (\text{or } f'*g). \tag{1.1}$$

We now define the functions $\operatorname{erf}_{2i,+}(|x|^{1/2})$ and $\operatorname{erf}_{2i,-}(|x|^{1/2})$ by

$$\operatorname{erf}_{2i,+}(|x|^{1/2}) = H(x)\operatorname{erf}_{2i}(|x|^{1/2}), \quad \operatorname{erf}_{2i,-}(|x|^{1/2}) = H(-x)\operatorname{erf}_{2i}(|x|^{1/2}),$$

for $i = 0, 1, 2, \ldots$.

The following results were proved in [11].

$$\begin{aligned} x_+^r * \operatorname{erf}_+(x) &= \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i \operatorname{erf}_i(x) x_+^{r-i+1}, \\ x_+^r * [x\exp_+(-x^2)] &= \frac{\sqrt{\pi}}{2(r+1)}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^{-i}\operatorname{erf}_i(x)x_+^{r-i+2} + \\ &\quad + \frac{\sqrt{\pi}}{2(r+1)}\sum_{i=1}^{r+1}\binom{r+1}{i}(-1)^{-i-1}\operatorname{erf}_i(x)x_+^{r-i+1}, \end{aligned}$$

for $r = 0, 1, 2, \ldots$, where $\exp_+(-x^2) = H(x)\exp(-x^2)$.

We now prove

Theorem 1. *The convolution* $x_+^r * \operatorname{erf}(x_+^{1/2})$ *exists and*

$$x_+^r * \operatorname{erf}(x_+^{1/2}) = \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i \operatorname{erf}_{2i}(x_+^{1/2})x_+^{r-i+1}, \tag{1.2}$$

$r = 0, 1, 2, \ldots$.

Proof. We note that if $x < 0$, then $x_+^r * \operatorname{erf}(x_+^{1/2}) = 0$. If $x > 0$, we have

$$\begin{aligned} x_+^r * \operatorname{erf}(x_+^{1/2}) &= \frac{2}{\sqrt{\pi}}\int_0^x (x-t)^r \operatorname{erf}(t^{1/2})\,dt \\ &= \frac{2}{\sqrt{\pi}}\int_0^x (x-t)^r \int_0^{t^{1/2}} \exp(-u^2)\,du\,dt \\ &= \frac{2}{\sqrt{\pi}}\int_0^{x^{1/2}} \exp(-u^2)\int_{u^2}^x (x-t)^r\,dt\,du \\ &= \frac{2}{\sqrt{\pi}(r+1)}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i x^{r-i+1}\int_0^{x^{1/2}} u^{2i}\exp(-u^2)\,du \\ &= \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i\operatorname{erf}_{2i}(x^{1/2})x^{r-i+1}, \end{aligned}$$

on using Lemma 1 Equation (1.2) follows. □

Replacing x by $-x$ in equation (1.2) gives

Corollary 1. *The convolution* $x_-^r * \operatorname{erf}(x_-^{1/2})$ *exists and*

$$x_-^r * \operatorname{erf}(x_-^{1/2}) = \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i\operatorname{erf}_{2i}(|x|^{1/2})x_-^{r-i+1}, \tag{1.3}$$

$r = 0, 1, 2, \ldots$.

Corollary 2. *The convolution* $x_+^r * [x_+^{-1/2}\exp(-|x|)]$ *exists and*

$$x_+^r * [x_+^{-1/2}\exp(-|x|)] = \sqrt{\pi}\sum_{i=0}^{r}\binom{r}{i}(-1)^i\operatorname{erf}_{2i}(x_+^{1/2})x_+^{r-i}, \tag{1.4}$$

$r = 0, 1, 2, \ldots$.

Proof. Differentiating equation (1.2) and using equation (1.1), we get

$$\begin{aligned} x_+^r * [\operatorname{erf}(x_+^{1/2})]' &= \frac{1}{\sqrt{\pi}} x_+^r * [x_+^{-1/2}\exp(-|x|)] \\ &= [x_+^r * \operatorname{erf}_+(|x|^{1/2})]' \\ &= r x_+^{r-1} * \operatorname{erf}(x_+^{1/2}) \\ &= \sum_{i=0}^r \binom{r}{i} (-1)^i \operatorname{erf}_{2i}(|x|^{1/2}) x_+^{r-i} \end{aligned}$$

and equation (1.4) follows for $r = 1, 2, \ldots$.

