ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **5** (2016), no.2, 269–272
ISSN 1857-8365 printed version
ISSN 1857-8438 electronic version

A NOTE ON SUBORDINATIONS FOR CERTAIN ANALYTIC FUNCTIONS

SHIGEYOSHI OWA

Abstract. There are many results for subordinations of functions $f(z)$ which are analytic in the open unit disk $\mathbb{U}$ with $f(0) = 0$ and $f'(0) = 1$. The object of the present paper is to discuss some interesting conditions for $f(z)$ that satisfy some subordinations in $\mathbb{U}$.

1. Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n ,$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ with $f(0) = 0$ and $f'(0) = 1$. If $f(z) \in \mathcal{A}$ satisfies $f(z_1) \neq f(z_2)$ for any $z_1 \in \mathbb{U}$ and $z_2 \in \mathbb{U}$ with $z_1 \neq z_2$, then $f(z)$ is said to be univalent in $\mathbb{U}$ and denoted by $f(z) \in \mathcal{S}$. If $f(z) \in \mathcal{A}$ satisfies the following inequality:

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}) ,$$

for some real α $(0 \leq \alpha < 1)$, then we say that $f(z)$ is starlike of order α in $\mathbb{U}$ and denoted by $f(z) \in \mathcal{S}^*(\alpha)$. If $\alpha = 0$, then we write that $\mathcal{S}^*(0) \equiv S^*$.

Let $f(z)$ and $g(z)$ be in the class $\mathcal{A}$. Then $f(z)$ is said to be subordinate to $g(z)$ if there exists an analytic function $w(z)$ in $\mathbb{U}$ satisfying $w(0) = 0, |w(z)| < 1\, (z \in \mathbb{U})$ and $f(z) = g(w(z))$. We denote this subordination by

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}) .$$

The basic tool for our paper is the following lemma due to Miller and Mocanu [2], [3] (also, due to Jack [1]).

Lemma 1. *Let $w(z)$ be analytic in $\mathbb{U}$ with $w(0) = 0$. Then, if $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathbb{U}$, then we have that*

$$z_0 w'(z_0) = k w(z_0) ,$$

where $k \geq 1$ is a real number.

2010 *Mathematics Subject Classification.* Primary 30C45.

Key words and phrases. Analytic function, subordination, starlike function of order α.

2. Main results

Let us consider a function $g(z)$ given by

$$g(z) = \frac{\alpha - z}{\alpha(1-z)} \qquad (z \in \mathbb{U})$$

for some real $\alpha > 0$ and $\alpha \neq 1$. Then $g(0) = 1$ and $g(z)$ is analytic in $\mathbb{U}$. Consider $z = e^{i\theta}$ for $g(z)$. Then

$$\operatorname{Re} g(z) = \operatorname{Re}\left(\frac{1}{\alpha} + \frac{\alpha-1}{\alpha}\frac{1}{(1-\cos\theta) - i\sin\theta}\right) = \frac{1+\alpha}{2\alpha}.$$

Therefore, if $0 < \alpha < 1$, then $\operatorname{Re} g(z) < \dfrac{1+\alpha}{2\alpha}$ $(z \in \mathbb{U})$ and if $\alpha > 1$, then $\operatorname{Re} g(z) > \dfrac{1+\alpha}{2\alpha}$ $(z \in \mathbb{U})$.

Now, we derive

Theorem 1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{1+3\alpha}{2\alpha(1+\alpha)} \qquad (z \in \mathbb{U}), \tag{2.1}$$

for $0 < \alpha < 1$, then

$$\frac{zf'(z)}{f(z)} \prec \frac{\alpha - z}{\alpha(1-z)} \qquad (z \in \mathbb{U}).$$

If $f(z) \in \mathcal{A}$ satisfies

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{1+3\alpha}{2\alpha(1+\alpha)} \qquad (z \in \mathbb{U}), \tag{2.2}$$

for $\alpha > 1$, then

$$\frac{zf'(z)}{f(z)} \prec \frac{\alpha - z}{\alpha(1-z)} \qquad (z \in \mathbb{U}).$$

Proof. We consider a function $w(z)$ which is analytic in $\mathbb{U}$, $w(0) = 0$, and given by

$$\frac{zf'(z)}{f(z)} = \frac{\alpha - w(z)}{\alpha(1-w(z))},$$

where $f(z)$ satisfies conditions in the theorem. It follows that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{\alpha - w(z)}{\alpha(1-w(z))} - \frac{zw'(z)}{\alpha - w(z)} + \frac{zw'(z)}{1-w(z)}.$$

Let us suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$Max_{|z| \le |z_0|}|w(z)| = |w(z_0)| = 1.$$