In the particular case $r = 0$, we have

$$\frac{1}{\sqrt{\pi}} x_+^0 * [x_+^{-1/2}\exp(-|x|)] = \delta(x) * \operatorname{erf}(x_+^{1/2}) = \operatorname{erf}(x_+^{1/2}),$$

giving

$$x_+^0 * [x_+^{-1/2}\exp(-|x|)] = \sqrt{\pi}\operatorname{erf}(x_+^{1/2}).$$

This proved equation (1.4) for the case $r = 0$. □

Replacing x by $-x$ in equation (1.4) gives

Corollary 3. *The convolution* $x_-^r * [x_-^{-1/2}\exp(-|x|)]$ *exists and*

$$x_-^r * [x_-^{-1/2}\exp(-|x|)] = \sqrt{\pi}\sum_{i=0}^r \binom{r}{i} (-1)^i \operatorname{erf}_{2i}(x_-^{1/2}) x_-^{r-i},$$

$r = 0, 1, 2, \ldots$.

2. Neutrix Convolution

We now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$.

Definition 2. *The* convolution $f * g$ *of two distributions* f *and* g *in* $\mathcal{D}'$ *is defined by the equation*

$$\langle (f * g)(x), \varphi\rangle = \langle f(y), \langle g(x), \varphi(x+y)\rangle\rangle$$

for arbitrary φ *in* $\mathcal{D}$ *provided* f *and* g *satisfy either of the conditions:*

(B1) *either* f *or* g *has bounded support,*
(B2) *the supports of* f *and* g *are bounded on the same side,*

see Gel'fand and Shilov [12] (or [14]). Note that if f and g are locally summable functions satisfying either of the above conditions and the classical convolution $f * g$ exists, then it is in agreement with Definition 1.

Now let f and g be two distributions on $\mathbb{R}$ such that $K = \operatorname{Supp}(f)$ and $K' = \operatorname{Supp}(g)$ satisfying the following conditions:

(a) for every bounded set $B \subset \mathbb{R}$, the set $(K \times K') \cap B^{\triangle}$ is bounded in $\mathbb{R}^2$,
(b) for every bounded set $B \subset \mathbb{R}$, the set $K \cap (B - K')$ is bounded in $\mathbb{R}$,
(c) for every bounded set $B \subset \mathbb{R}$, the set $(B - K) \cap K'$ is bounded in $\mathbb{R}$,
(d) if $x_n \in K, y_n \in K'$ and $|x_n| + |y_n| \to \infty$, then $|x_n + y_n| \to \infty$,

where $B^{\triangle} = \{(x, y) \in \mathbb{R}^2 : x + y \in B\}$, then the convolution product $f * g$ of f and g exists and is defined as in Definition 2.

The condition (a)-(d) are well known see [9, 17]. Condition (d) was introduced by J. Mikusinski in [1]. If the supports of distributions f and g satisfy conditions (B1) or (B2), then they fulfill conditions (a)-(d).

In [9], two pairs of distributions S, T and f, g were given, which did not satisfy the conditions (B1) or (B2), but the convolution $S * T$ and $f * g$ existed and conditions (a)-(d) were satisfied.

The convolution product of distributions may be defined in a more general way without any restriction on the supports. The most well-known is given by Jones, see [16]. However, there still exist many pairs of distributions such that the convolution products do not exist in the sense of these definitions.

The method of neglecting appropriately defined infinite quantities was devised by Hadamard and the resulting finite value extracted from the divergent integral is usually referred to as the Hadamard finite part. Using the concepts of the neutrix and the neutrix limit due to van der Corput [4], Fisher gave the general principle for the discarding of unwanted infinite quantities from asymptotic expansions and this has been exploited in context of distributions, particularly in connection with convolution product and distributional multiplication see [6, 7, 8, 15, 18].

In order to introduce Fisher's definition of neutrix convolution product, we first of all let τ be a function in $\mathcal{D}$ satisfying the following properties :

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \le \tau(x) \le 1$,
(iii) $\tau(x) = 1$ for $|x| \le \frac{1}{2}$,
(iv) $\tau(x) = 0$ for $|x| \ge 1$.

The function τ_n is now defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \le n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases}$$

for $n = 1, 2, \ldots$.

Definition 3. *Let f and g be distributions in $\mathcal{D}\mathcal{D}'$ and let $f_n = f\tau_n$ for $n = 1, 2, \ldots$. Then the neutrix convolution $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_n * g\}$, provided that the limit h exists in the sense that*

$$\underset{n\to\infty}{\text{N}-\lim}\langle f_n * g, \varphi\rangle = \langle h, \varphi\rangle,$$

for all φ in $\mathcal{D}\mathcal{D}$, where N is the neutrix, see van der Corput [4], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range N'' the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n,\ \ln^r n \quad (\lambda > 0,\ r = 1, 2, \ldots)$$

and all functions which converge to zero in the usual sense as n tends to infinity.

In this definition the convolution product $f_n * g$ exists since the distribution f_n having bounded support. Note that because of the lack of symmetry in the definition of $f \circledast g$, the neutrix convolution is in general non-commutative.

The following two theorems were proved in [7], showing that the neutrix convolution is a generalization of the convolution.

Theorem 2. *Let f and g be distributions in $\mathcal{D}\mathcal{D}'$ satisfying either condition $(B1)$ or condition $(B2)$ of Definition 2. Then the neutrix convolution $f \circledast g$ exists and*

$$f \circledast g = f * g.$$

Theorem 3. *Let f and g be distributions in $\mathcal{D}\mathcal{D}'$ and suppose that $f \circledast g$ exists, then the neutrix convolution $f \circledast g'$ exists and*

$$(f \circledast g)' = f \circledast g'. \tag{2.1}$$

Note however that equation (1.1) does not necessarily hold for the neutrix convolution product and that $(f \circledast g)'$ is not necessarily equal to $f' \circledast g$.

In order to prove our next results we need to extend our set of negligible functions given in Definition 3 to also include finite linear sums of the function

$$n^r \operatorname{erf}[(x+n)^{1/2})], \qquad r = 1, 2, \ldots.$$

We now prove

Theorem 4. *The neutrix convolution $x^r \circledast \operatorname{erf}(x_+^{1/2})$ exists and*

$$x^r \circledast \operatorname{erf}(x_+^{1/2}) = \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} \frac{(-1)^i (2i)!}{2^{2i} i!} x^{r-i+1}, \tag{2.2}$$

for $r = 0, 1, 2, \ldots$.

Proof. We put $(x^r)_n = x^r \tau_n(x)$ for $n = 1, 2, \ldots$. Since $(x^r)_n$ has compact support, the classical convolution $(x^r)_n * \operatorname{erf}(x_+^{1/2})$ exists and

$$\begin{aligned}
(x^r)_n * \operatorname{erf}(x_+^{1/2}) &= \frac{2}{\sqrt{\pi}} \int_0^{x+n} (x-t)^r \operatorname{erf}(t^{1/2})\, dt + \\
&\qquad + \frac{2}{\sqrt{\pi}} \int_{x+n}^{x+n+n^{-n}} (x-t)^r \tau_n(x-t) \operatorname{erf}(t^{1/2})\, dt \\
&= I_1 + I_2.
\end{aligned}$$

It is easily seen that

$$\lim_{n \to \infty} I_2 = 0.$$

Further,

$$\begin{aligned}
I_1 &= \frac{2}{\sqrt{\pi}} \int_0^{x+n} (x-t)^r \int_0^{t^{1/2}} \exp(-u^2)\, du\, dt \\
&= \frac{2}{\sqrt{\pi}} \int_0^{(x+n)^{1/2}} \exp(-u^2) \int_{u^2}^{x+n} (x-t)^r\, dt\, du \\
&= \frac{2}{\sqrt{\pi}(r+1)} \int_0^{(x+n)^{1/2}} (x-u^2)^{r+1} \exp(-u^2)\, du - \frac{2(-n)^{r+1}}{\sqrt{\pi}(r+1)} \int_0^{(x+n)^{1/2}} \exp(-u^2)\, du \\
&= \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^i \operatorname{erf}_{2i}[(x+n)^{1/2})] x^{r-i+1} \\
&\qquad - \frac{(-n)^{r+1}}{r+1} \operatorname{erf}[(x+n)^{1/2}].
\end{aligned}$$

It follows easily from Lemma 1 and on noting that $\operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty \exp(-u^2)\,du = 1$, we have

$$\mathrm{N}-\lim_{n\to\infty} \operatorname{erf}_{2i}[(x+n)^{1/2}] = \frac{(2i)!}{2^{2i} i!},$$

and so

$$\mathrm{N}-\lim_{n\to\infty} I_1 = \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} \frac{(-1)^i (2i)!}{2^{2i} i!} x^{r-i+1}$$

for $i = 0, 1, 2, \ldots$. Equation (2.2) follows. □

Replacing x by $-x$ in equation (2.2), we get

Corollary 4. *The neutrix convolution* $x^r \circledast \operatorname{erf}(x_-^{1/2})$ *exists and*

$$x^r \circledast \operatorname{erf}(x_-^{1/2}) = -\frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} \frac{(2i)!}{2^{2i} i!} x^{r-i+1} \tag{2.3}$$

for $r = 0, 1, 2, \ldots$.