Then, applying Lemma 1, we can write that

$$z_0 w'(z_0) = k w(z_0) \qquad (k \geq 1).$$

Noting that $w(z_0) = e^{i\theta}$, we know that

$$\begin{aligned}
\operatorname{Re}\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) &= \operatorname{Re}\left\{\frac{\alpha - w(z_0)}{\alpha(1-w(z_0)} - \frac{z_0 w'(z_0)}{\alpha - w(z_0)} + \frac{z_0 w'(z_0)}{1-w(z_0)}\right\} \\
&= \operatorname{Re}\left(\frac{\alpha - e^{i\theta}}{\alpha(1-e^{i\theta})} - \frac{ke^{i\theta}}{\alpha - e^{i\theta}} + \frac{ke^{i\theta}}{1-e^{i\theta}}\right) \\
&= \frac{1+\alpha}{2\alpha} + \frac{k(1-\alpha\cos\theta)}{1+\alpha^2 - 2\alpha\cos\theta} - \frac{k}{2}.
\end{aligned}$$

Let us define the function $p(t)$ by

$$p(t) = \frac{1-\alpha t}{1+\alpha^2-2\alpha t} \qquad (t=\cos\theta)\,.$$

Then, we have that

$$p'(t) = \frac{\alpha(1+\alpha)(1-\alpha)}{(1+\alpha^2-2\alpha t)^2}\,.$$

Thus, $p'(t) > 0$ for $0<\alpha<1$ and $p'(t)<0$ for $\alpha>1$. This gives us that

$$\begin{aligned}\mathrm{Re}\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right) &\geq \frac{1+\alpha}{2\alpha}+\frac{k}{1+\alpha}-\frac{k}{2}\\ &= \frac{1+\alpha}{2\alpha}+\frac{k(1-\alpha)}{2(1+\alpha)} \geq \frac{1+3\alpha}{2\alpha(1+\alpha)}\,,\end{aligned}$$

for $0<\alpha<1$, and that

$$\begin{aligned}\mathrm{Re}\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right) &\leq \frac{1+\alpha}{2\alpha}+\frac{k}{1+\alpha}-\frac{k}{2}\\ &= \frac{1+\alpha}{2\alpha}+\frac{k(1-\alpha)}{2(1+\alpha)} \leq \frac{1+3\alpha}{2\alpha(1+\alpha)}\,,\end{aligned}$$

for $\alpha>1$. This contradicts the conditions (2.1) and (2.2) of the theorem. Therefore, there is no $w(z)$ such that $w(0)=0$ and $|w(z_0)|=1$ for any $z_0\in\mathbb{U}$. It follows that $|w(z)|<1$ for all $z\in\mathbb{U}$. With this fact, we have that

$$|w(z)| = \left|\frac{\alpha\left(1-\frac{zf'(z)}{f(z)}\right)}{1-\alpha\frac{zf'(z)}{f(z)}}\right| < 1 \qquad (z\in\mathbb{U})\,.$$

It follows that

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) < \frac{1+\alpha}{2\alpha}\ (z\in\mathbb{U})\,,$$

for $0<\alpha<1$ and that

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \frac{1+\alpha}{2\alpha}\ (z\in\mathbb{U})\,,$$

for $\alpha>1$. Consequently, we have that

$$\frac{zf'(z)}{f(z)} \prec \frac{\alpha-z}{\alpha(1-z)} \qquad (z\in\mathbb{U})\,.$$

Finally, if we take a function $f(z)$ given by

$$f(z) = z(1-z)^{\frac{1-\alpha}{\alpha}} \qquad (z\in\mathbb{U})\,, \tag{2.3}$$

then we know that

$$\frac{zf'(z)}{f(z)} = \frac{\alpha-z}{\alpha(1-z)}\,.$$

Thus, the result is sharp for $f(z)$ which is defined by (2.3). □

Example 1. Letting $\alpha=\dfrac{1}{3}$ and considering

$$f(z) = z(1-z)^2 = z-2z^2+z^3\,,$$

we have that

$$\frac{zf'(z)}{f(z)} = \frac{1-3z}{1-z} \qquad (z\in\mathbb{U})\,,$$

which gives us that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < 2 \qquad (z \in \mathbb{U}).$$

Example 2. Letting $\alpha = 2$ and considering

$$f(z) = \frac{z}{\sqrt{1-z}} = z + \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^{n-1}(n-1)!} z^n ,$$

we have that

$$\frac{zf'(z)}{f(z)} = \frac{z(3-2z)}{2(1-z)} \qquad (z \in \mathbb{U}),$$

which gives us that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > -\frac{5}{4} \qquad (z \in \mathbb{U}).$$

References

[1] I.S.Jack: *Functions starlike and convex of order α*, J. London Math. Soc., **3**(1971), 469 – 474.

[2] S.S.Miller, P.T.Mocanu: *Second-order differential inequalities in the complex plane*, J. Math. Anal. Appl., **65**(1978), 289 – 305.

[3] S.S.Miller, P.T.Mocanu: *Differential Subordination: Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, No. **225**, Marcel Dekker Incorporated, New York and Basel, 2000.

Department of Mathematics
Faculty of Education
Yamato University
Katayama 2-5-1, Suita, Osaka 564-0082
Japan
E-mail address: owa.shigeyoshi@yamato-u.ac.jp