Corollary 5. *The neutrix convolution* $x^r \circledast \operatorname{erf}(|x|^{1/2})$ *exists and*

$$x^r \circledast \operatorname{erf}(|x|^{1/2}) = \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} \frac{[(-1)^i - 1](2i)!}{2^{2i} i!} x^{r-i+1} \tag{2.4}$$

for $r = 0, 1, 2, \ldots$.

Proof. Note that

$$x^r \circledast \operatorname{erf}(|x|^{1/2}) = x^r \circledast \operatorname{erf}(x_+^{1/2}) + x^r \circledast \operatorname{erf}(x_-^{1/2}).$$

Then equation (2.4) follows from Equations (2.2) and (2.3). □

Theorem 5. *The neutrix convolution* $x_+^r \circledast \operatorname{erf}(|x|^{1/2})$ *exists and*

$$\begin{aligned} x_+^r \circledast \operatorname{erf}(|x|^{1/2}) &= \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^i [x_+^{r-i+1} - (-1)^r x_-^{r-i+1}] \operatorname{erf}_{2i}(|x|^{1/2}) \\ &\quad - \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} \frac{(2i)!}{2^{2i} i!} x^{r-i+1} \end{aligned} \tag{2.5}$$

for $r = 0, 1, 2, \ldots$.

Proof. Using equation (2.2), we have

$$\begin{aligned} x_+^r \circledast \operatorname{erf}(|x|^{1/2}) &= x_+^r * \operatorname{erf}(x_+^{1/2}) + x_+^r \circledast \operatorname{erf}(x_-^{1/2}) \\ &= \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^i x_+^{r-i+1} \operatorname{erf}_{2i}(|x|^{1/2}) \\ &\quad + x_+^r \circledast \operatorname{erf}(x_-^{1/2}). \end{aligned} \tag{2.6}$$

Using equations (1.3) and (2.3), we have

$$\begin{aligned} x^r \circledast \operatorname{erf}(x_-^{1/2}) &= x_+^r \circledast \operatorname{erf}(x_-^{1/2}) + (-1)^r x_-^r * \operatorname{erf}(x_-^{1/2}) \\ &= x_+^r \circledast \operatorname{erf}(x_-^{1/2}) \\ &\quad + \frac{(-1)^r}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i x_-^{r-i+1}\operatorname{erf}_{2i}(|x|^{1/2}) \\ &= -\frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}\frac{(2i)!}{2^{2i}i!}x^{r-i+1}. \end{aligned} \tag{2.7}$$

Equation (2.5) now follows from equations (2.6) and (2.7). □

Replacing x by $-x$, we get

Corollary 6. *The neutrix convolution* $x_-^r \circledast \operatorname{erf}(|x|^{1/2})$ *exists and*

$$\begin{aligned} x_-^r \circledast \operatorname{erf}(|x|^{1/2}) &= \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i[x_-^{r-i+1} - (-1)^r x_+^{r-i+1}]\operatorname{erf}_{2i}(|x|^{1/2}) \\ &\quad + \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^{r-i}\frac{(2i)!}{2^{2i}i!}x^{r-i+1} \end{aligned}$$

for $r = 0, 1, 2, \ldots$.

Theorem 6. *The neutrix convolution* $x^r \circledast [x_+^{-1/2}\exp(-|x|)]$ *exists and*

$$x^r \circledast [x_+^{-1/2}\exp(-|x|)] = \frac{\sqrt{\pi}}{r+1}\sum_{i=0}^{r}\binom{r+1}{i}\frac{(-1)^i(r-i+1)(2i)!}{2^{2i}i!}x^{r-i}, \tag{2.8}$$

for $r = 0, 1, 2, \ldots$.

Proof. Differentiating equation (2.2) using equation (2.1), we get

$$\begin{aligned} [x^r \circledast \operatorname{erf}(x_+^{1/2})]' &= \frac{1}{\sqrt{\pi}}x^r \circledast [x_+^{-1/2}\exp(-|x|)] \\ &= \frac{1}{r+1}\sum_{i=0}^{r}\binom{r+1}{i}\frac{(-1)^i(r-i+1)(2i)!}{2^{2i}i!}x^{r-i}, \end{aligned}$$

and equation (2.8) follows for $r = 0, 1, 2, \ldots$. □

Replacing x by $-x$ in equation (2.8), we get

Corollary 7. *The neutrix convolution* $x^r \circledast [x_-^{-1/2}\exp(-|x|)]$ *exists and*

$$x^r \circledast [x_-^{-1/2}\exp(-|x|)] = \frac{\sqrt{\pi}}{r+1}\sum_{i=0}^{r}\binom{r+1}{i}\frac{(r-i+1)(2i)!}{2^{2i}i!}x^{r-i}, \tag{2.9}$$

for $r = 0, 1, 2, \ldots$.

Adding equations (2.8) and (2.9) we get

Corollary 8. *The neutrix convolution* $x^r \circledast [|x|^{-1/2}\exp(-|x|)]$ *exists and*

$$\begin{aligned} x^r \circledast [|x|^{-1/2}\exp(-|x|)] &= \\ &= \frac{\sqrt{\pi}}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}\frac{[1+(-1)^i](r-i+1)(2i)!}{2^{2i}i!}x^{r-i} \end{aligned}$$

for $r = 0, 1, 2, \dots$.

Taking equation (2.9) from equation (2.8), we get

Corollary 9. *The neutrix convolution* $x^r \circledast [\operatorname{sgn} x.|x|^{-1/2}\exp(-|x|)]$ *exists and*

$$x^r \circledast [\operatorname{sgn} x.|x|^{-1/2}\exp(-|x|)] = = \frac{\sqrt{\pi}}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}\frac{[(-1)^i-1](r-i+1)(2i)!}{2^{2i}i!}x^{r-i}$$

for $r = 0, 1, 2, \dots$.

Theorem 7. *The neutrix convolution* $x_+^r \circledast [\operatorname{sgn} x.|x|^{-1/2}\exp(-|x|)]$ *exists and*

$$\begin{aligned} x_+^r \circledast [\operatorname{sgn} x.|x|^{-1/2}\exp(-|x|)] &= \\ = \frac{\sqrt{\pi}}{r+1}\sum_{i=0}^{r}\binom{r+1}{i}(-1)^i(r-i+1)[x_+^{r-i}-(-1)^r x_-^{r-i}]\operatorname{erf}_{2i}(|x|^{1/2}) \\ +\frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i[x_+^{r+1}+(-1)^r x_-^{r+1}]\exp(-|x|) \\ -\frac{\sqrt{\pi}}{r+1}\sum_{i=0}^{r}\binom{r+1}{i}\frac{(r-i+1)(2i)!}{2^{2i}i!}x^{r-i} \end{aligned} \tag{2.10}$$

for $r = 0, 1, 2, \dots$.

Proof. Differentiating equation (2.5) using equation (2.1), we get

$$\begin{aligned} [x_+^r \circledast \operatorname{erf}(|x|^{1/2})]' &= \frac{1}{\sqrt{\pi}}x_+^r \circledast [\operatorname{sgn} x.|x|^{-1/2}\exp(-|x|)] \\ &= \frac{1}{r+1}\sum_{i=0}^{r}\binom{r+1}{i}(-1)^i(r-i+1)[x_+^{r-i}+(-1)^r x_-^{r-i}]\operatorname{erf}_{2i}(|x|^{1/2}) \\ &\quad +\frac{1}{\sqrt{\pi}(r+1)}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i[x_+^{r-i+1}-(-1)^r x_-^{r-i+1}]x^i\exp(-|u|) \\ &\quad -\frac{1}{r+1}\sum_{i=0}^{r}\binom{r+1}{i}\frac{(r-i+1)(2i)!}{2^{2i}i!}x^{r-i} \end{aligned}$$

and equation (2.10) follows. □

Replacing x by $-x$, we get:

Corollary 10. *The neutrix convolution* $x_-^r \circledast [\operatorname{sgn} x.|x|^{-1/2}\exp(-|x|)]$ *exists and*

$$\begin{aligned} x_-^r \circledast [\operatorname{sgn} x.|x|^{-1/2}\exp(-|x|)] &= \\ = -\frac{\sqrt{\pi}}{r+1}\sum_{i=0}^{r}\binom{r+1}{i}(-1)^i(r-i+1)[x_-^{r-i}+(-1)^r x_+^{r-i}]\operatorname{erf}_{2i}(|x|^{1/2}) \\ +\frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i[x_-^{r+1}-(-1)^r x_+^{r+1}]\exp(-|x|) \\ -\frac{\sqrt{\pi}}{r+1}\sum_{i=0}^{r}\binom{r+1}{i}\frac{(-1)^{r-i}(r-i+1)(2i)!}{2^{2i}i!}x^{r-i} \end{aligned}$$

for $r = 0, 1, 2, \ldots$.

For further related results, see [10, 11].

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